“Practical rules” and indeterminate forms, cont’d

Compute the following limit

$$\lim_{n\to\infty}(n^2 - n)$$

Solution

We have an indeterminate form $+\infty - \infty$.

To find the limit, we put in evidence the highest power:

$$n^2 - n \quad = \quad \underbrace{n^2}_{\to+\infty}\overbrace{\left(1 - \frac{1}{n}\right)}^{\to 1} \to +\infty$$

We use the same technique to compute the limit of polynomials of higher degree:

$$n^5 - n^3 + 2n - 5 \quad = \quad n^5\left(1 - \frac{1}{n^2} + \frac{2}{n^4} - \frac{5}{n^5}\right) \to +\infty$$

“Practical rules” and indeterminate forms, cont’d

Compute the following limit

$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$$

Solution

We have $\sqrt{n^2+n}\to+\infty$ so the limit is of the form $+\infty-\infty$ and is undetermined.

To solve it, we use the following trick:

$$\begin{aligned}
\left(\sqrt{n^2+n}-n\right) &= \frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n}\\
&= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\frac{n}{\sqrt{n^2+n}+n}\\
&= \frac{n}{n\left(\sqrt{1+\frac{1}{n}}+1\right)}=\frac{1}{\left(\sqrt{1+\frac{1}{n}}+1\right)}\to\frac{1}{2}
\end{aligned}$$

"Practical rules" and indeterminate forms, cont'd

Compute the following limit

$$\lim_{n\to\infty}\left(\sqrt{n^2+100n}-n\right)$$

Solution

As in the previous exercise, we have an indeterminate form $+\infty-\infty$.

To find the limit, we use the same trick:

$$\begin{aligned}
\left(\sqrt{n^2+100n}-n\right) &= \frac{\left(\sqrt{n^2+100n}-n\right)\left(\sqrt{n^2+100n}+n\right)}{\sqrt{n^2+100n}+n} \\
&= \frac{n^2+100n-n^2}{\sqrt{n^2+100n}+n} = \frac{100n}{\sqrt{n^2+100n}+n} \\
&= \frac{100n}{n\left(\sqrt{1+\frac{100}{n}}+1\right)} = \frac{100}{\left(\sqrt{1+\frac{100}{n}}+1\right)} \to 50
\end{aligned}$$

“Practical rules” and indeterminate forms, cont’d

$$\lim_{n\to\infty} \frac{3n^7 - 8n^6 + 15n^3 - 10n}{12n - 4n^7} = ?$$

Solution

$$\begin{aligned} \frac{3n^7 - 8n^6 + 15n^3 - 10n}{12n - 4n^7} &= \frac{n^7\left(3 - 8n^{-1} + 15n^{-4} - 10n^{-6}\right)}{n^7\left(12n^{-6} - 4\right)} \\ &= \frac{\left(3 - 8\left(\frac{1}{n}\right) + 15\left(\frac{1}{n}\right)^4 - 10\left(\frac{1}{n}\right)^6\right)}{\left(12\left(\frac{1}{n}\right)^6 - 4\right)} \end{aligned}$$

The terms in $1/n$ tend to zero. Thus, we have:

$$\lim_{n\to\infty} \frac{3n^7 - 8n^6 + 15n^3 - 10n}{12n - 4n^7} = -\frac{3}{4}$$

“Practical rules” and indeterminate forms, cont’d

$$\lim_{n\to\infty} \frac{3n^8 - 8n^6 + 15n^3 - 10n}{12n - 4n^7} = ?$$

Solution

$$\begin{aligned}
\frac{3n^8 - 8n^6 + 15n^3 - 10n}{12n - 4n^7} &= \frac{n^8\left(3 - 8n^{-2} + 15n^{-5} - 10n^{-7}\right)}{n^7\left(12n^{-6} - 4\right)} \\
&= n \times \frac{\left(3 - 8\left(\frac{1}{n}\right)^2 + 15\left(\frac{1}{n}\right)^5 - 10\left(\frac{1}{n}\right)^7\right)}{\left(12\left(\frac{1}{n}\right)^6 - 4\right)}
\end{aligned}$$

We get a limit of the form $(+\infty) \times \alpha$, with $\alpha = -\frac{3}{4}$ and thus:

$$\lim_{n\to\infty} \frac{3n^8 - 8n^6 + 15n^3 - 10n}{12n - 4n^7} = -\infty$$

"Practical rules" and indeterminate forms, cont'd

$$\lim_{n\to\infty} \frac{-5n^6 + 15n^3 - 10n}{12n - 4n^7} = ?$$

Solution

$$\begin{aligned}
\frac{-5n^6 + 15n^3 - 10n}{12n - 4n^7} &= \frac{n^6\left(-5 + 15n^{-3} - 10n^{-5}\right)}{n^7\left(12n^{-6} - 4\right)} \\
&= \frac{1}{n} \times \frac{\left(-5 + 15\left(\frac{1}{n}\right)^3 - 10\left(\frac{1}{n}\right)^5\right)}{\left(12\left(\frac{1}{n}\right)^6 - 4\right)}
\end{aligned}$$

We get a limit of the form $0 \times \alpha$, with $\alpha = \frac{5}{4}$ and thus:

$$\lim_{n\to\infty} \frac{-5n^6 + 15n^3 - 10n}{12n - 4n^7} = 0$$

The comparison theorem

Theorem (Comparison Theorem)

Let $(a_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ *be three sequences such that*

$$a_n \leq s_n \leq b_n$$

The following three results hold:

- *If* $a_n \to \ell$ *and* $b_n \to \ell$ *then also* $s_n \to \ell$
- *If* $a_n \to +\infty$ *then also* $s_n \to +\infty$
- *If* $b_n \to -\infty$ *then also* $s_n \to -\infty$

The notable limit: $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$

Theorem

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1$$

Proof:

Notice that we have an indeterminate form $+\infty \times 0$.

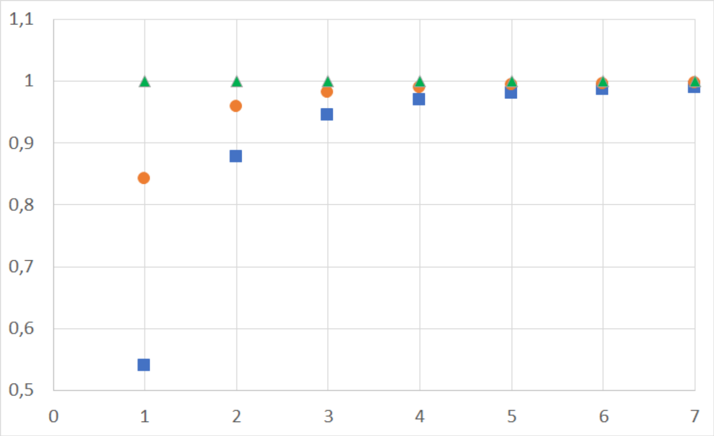
We exploit the following **key inequality**:

$$\cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1$$

Let's set $a_n = \cos\left(\frac{1}{n}\right)$, $s_n = n \sin\left(\frac{1}{n}\right)$ and $b_n = 1$.

Since $a_n \to 1$ and $b_n \to 1$, the comparison theorem implies that $s_n \to 1$.

The notable limit: $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$



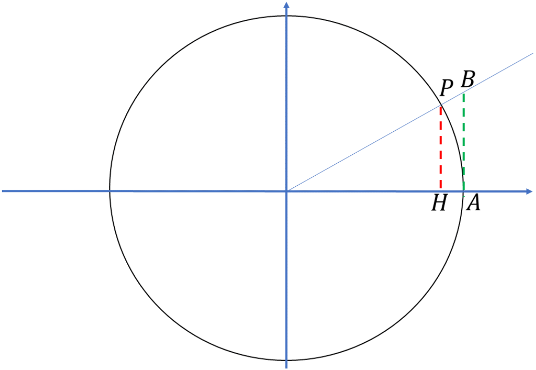
- Green sequence $b_n = 1$
- Orange sequence $s_n = n \sin\left(\frac{1}{n}\right)$
- Blue sequence $a_n = \cos\left(\frac{1}{n}\right)$

The key inequality: $\cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1$

For small angles θ, the length of the arc $\overset{\frown}{PA}$ is between the length of segments $\overline{PH}$ and $\overline{BA}$:

$$\overline{PH} \leq \overset{\frown}{PA} \leq \overline{BA}$$

$$\sin(\theta) \leq \theta \leq \tan(\theta)$$



The key inequality: $\cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1$

Dividing all members by $\sin(\theta)$ leads to

$$1 \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)}$$

Taking the reciprocals we obtain

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1$$

Finally, we set $\theta = \frac{1}{n}$ and get

$$\cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1$$

This is the end of the proof.

Notable limits that follow from $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$

- $\displaystyle \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1$
- $\displaystyle \lim_{n\to\infty} n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1}{2}$
- If $s_n \to 0$, then $\displaystyle \lim_{n\to\infty} s_n \sin\left(\frac{1}{s_n}\right) = 1$
- If $s_n \to 0$, then $\displaystyle \lim_{n\to\infty} (s_n)^2 \left(1 - \cos\left(\frac{1}{s_n}\right)\right) = \frac{1}{2}$

Notable limits

Let $a \in \mathbb{R}$, $a \neq 1$. Then:

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ +\infty & \text{if } a > 1 \\ \nexists & \text{if } a \leq -1 \end{cases}$$

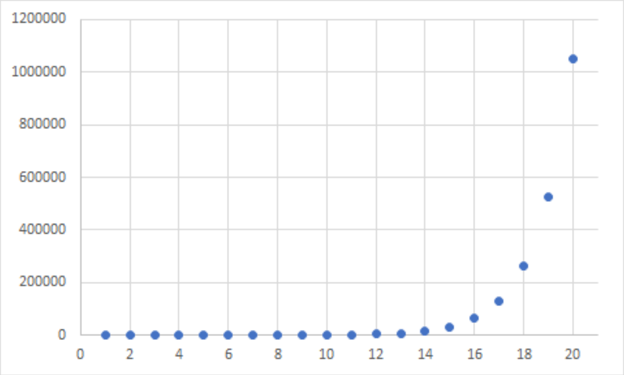
Notice that for the case $a = -1$ we obtain the well known limit

$$\lim_{n\to+\infty} (-1)^n$$

which does not exist!

