Limits of functions: the intuition

Consider the following function:

$$f(x) = \frac{2x^2 - 8}{x - 2}$$

The domain is $D = \mathbb{R} \setminus \{2\}$.

The function is not defined in $x = 2$, but we can compute its value in a neighbourhood of $x = 2$:

x approaching 2 from right	
$x = 2.1$	$f(x) = 8.2$
$x = 2.01$	$f(x) = 8.02$
$x = 2.001$	$f(x) = 8.002$
$x = 2.0004$	$f(x) = 8.0008$

x approaching 2 from left	
$x = 1.9$	$f(x) = 7.8$
$x = 1.95$	$f(x) = 7.9$
$x = 1.995$	$f(x) = 7.99$
$x = 1.9991$	$f(x) = 7.9982$

Limits of functions: the intuition, cont'd

The function $f(x)$ gets closer and closer to the value 8 as x gets closer and closer to 2.

Put in other words, if x is sufficiently close to s, then the distance between $f(x)$ and 8 is small.

Thus, for any $\epsilon > 0$ arbitrarily small, we can find a $\delta > 0$ such that:

$$|f(x) - 8| < \varepsilon$$

provided that

$$|x - 2| < \delta$$

Importance of limits

Limits serve to answer the following questions:

- How does a function behave when x gets closer and closer to a point x_0?
- How does a function behave when x gets larger and larger?

Interesting cases are particular points of the domain:

- Points where the function is not defined, i.e. **outside its domain**, but on the boundary
- $+\infty/-\infty$, if the domain is **unbounded from above/below**

We will consider four cases:

- "Finite limit at a point": $\lim_{x\to x_0} f(x) = \ell$
- "Finite limit at infinity": $\lim_{x\to \pm\infty} f(x) = \ell$
- "Infinite limit at a point": $\lim_{x\to x_0} f(x) = \pm\infty$
- "Infinite limit at infinite": $\lim_{x\to \pm\infty} f(x) = \pm\infty$

Finite limit at a point

Intuitively, we say that

$$\lim_{x \to x_0} f(x) = \ell$$

if $f(x)$ gets close to l when x approaches x
Remark: Approach... How?
Through points of the domain!

$$f(x) = \sqrt{x} \Rightarrow D = [0, +\infty)$$

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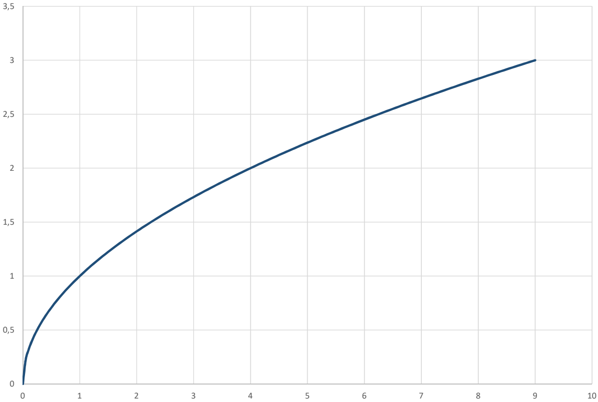
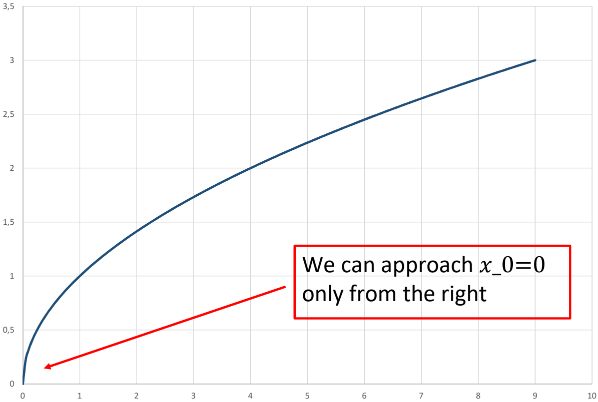