Examples

Consider the sequence $(s_n)_{n \in \mathbb{N}}$ with $s_n = 2^n$.



It is easy to see that

$$\lim_{n \to \infty} 2^n = +\infty$$

Exercise Prove the above limit using the definition.

Examples

Consider the sequences $(s_n)_{n\in\mathbb{N}}$ with $s_n = \left(\frac{1}{2}\right)^n$ and $(p_n)_{n\in\mathbb{N}}$ with $p_n = \left(-\frac{1}{2}\right)^n$
.

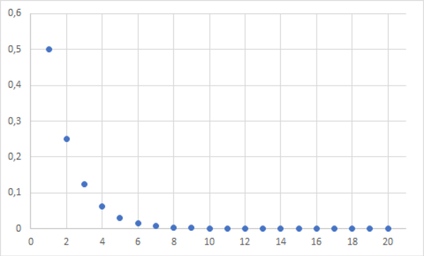


Figure: $s_n = \left(\frac{1}{2}\right)^n$

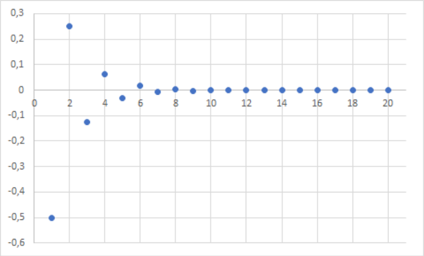


Figure: $p_n = \left(-\frac{1}{2}\right)^n$

Then, we get that

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = 0$$

Exercise Prove the above limits using the definition.

Examples

Consider the sequences $(s_n)_{n\in\mathbb{N}}$ with $s_n = (-2)^n$.

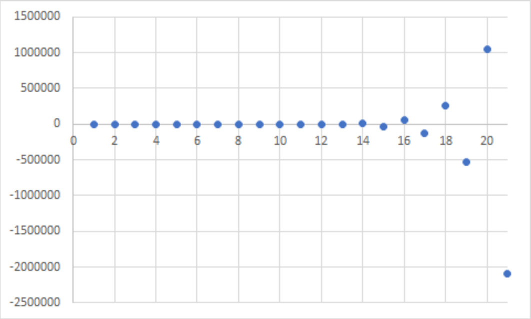


Figure: $s_n = (-2)^n$

n	s_n
1	-2
2	4
3	-8
4	16
5	-32
6	64
7	-128
8	256
9	-512
10	1024
11	-2048
12	4096
13	-8192
14	16384
15	-32768
16	65536
17	-131072
18	262144
19	-524288
20	1048576
21	-2097152

Then, we get that

$$\nexists \lim_{n\to\infty} (-2)^n$$

Exercise Prove that the above limit does not exist.

The factorial

Definition

Let $n \in \mathbb{N}$. The factorial is defined as:

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

and $0! = 1$

Examples

- $1! = 1$
- $2! = 1 \cdot 2 = 2$
- $3! = 1 \cdot 2 \cdot 3 = 6$
- $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$
- $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

Notable limits, cont'd

The following notable limits hold:

- $\forall b > 0$, $\lim_{n\to\infty} \frac{log_a(n)}{n^b} = 0$
- $\forall b > 0$ and $a > 1$, $\lim_{n\to\infty} \frac{n^b}{a^n} = 0$
- $\forall a > 1$, $\lim_{n\to\infty} \frac{a^n}{n!} = 0$
- $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Note that the above results imply:

- $\forall b > 0$, $\lim_{n\to\infty} \frac{n^b}{log_a(n)} = \lim_{n\to\infty} \frac{1}{\frac{log_a(n)}{n^b}} = +\infty$
- $\forall b > 0$ and $a > 1$, $\lim_{n\to\infty} \frac{a^n}{n^b} = \lim_{n\to\infty} \frac{1}{\frac{n^b}{a^n}} = +\infty$
- $\forall a > 1$, $\lim_{n\to\infty} \frac{n!}{a^n} = \lim_{n\to\infty} \frac{1}{\frac{a^n}{n!}} = +\infty$
- $\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{1}{\frac{n!}{n^n}} = +\infty$

Notable limits, cont'd

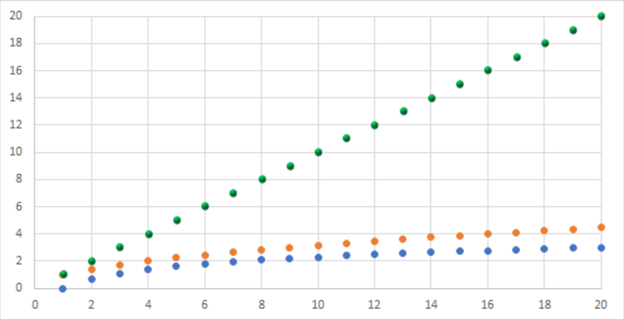
The above results can be interpreted in terms of "speed" of divergence:

- $\lim_{n\to\infty} \frac{log_a(n)}{n^b} = 0 \quad \Rightarrow$ The power diverges faster than the logarithm
- $\lim_{n\to\infty} \frac{n^b}{a^n} = 0 \quad \Rightarrow$ The exponential diverges faster than the power
- $\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad \Rightarrow$ The factorial diverges faster than the exponential
- $\lim_{n\to\infty} \frac{n!}{n^n} = 0 \quad \Rightarrow$ n^n diverges faster than the factorial

At $+\infty$ we have the following "order of infinity"

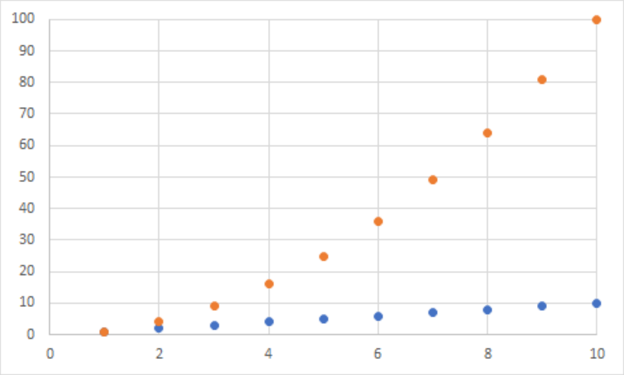
$$\log(n) < \sqrt[3]{n} < \sqrt{n} < n < n^2 < n^3 < \cdots < 2^n < e^n < 3^n < \cdots < n! < n^n$$

Examples



- $a_n = \log(n)$
- $b_n = \sqrt{n}$
- $c_n = n$

Examples



- $a_n = n$
- $b_n = n^2$

$\log(n)$	$\sqrt{n}$	n	n^2	2^n	$n!$
0	1	1	1	2	1
0,69	1,41	2	4	4	2
1,09	1,73	3	9	8	6
1,38	2	4	16	16	24
1,60	2,23	5	25	32	120
1,79	2,44	6	36	64	720
1,94	2,64	7	49	128	5040
2,07	2,82	8	64	256	40320
2,19	3	9	81	512	362880
2,30	3,16	10	100	1024	3628800
2,39	3,31	11	121	2048	39916800
2,48	3,46	12	144	4096	4,79E+08
2,56	3,60	13	169	8192	6,23E+09
2,63	3,74	14	196	16384	8,72E+10
2,70	3,87	15	225	32768	1,31E+12
2,77	4	16	256	65536	2,09E+13
2,83	4,12	17	289	131072	3,56E+14
2,89	4,24	18	324	262144	6,4E+15
2,94	4,35	19	361	524288	1,22E+17
2,99	4,47	20	400	1048576	2,43E+18

Examples, cont'd

Examples

$$\lim_{n\to\infty} \frac{3^n + \log_2(n)}{n!} = \lim_{n\to\infty} 3^n \frac{1 + \overbrace{\frac{\log_2(n)}{3^n}}^{\to 0}}{n!} = 0$$

$$\lim_{n\to\infty} \frac{n^{\frac{1}{10}}}{\log_2(n^{100})} = \lim_{n\to\infty} \frac{1}{100} \overbrace{\frac{n^{\frac{1}{10}}}{\log_2(n)}}^{\to +\infty} = +\infty$$

$$\lim_{n\to\infty} \frac{n! + n^{200000}}{n^n} = n! \frac{1 + \overbrace{\frac{n^{200000}}{n!}}^{\to 0}}{n^n} = 0$$

Put always in evidence the term diverging faster!

Increasing/Decreasing sequences

Definition

A sequence $(s_n)_{n\in\mathbb{N}}$ is said to be

- **strictly increasing** if $s_n < s_{n+1}$ for all n.
- **increasing** if $s_n \leq s_{n+1}$ for all n.
- **strictly decreasing** if $s_n > s_{n+1}$ for all n.
- **decreasing** if $s_n \geq s_{n+1}$ for all n.

Theorem

(i) *Let $(s_n)_{n\in\mathbb{N}}$ be increasing. Then $s_n \to \ell$ if and only if $\exists M$ such that $s_n \leq M$, for all n.*

(ii) *Let $(s_n)_{n\in\mathbb{N}}$ be decreasing. Then $s_n \to \ell$ if and only if $\exists M$ such that $s_n \geq M$, for all n.*

The Euler sequence

Definition

The sequence $(s_n)_{n\in\mathbb{N}}$ with

$$s_n = \left(1 + \frac{1}{n}\right)^n$$

is called the Euler sequence.

Sequences: the Euler sequence

Properties

- Euler sequence is defined by
$$s_n = \left(1 + \frac{1}{n}\right)^n .$$

It can be proved that:

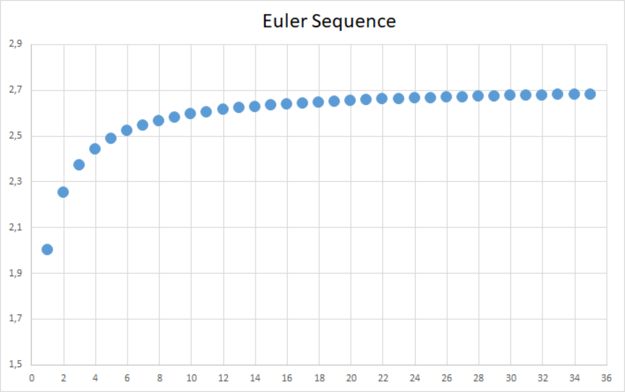
- $s_n < s_{n+1}$ for all n, i.e. Euler sequence is (strictly) **increasing** (For a rigorous proof see Lecture Notes).
- $2 \leq s_n < 3$ for all n, i.e. Euler sequence is bounded from **above** and **below**.

In particular

$$\text{Increasing } + \text{ bounded from above } \Rightarrow \text{ There exists } \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$$

Besides, $2 < e < 3$. The number e is called the **Euler's number** or **Neper's number**.

The Euler sequence



The Euler sequence

Exercise Compute

$$\lim_{n\to\infty}\left(1+\frac{7}{n^2}\right)^{n^2}.$$

Solution. Recall that $\lim_{m\to+\infty}\left(1+\frac{1}{m}\right)^m = e$.
In order to recover this expression we set $\frac{1}{m}=\frac{7}{n^2}$, then

$$m=\frac{n^2}{7} \qquad 7m=n^2 \qquad \text{and} \qquad n\to\infty \text{ implies } m\to\infty.$$

Then we make a change of variable and get

$$\lim_{n\to\infty}\left(1+\frac{7}{n^2}\right)^{n^2} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{7\,m} \lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^m\right)^7 = e^7.$$

The Geometric sum

Let $x \in \mathbb{R}$ and consider the following sum:

$$S_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$$

Recall that $S_n(x)$ can be re-written using the "summation" symbol:

$$S_n(x) = \sum_{k=0}^{n} x^k$$

The above sum is called **Geometric sum**.

Question: What is the result of this sum?

The Geometric sum

The result can be easily found. Recall that:

$$S_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$$

Multiply both sides of the above equation by x:

$$xS_n(x) = x + x^2 + x^3 + x^4 + \cdots + x^{n+1}$$

Now, subtract the first equation from the second:

$$xS_n(x) - S_n(x) = x^{n+1} - 1$$

and therefore:

$$S_n(x) = \frac{x^{n+1} - 1}{x - 1}$$

Thus, we can write:

$$S_n(x) = \sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}$$

The Geometric sum: examples

Examples

- Assume that we want to compute:

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{10}$$

  In this case, $x = \frac{1}{2}$ and $n = 10$. Thus:

$$S_{10}\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^{10+1} - 1}{\frac{1}{2} - 1} = 1.9990234375$$

- Now, let's compute:

$$1 + 3 + 3^2 + 3^3 + 3^4 + 3^5$$

  In this case, $x = 3$ and $n = 5$. Thus:

$$S_5(3) = \frac{3^{5+1} - 1}{3 - 1} = 364$$

The Geometric series

Assume now that we want to compute $S_n(x)$ when n becomes very large.

Observe that $S_n(x)$ is a sequence, and thus we can try to compute its limit, that we denote by:

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} S_n(x)$$

This limit is called **Geometric series**.

Using the result of the previous slide, we have:

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \frac{x^{n+1} - 1}{x - 1}$$

We know how to compute this limit: it is the limit of the power function.

The Geometric series

Remind the following result:

Let $x \in \mathbb{R}$, $x \neq 1$. Then:

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } |x| < 1 \\ +\infty & \text{if } x > 1 \\ \nexists & \text{if } x \leq -1 \end{cases}$$

Thus, the limit we want to compute is given by:

$$\sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x} & \text{if } |x| < 1 \\ +\infty & \text{if } x > 1 \\ \nexists & \text{if } x \leq -1 \end{cases}$$

What if $x = 1$? Observe that $\sum_{k=0}^{n} 1^k = \underbrace{1 + \cdots + 1}_{n+1 \text{ times}} = n + 1 \to +\infty$. It follows that $\sum_{k=0}^{\infty} x^k = +\infty$ for $x = 1$.

The Geometric series: examples

Examples

$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{1}{1-\frac{1}{2}} = 2$$

$$\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k = 1 - \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 + \cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$$

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 2^2 + 2^3 + \cdots = +\infty$$

$$\sum_{k=0}^{\infty} (-2)^k = 1 - 2 + 2^2 - 2^3 + \cdots = \nexists$$

Limits of functions: the intuition

Consider the following function:

$$f(x) = \frac{2x^2 - 8}{x - 2}$$

The domain is $D = \mathbb{R} \setminus \{2\}$.

The function is not defined in $x = 2$, but we can compute its value in a neighbourhood of $x = 2$:

x approaching 2 from right	
$x = 2.1$	$f(x) = 8.2$
$x = 2.01$	$f(x) = 8.02$
$x = 2.001$	$f(x) = 8.002$
$x = 2.0004$	$f(x) = 8.0008$

x approaching 2 from left	
$x = 1.9$	$f(x) = 7.8$
$x = 1.95$	$f(x) = 7.9$
$x = 1.995$	$f(x) = 7.99$
$x = 1.9991$	$f(x) = 7.9982$

Limits of functions: the intuition, cont'd

The function $f(x)$ gets closer and closer to the value 8 as x gets closer and closer to 2.

Put in other words, the distance between $f(x)$ and 8 can be made as small as we want by "choosing" an x sufficiently close to 2.

Thus, for any $\epsilon > 0$ arbitrarily small, we can find a $\delta > 0$ such that:

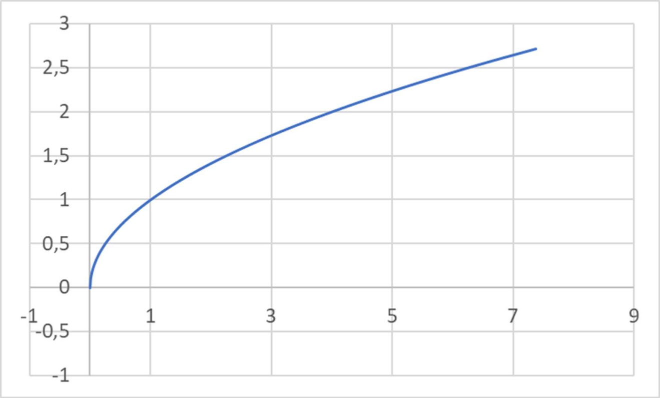
$$|f(x) - 8| < \varepsilon$$

provided that

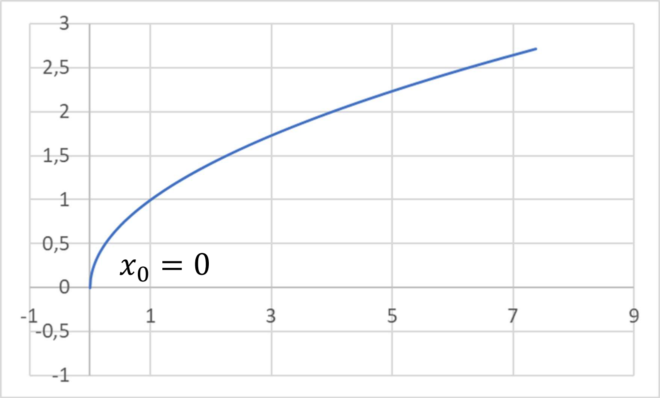
$$|x - 2| < \delta$$

Remark: Note that it must be possible to approach x_0 as much as we want through the points of the domain.

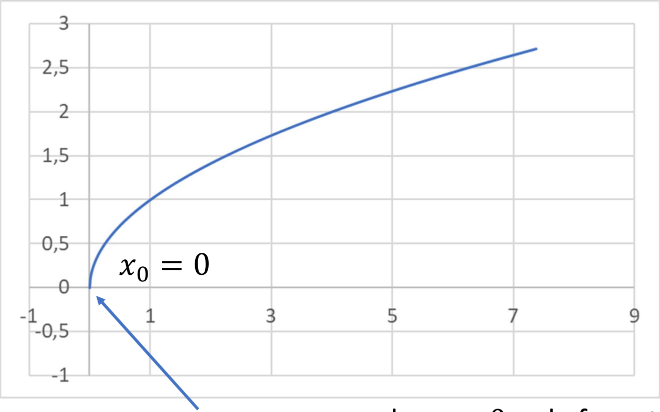
$$f(x) = \sqrt{x}$$



$$f(x) = \sqrt{x}$$

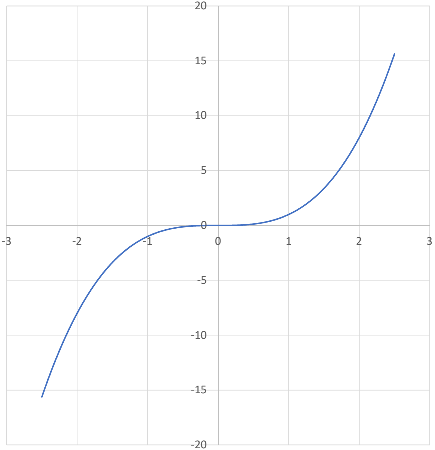


$f(x) = \sqrt{x}$

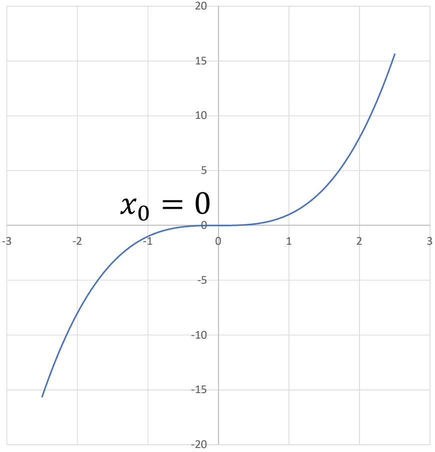


x can approach $x_0 = 0$ only from the right

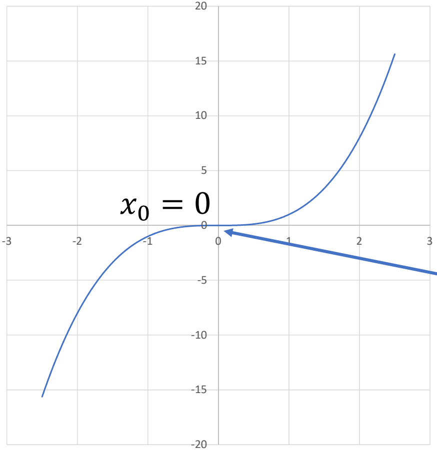
$$f(x) = x^3$$



$$f(x) = x^3$$



$f(x) = x^3$



$x_0 = 0$

x can approach 0 from the left and from the right.