$$f(x) = \sqrt{x} \Rightarrow D = [0, +\infty)$$



$$f(x) = \frac{1}{x} \Rightarrow D = (-\infty, 0) \cup (0, +\infty)$$

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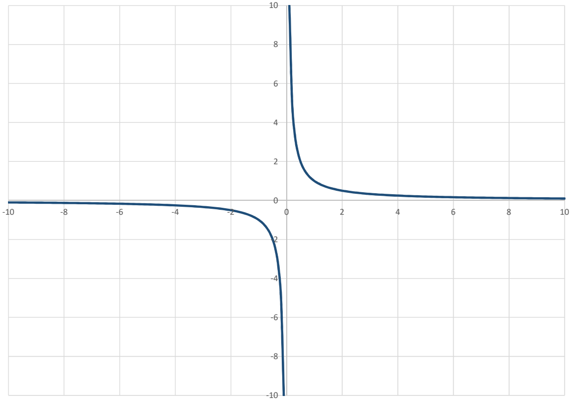
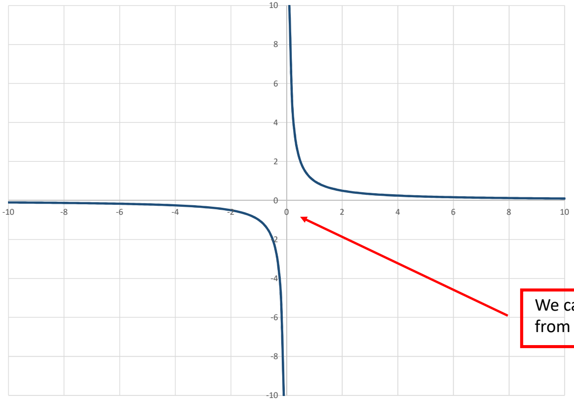


$$f(x) = \frac{1}{x} \Rightarrow D = (-\infty, 0) \cup (0, +\infty)$$



We can approach $x_0 = 0$ from left and from right!!

"Finite limit at a point": $\lim_{x\to x_0} f(x) = \ell$

Definition

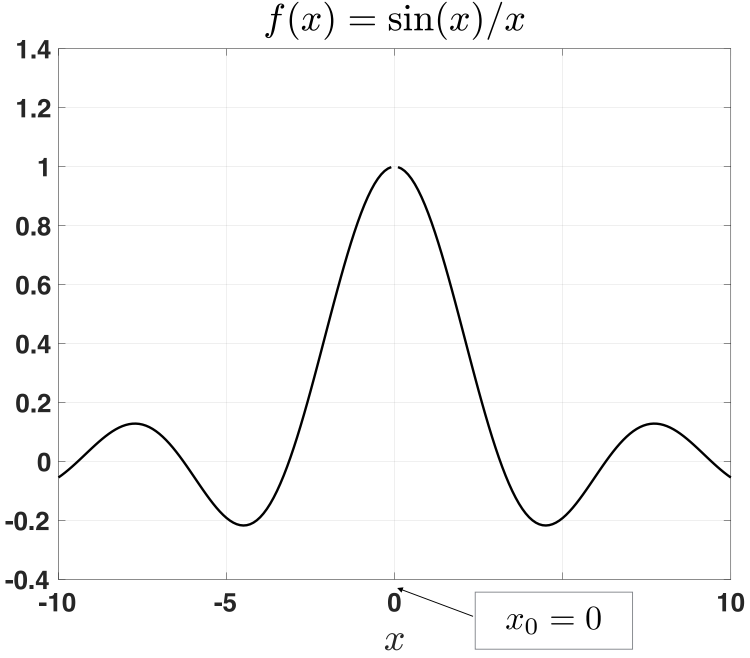
Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