Importance of limits

Limits serve to answer the following questions:

- How does a function behave when x gets closer and closer to a point x_0?
- How does a function behave when x gets larger and larger?

Interesting cases are particular points of the domain:

- Points where the function is not defined, i.e. **outside its domain**, but on the boundary
- $+\infty / -\infty$, if the domain is **unbounded from above/below**

We will consider four cases:

- “Finite limit at a point”: $\lim_{x \to x_0} f(x) = L$
- “Finite limit at infinity”: $\lim_{x \to \pm\infty} f(x) = L$
- “Infinite limit at a point”: $\lim_{x \to x_0} f(x) = \pm\infty$
- “Infinite limit at infinite”: $\lim_{x \to \pm\infty} f(x) = \pm\infty$

"Finite limit at a point"

Let's first consider the case $\lim_{x \to x_0} f(x) = L$.

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0} f(x) = L$$

if

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \forall x \in D : |x - x_0| < \delta,\ x \neq x_0 \ \Rightarrow \ |f(x) - L| < \epsilon$$

Remark: Note that the function is not required to be defined in x_0!

"Finite limit at a point", cont'd

Note that the condition $|x - x_0| < \delta$ can be rewritten as:

$$|x - x_0| < \delta \iff -\delta < x - x_0 < \delta \iff x_0 - \delta < x < x_0 + \delta \iff x \in N_\delta(x_0)$$

Similarly, the condition $|f(x) - L| < \epsilon$ can be rewritten as:

$$|f(x) - L| < \epsilon \iff -\epsilon < f(x) - L < \epsilon \iff L - \epsilon < f(x) < L + \epsilon \iff f(x) \in N_\epsilon(L)$$

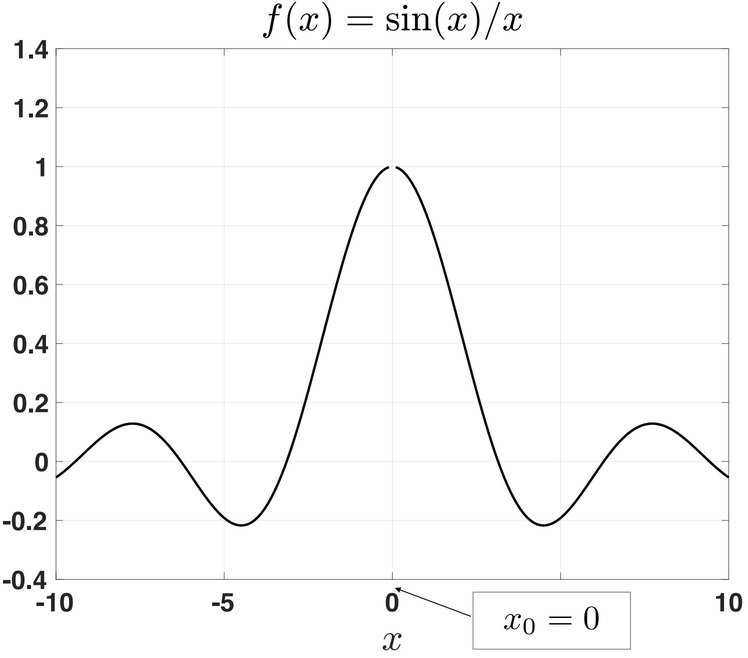
Thus, an equivalent definition of limit of a function is:

Definition

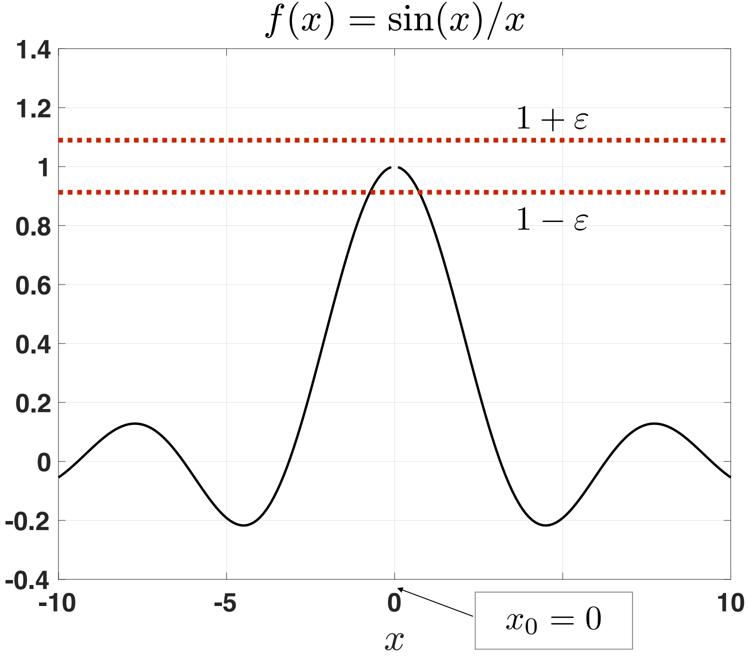
Let $f : D \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

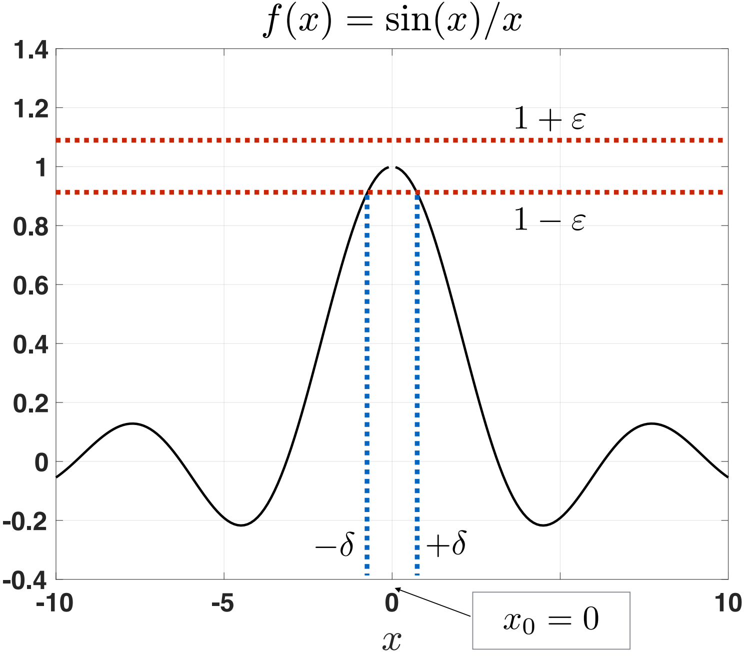
$$\lim_{x \to x_0} f(x) = L$$

if, for any neighborhood $N_\epsilon(L)$, we can find a neighborhood $N_\delta(x_0)$, such that, $\forall x \in D$: $x \in N_\delta(x_0)$, $x \neq x_0$, $f(x) \in N_\epsilon(L)$.



$f(x) = \sin(x)/x$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-10
-5
0
10
x
$x_0 = 0$


$f(x) = \sin(x)/x$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
$1+\varepsilon$
$1-\varepsilon$
-10
-5
0
10
x
$x_0 = 0$



$f(x) = \sin(x)/x$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-10
-5
0
5
10
x
$1+\varepsilon$
$1-\varepsilon$
$-\delta$
$+\delta$
$x_0 = 0$

Piecewise-defined functions

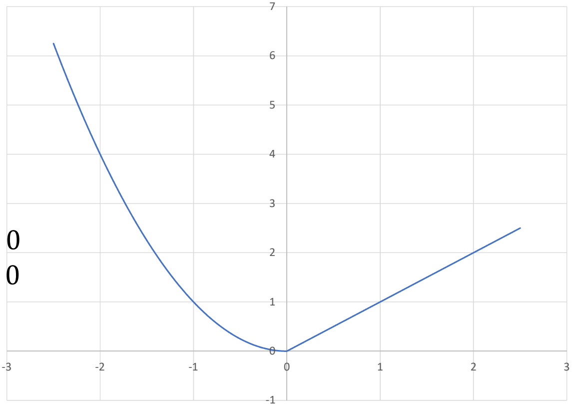
Definition

A piecewise-defined function is a function defined by multiple sub-functions on different intervals.

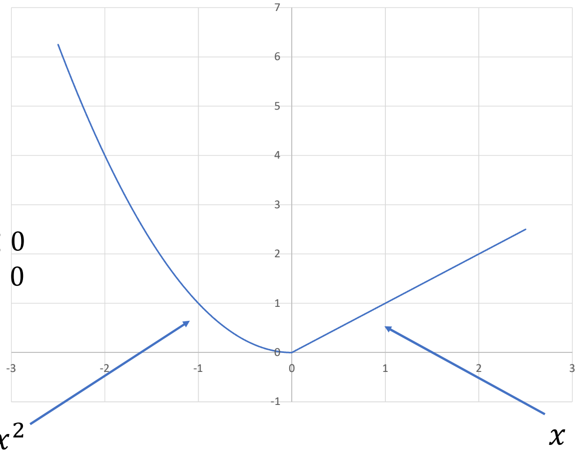
Example:

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$



$$f(x) = \begin{cases} x^2 & if\ x < 0 \\ x & if\ x \geq 0 \end{cases}$$

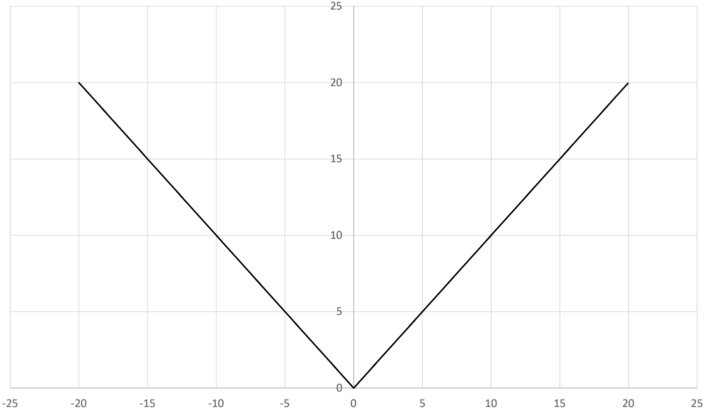


Piecewise-defined functions: the absolute value

A common example is the absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

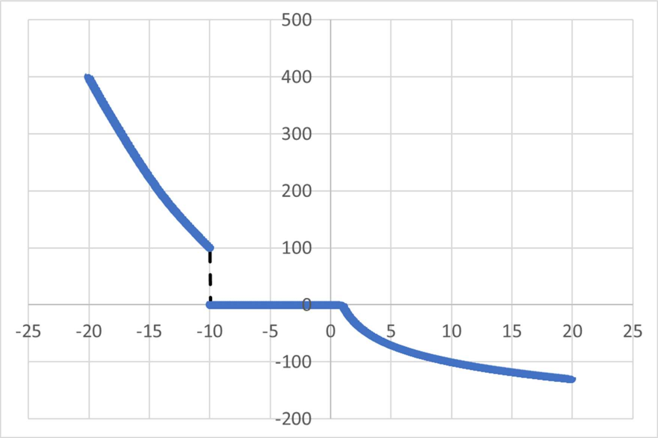
$$f(x) = |x|$$



Piecewise-defined functions: other examples

Nevertheless, there is no limit in creating a piecewise-defined function...

$$f(x) = \begin{cases} x^2 + \sin(x) & \text{if } x < -10 \\ 0 & \text{if } -10 \leq x \leq 1 \\ -\ln(x) & \text{if } x > 1 \end{cases}$$



500
400
300
200
100
0
-100
-200
-25
-20
-15
-10
-5
0
5
10
15
20
25

Right and left limits: the intuition

Right and left limits refer to the fact that x approaches x_0 from the right or from the left. See for instance the example in slide 2.

We say that:

$$\lim_{x \to x_0^+} f(x) = L$$

if $f(x)$ approaches L when x approaches x_0 from the right, i.e. $x > x_0$.

We say that:

$$\lim_{x \to x_0^-} f(x) = L$$

if $f(x)$ approaches L when x approaches x_0 from the left, i.e. $x < x_0$.

Right and left limits: the definition

Definition

Let $f : D \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0^+} f(x) = L$$

if

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \forall x \in D : 0 < x - x_0 < \delta,\ x \neq x_0 \ \Rightarrow \ |f(x) - L| < \epsilon$$

similarly, we say that

$$\lim_{x \to x_0^-} f(x) = L$$

if

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \forall x \in D : -\delta < x - x_0 < 0,\ x \neq x_0 \ \Rightarrow \ |f(x) - L| < \epsilon$$

Right and left limits: an important theorem

Why are right and left limits important?
Theorem Let $f : D \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ be a function. Let x_0 be a limit point of D. Then the limit

$$\lim_{x \to x_0} f(x) = L$$

exists ***if and only if***:

$$\lim_{x \to x_0^+} f(x) = L \quad \textbf{and} \quad \lim_{x \to x_0^-} f(x) = L.$$

This means that, if the right and left limits exist and are different, then we can conclude that the limit does not exist.

Right and left limits: an example

The $\operatorname{sign}(x)$ function is defined on $\mathbb{R}\backslash\{0\}$ by:

$$\operatorname{sign}(x) = \frac{|x|}{x} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

If $x > 0$, then $\operatorname{sign}(x) = +1$ so that:

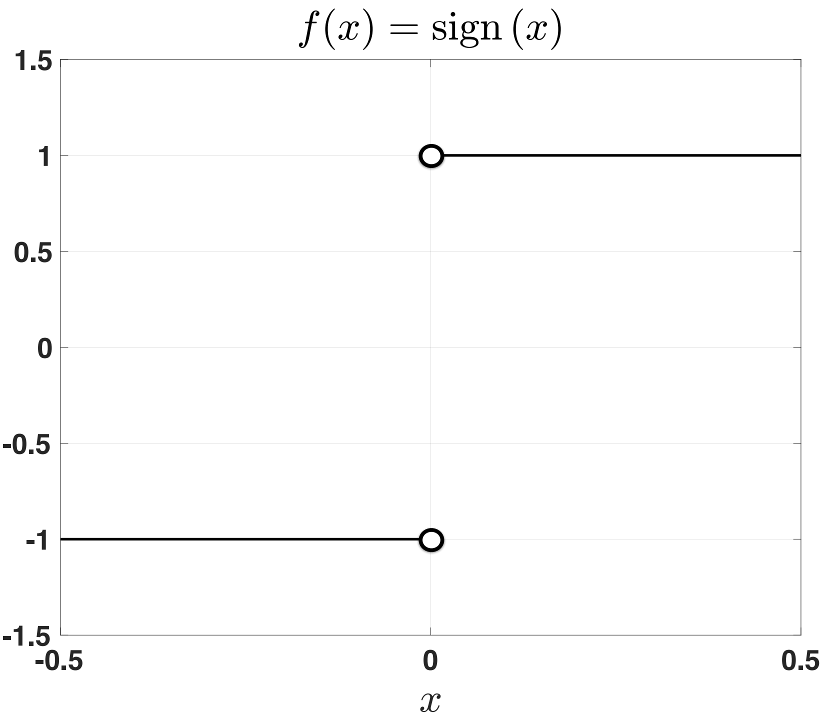
$$\lim_{x \to 0^+} \operatorname{sign}(x) = +1$$

If $x < 0$, then $\operatorname{sign}(x) = -1$ so that:

$$\lim_{x \to 0^-} \operatorname{sign}(x) = -1$$

Since the right and left limits are different, the limit does not exist:

$$\nexists \lim_{x \to 0} \operatorname{sign}(x) \, .$$


$f(x) = \mathrm{sign}\,(x)$
1.5
1
0.5
0
-0.5
-1
-1.5
-0.5
0
0.5
x

“Finite limit at infinity”

We now consider the case $\lim_{x\to\pm\infty} f(x) = L$

Definition

Let $f : D \to \mathbb{R}$ be a function and let D be unbounded from above, e.g. $D = (a, +\infty)$ or $D = [a, +\infty)$. We say that

$$\lim_{x\to+\infty} f(x) = L$$

if

$$\forall \epsilon > 0 \quad \exists M > 0 : \quad \forall x \in D : x > M \ \Rightarrow \ |f(x) - L| < \epsilon$$

We say that $f(x)$ has an **horizontal asymptote** at $y = L$.

Very similar to the analogous limit in the case of sequences, see Class 6.

“Finite limit at infinity”

Definition

Let $f : D \to \mathbb{R}$ be a function and let D be unbounded from below, e.g. $D = (-\infty, a)$ or $D = (-\infty, a]$. We say that