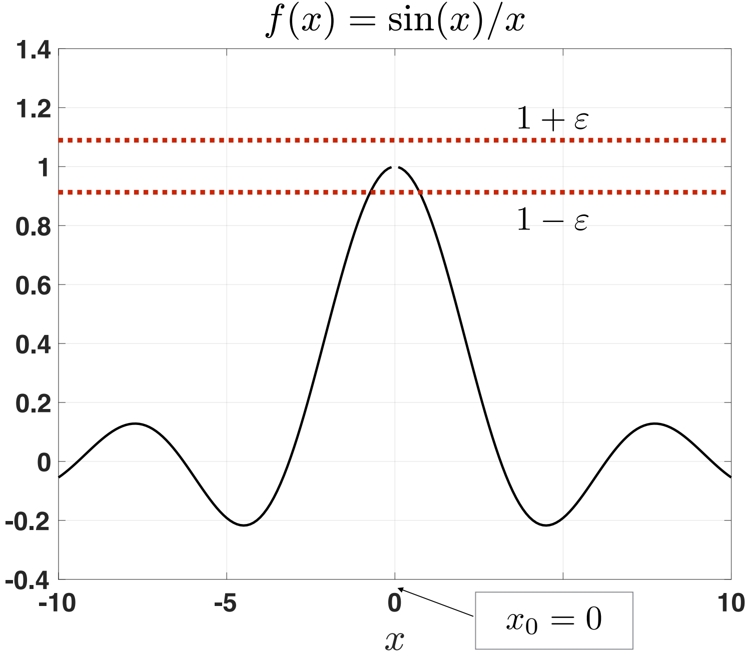
$$\lim_{x\to x_0} f(x) = \ell$$

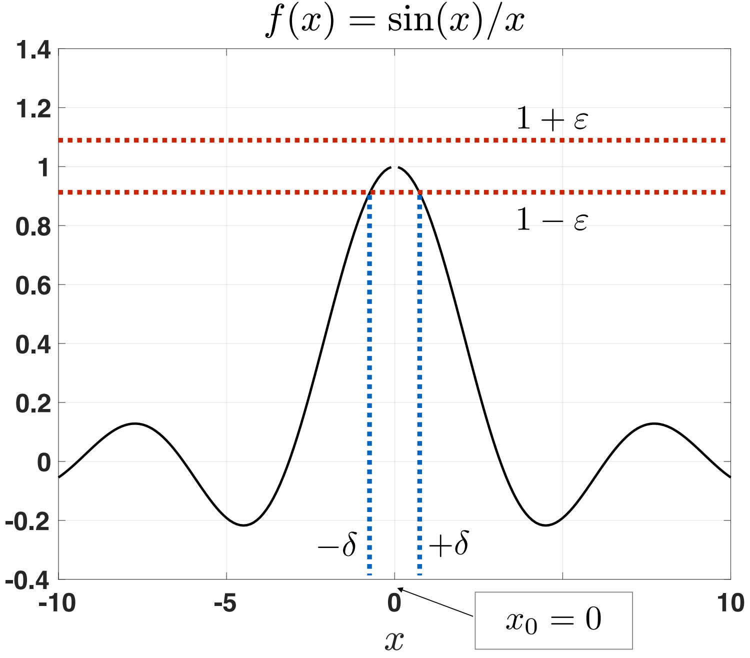
if

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \forall x \in D : |x - x_0| < \delta,\ x \neq x_0 \ \Rightarrow\ |f(x) - \ell| < \epsilon$$

Remark: Note that the function is not required to be defined in x_0!


$f(x) = \sin(x)/x$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-10
-5
0
10
x
$x_0 = 0$


$f(x) = \sin(x)/x$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
$1+\varepsilon$
$1-\varepsilon$
-10
-5
0
10
x
$x_0 = 0$


$f(x) = \sin(x)/x$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
$1+\varepsilon$
$1-\varepsilon$
$-\delta$
$+\delta$
-10
-5
0
10
x
$x_0 = 0$

Piecewise-defined functions

Definition

A piecewise-defined function is a function defined by multiple sub-functions on different intervals.

Example:

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

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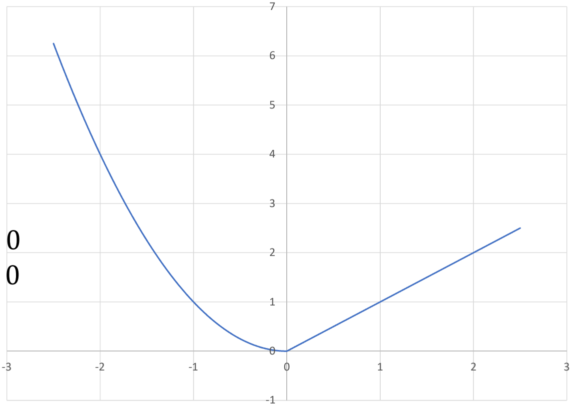
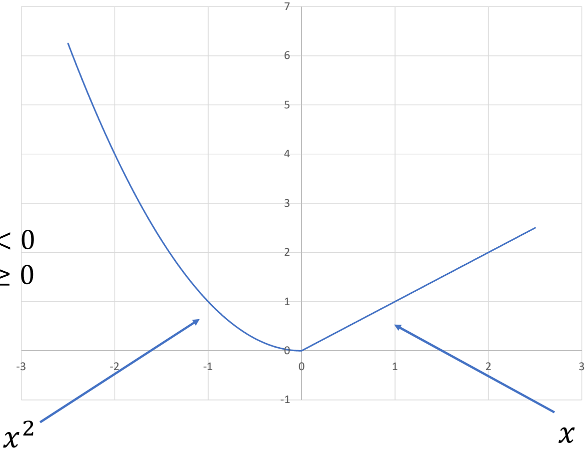