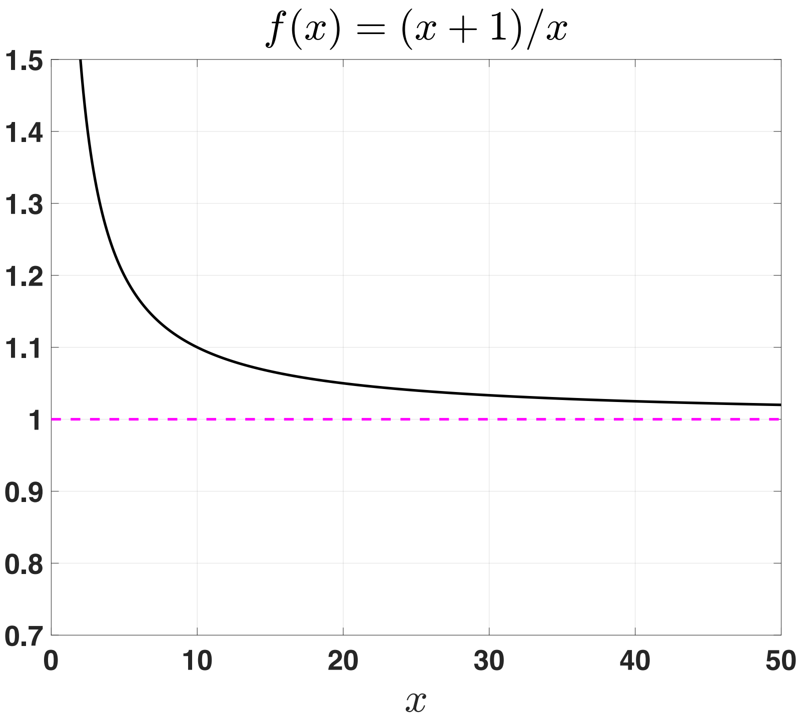
$$\lim_{x \to -\infty} f(x) = L$$

if

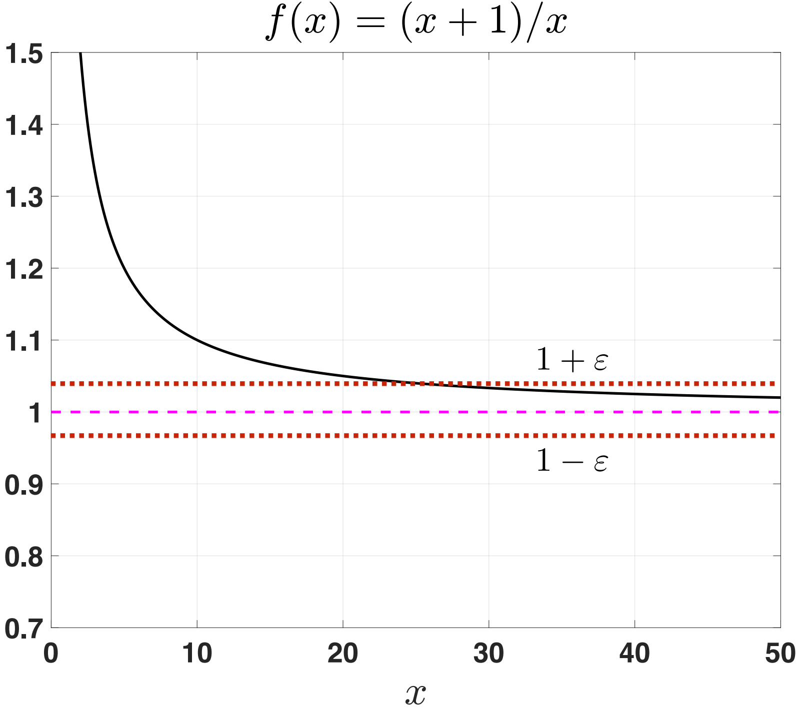
$$\forall \epsilon > 0 \quad \exists M > 0 : \quad \forall x \in D : x < -M \ \Rightarrow \ |f(x) - L| < \epsilon$$

We say that $f(x)$ has an **horizontal asymptote** at $y = L$.

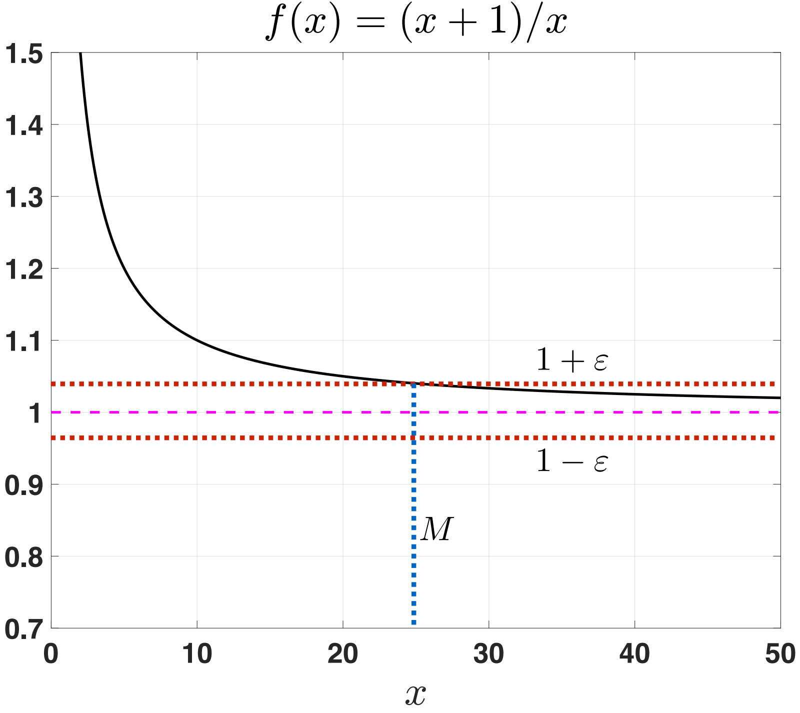
Very similar to the analogous limit in the case of sequences, see Class 6.



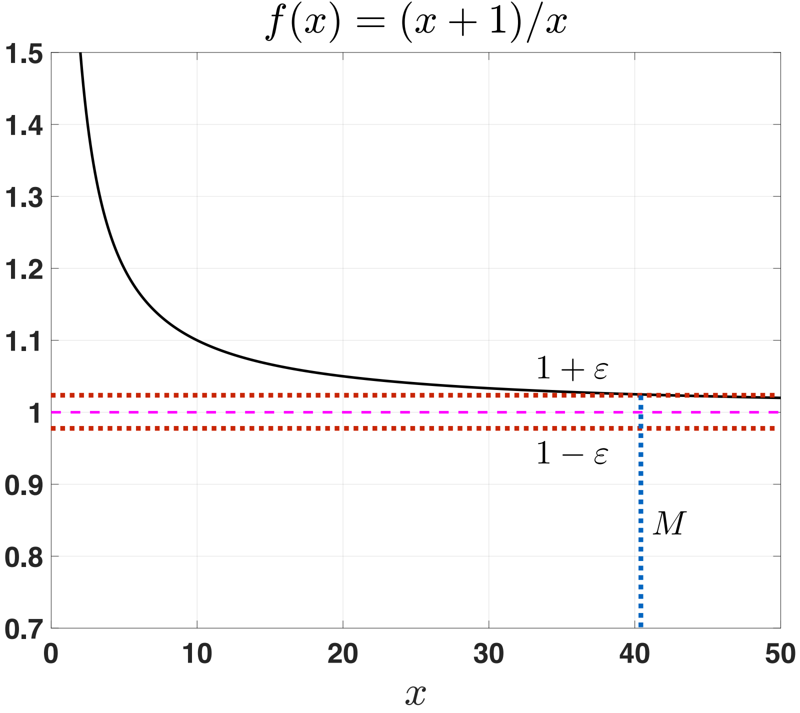
$f(x) = (x+1)/x$
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
10
20
30
40
50
x



$f(x) = (x+1)/x$
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
10
20
30
40
50
$1+\varepsilon$
$1-\varepsilon$
x



$f(x) = (x+1)/x$
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
10
20
30
40
50
x
$1+\varepsilon$
$1-\varepsilon$
M



$f(x) = (x+1)/x$
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
$1+\varepsilon$
$1-\varepsilon$
M
0
10
20
30
40
50
x

“Infinite limit at a point”

It may happen that a function becomes larger and larger when x approaches a point x_0.

$$f(x) = \frac{1}{(x+5)^2}$$

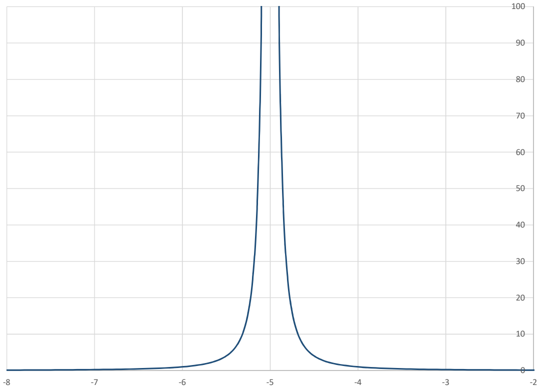
The domain is $D = \mathbb{R} \setminus \{-5\}$. What happens when x approaches -5?

x approaching –5 from right	
–4.9	100
–4.95	400
–4.99	1000
–4.999	100000
–4.9999	100000000

x approaching –5 from left	
–5.1	100
–5.05	400
–5.005	40000
–5.0005	4000000
–5.0001	100000000

$$f(x) = \frac{1}{(x+5)^2} \Rightarrow D = (-\infty, -5) \cup (-5, +\infty)$$

$$f(x) = \frac{1}{(x+5)^2} \Rightarrow D = (-\infty, -5) \cup (-5, +\infty)$$



"Infinite limit at a point"

We now consider the case $\lim_{x\to x_0} f(x) = \pm\infty$

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let x_0 be a limit point of D. We say that

$$\lim_{x\to x_0} f(x) = +\infty$$

if

$$\forall K > 0 \quad \exists \delta > 0 : \quad \forall x \in D : |x - x_0| < \delta,\ x \neq x_0 \ \Rightarrow\ f(x) > K$$

We say that $f(x)$ has an **vertical asymptote** at $x = x_0$.

Intuitively, the definition means that $f(x)$ can be made larger than any positive number K, provided that x is sufficiently close to x_0.

“Infinite limit at a point”

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let x_0 be a limit point of D. We say that

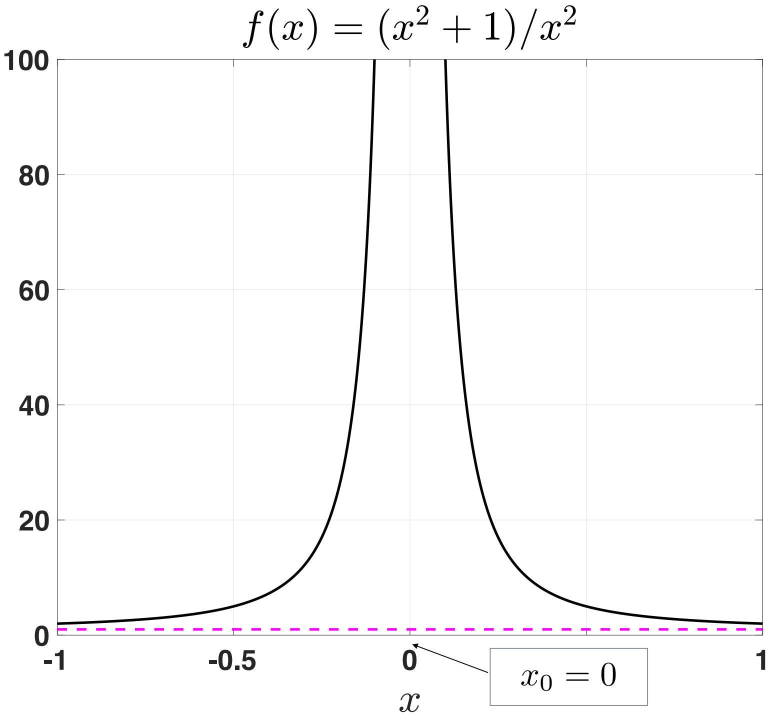
$$\lim_{x \to x_0} f(x) = -\infty$$

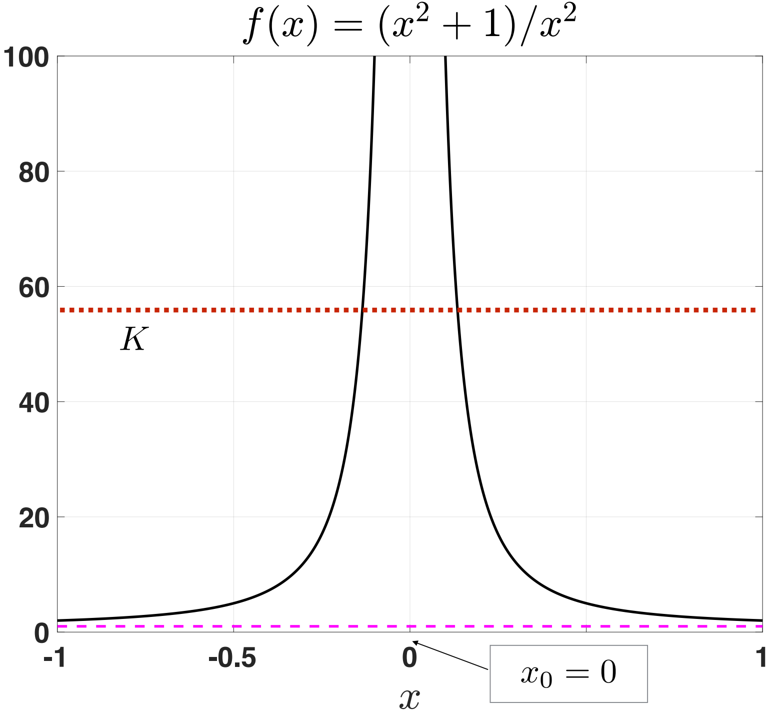
if

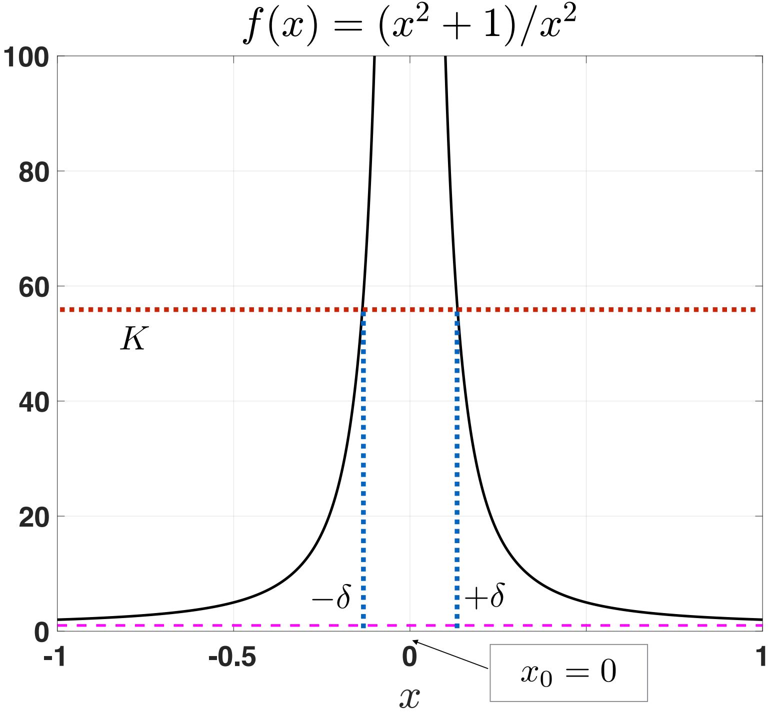
$$\forall K > 0 \quad \exists \delta > 0 : \quad \forall x \in D : |x - x_0| < \delta, \; x \neq x_0 \; \Rightarrow \; f(x) < -K$$

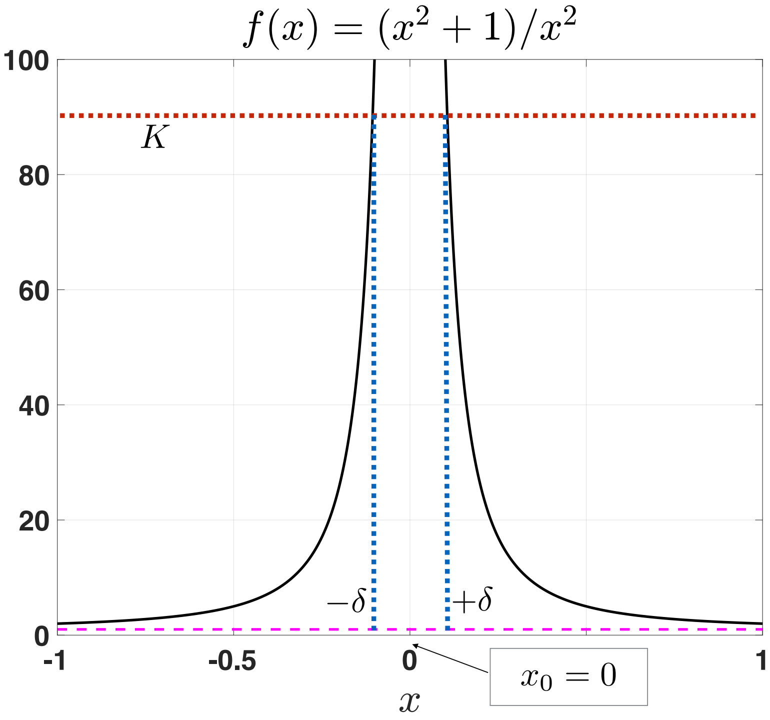
We say that $f(x)$ has an **vertical asymptote** at $x = x_0$.

Intuitively, the definition means that $f(x)$ can be made lower than any negative number $-K$, provided that x is sufficiently close to x_0.


$f(x) = (x^2 + 1)/x^2$
100
80
60
40
20
0
-1
-0.5
0
1
x
$x_0 = 0$


$f(x) = (x^2 + 1)/x^2$
100
80
60
40
20
0
-1
-0.5
0
1
K
$x_0 = 0$
x


$f(x) = (x^2 + 1)/x^2$
100
80
60
40
20
0
-1
-0.5
0
1
K
$-\delta$
$+\delta$
$x_0 = 0$
x



$f(x) = (x^2 + 1)/x^2$
100
80
60
40
20
0
-1
-0.5
0
1
K
$-\delta$
$+\delta$
$x_0 = 0$
x

"Infinite limit at a point": right and left limits

The concept of right and left limits can be introduced even in the "infinite limit at a point" case. For instance, in the case $\lim_{x \to x_0^+} f(x) = +\infty$, we have:

Definition

Let $f : D \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0^+} f(x) = +\infty$$

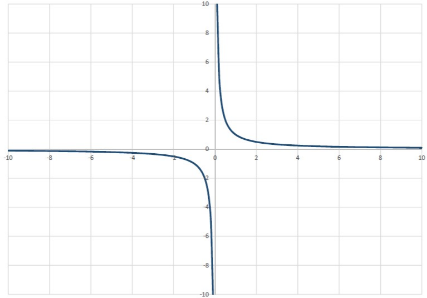
if

$$\forall K > 0 \quad \exists \delta > 0 : \quad \forall x \in D : 0 < x - x_0 < \delta, \; x \neq x_0 \;\Rightarrow\; f(x) > K$$

Remark: Even in this case, **the limit exists if and only if the right and left limits exist and are equal**. If the limit does not exists, but at least one between the right and left limits are $\pm\infty$, **we still say that the function has a vertical asymptote**.

“Infinite limit at a point”: right and left limits

$$f(x) = \frac{1}{x} \Rightarrow D = (-\infty, 0) \cup (0, +\infty)$$



We have: $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$. Therefore, the limit does not exists, however we say that the function has a vertical asymptote in $x = 0$.

Infinite limit at infinity

This case covers four sub-cases:

1. $\lim_{x \to +\infty} f(x) = +\infty$
2. $\lim_{x \to -\infty} f(x) = -\infty$
3. $\lim_{x \to +\infty} f(x) = -\infty$
4. $\lim_{x \to -\infty} f(x) = +\infty$

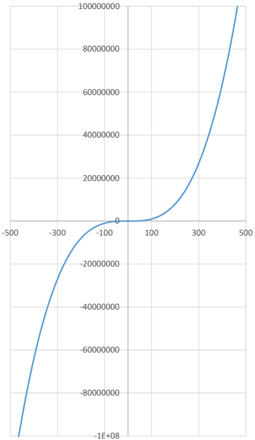
Definitions are very intuitive. For instance, let's consider the case $\lim_{x \to +\infty} f(x) = +\infty$

Definition

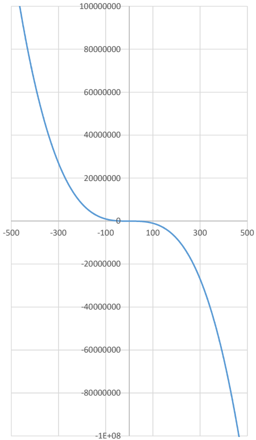
Let $f : D \to \mathbb{R}$ be a function and let D be unbounded from above. We say that: $\lim_{x \to +\infty} f(x) = +\infty$ if

$$\forall K > 0 \quad \exists M > 0 : \quad \forall x \in D : x > M \;\Rightarrow\; f(x) > K$$

$$f(x) = x^3$$


100000000
80000000
60000000
40000000
20000000
0
-20000000
-40000000
-60000000
-80000000
-1E+08
-500
-300
-100
100
300
500

$$g(x) = -x^3$$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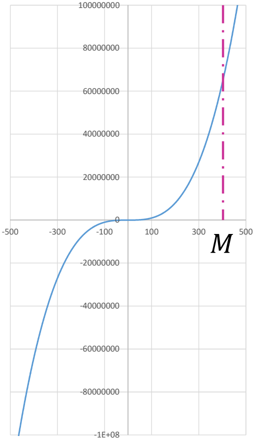
500

$$f(x) = x^3$$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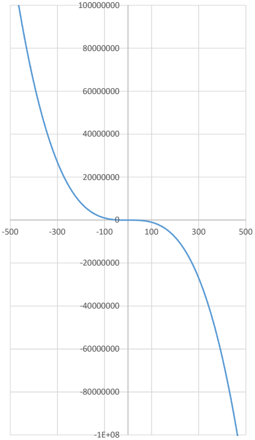
500
M

$$g(x) = -x^3$$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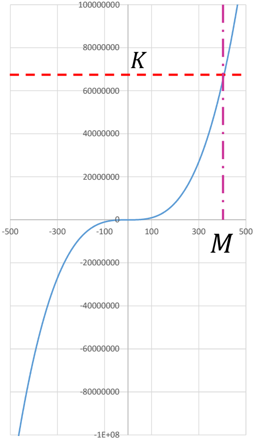
-40000000
-60000000
-80000000
-1E+08
-500
-300
-100
100
300
500

$$f(x) = x^3$$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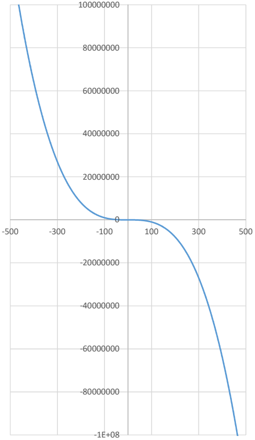
40000000
20000000
0
-20000000
-40000000
-60000000
-80000000
-1E+08
-500
-300
-100
100
300
500
K
M

$$g(x) = -x^3$$


100000000
80000000
60000000
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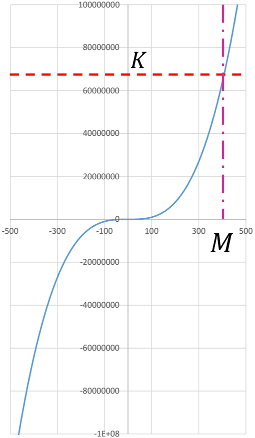
$$f(x) = x^3$$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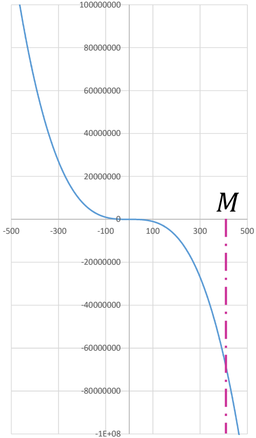
$$g(x) = -x^3$$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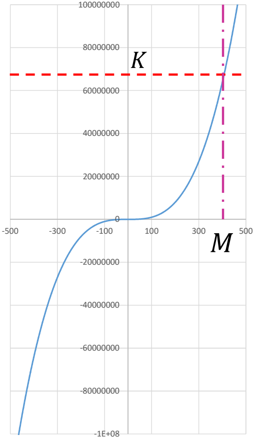
$f(x) = x^3$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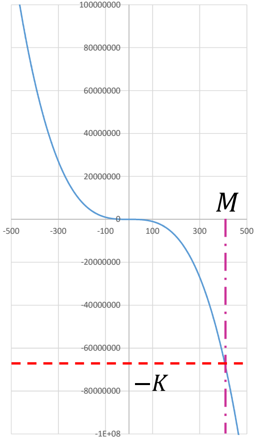
$g(x) = -x^3$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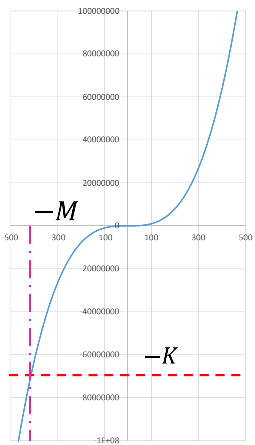
$$f(x) = x^3$$



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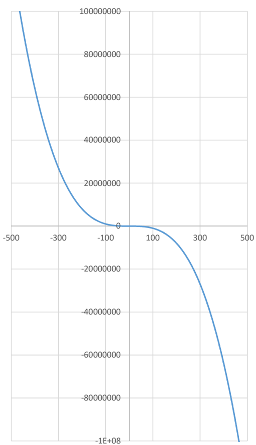
$$g(x) = -x^3$$



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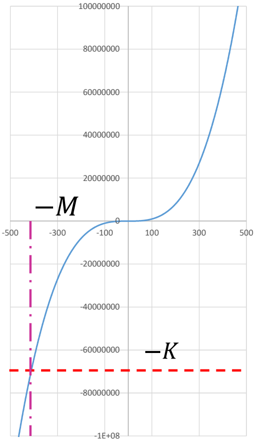
$$f(x) = x^3$$



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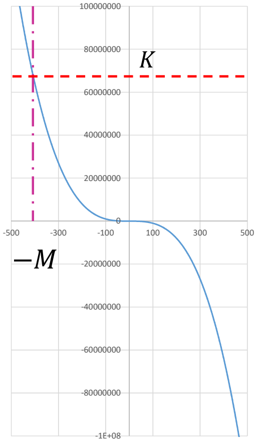
$$g(x) = -x^3$$



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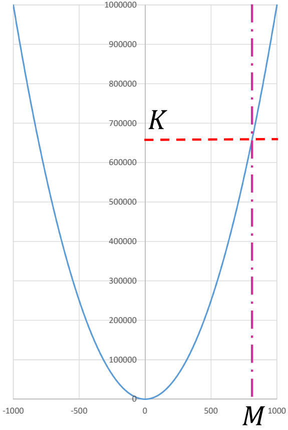
$$f(x) = x^2$$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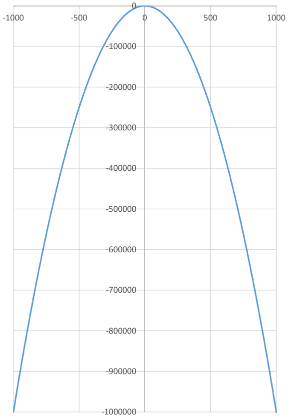
$$g(x) = -x^2$$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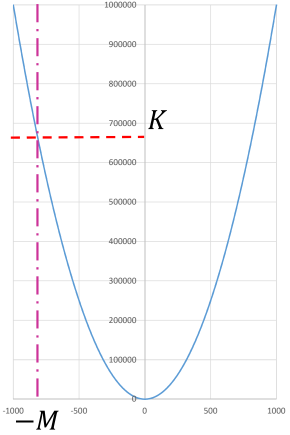
$f(x) = x^2$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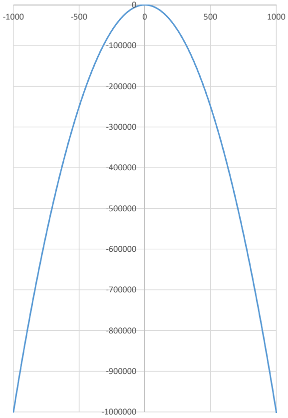
$g(x) = -x^2$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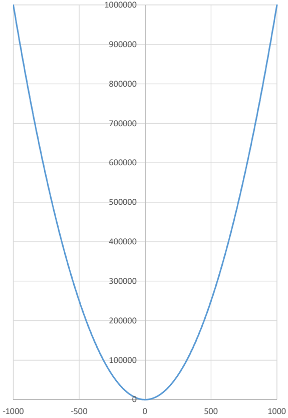
$$f(x) = x^2$$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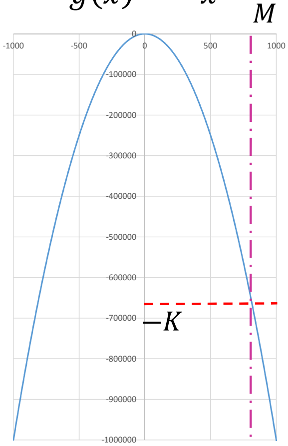
$$g(x) = -x^2$$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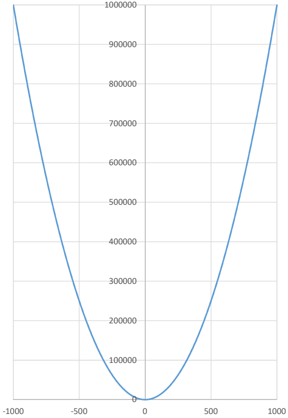
$$f(x) = x^2$$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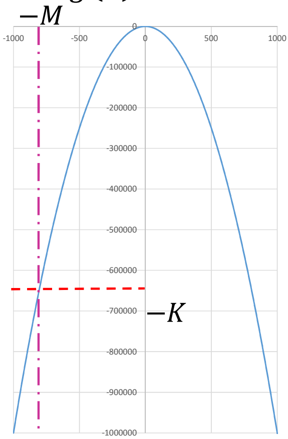
$$g(x) = -x^2$$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