$$f(x) = \begin{cases} x^2 & if\ x < 0 \\ x & if\ x \geq 0 \end{cases}$$

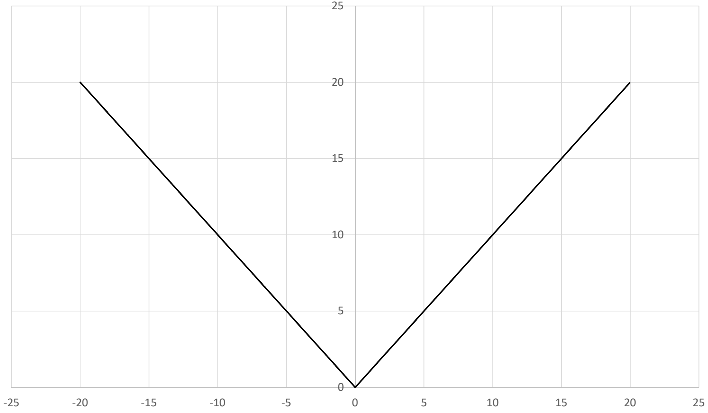


Piecewise-defined functions: the absolute value

A common example is the absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

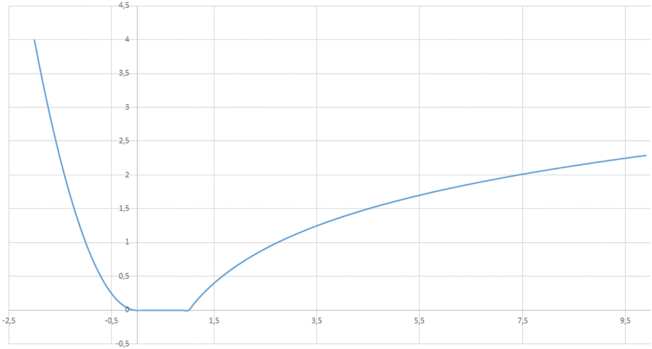
$$f(x) = |x|$$



Piecewise-defined functions: other examples

Nevertheless, there is no limit in creating a piecewise-defined function...

$$f(x) = \begin{cases} x^2 & if\ x < 0 \\ 0 & if\ 0 \le x < 1 \\ \log(x) & if\ x \ge 1 \end{cases}$$



Right and left limits: the intuition

Right and left limits refer to the fact that x approaches x_0 from the right or from the left.

We say that:

$$\lim_{x \to x_0^+} f(x) = \ell$$

if $f(x)$ approaches ℓ when x approaches x_0 from the right, i.e. $x > x_0$.

We say that:

$$\lim_{x \to x_0^-} f(x) = \ell$$

if $f(x)$ approaches L when x approaches x_0 from the left, i.e. $x < x_0$.

Right and left limits: the definition

Definition

Let $f : D \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0^+} f(x) = \ell$$

if

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \forall x \in D : 0 < x - x_0 < \delta,\ x \neq x_0 \ \Rightarrow\ |f(x) - \ell| < \epsilon$$

similarly, we say that

$$\lim_{x \to x_0^-} f(x) = \ell$$

if

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \forall x \in D : -\delta < x - x_0 < 0,\ x \neq x_0 \ \Rightarrow\ |f(x) - \ell| < \epsilon$$

Right and left limits: an important theorem

Why are right and left limits important?

Theorem

Let $f : D \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ be a function. Let x_0 be a limit point of D. Then the limit

$$\lim_{x \to x_0} f(x) = \ell$$

exists **if and only if***:*

$$\lim_{x \to x_0^+} f(x) = \ell \quad \textbf{and} \quad \lim_{x \to x_0^-} f(x) = \ell.$$

This means that, if the right and left limits exist and are different, then we can conclude that the limit does not exist.

Right and left limits: an example

The $\operatorname{sign}(x)$ function is defined on $\mathbb{R}\backslash\{0\}$ by:

$$\operatorname{sign}(x) = \frac{|x|}{x} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

If $x > 0$, then $\operatorname{sign}(x) = +1$ so that:

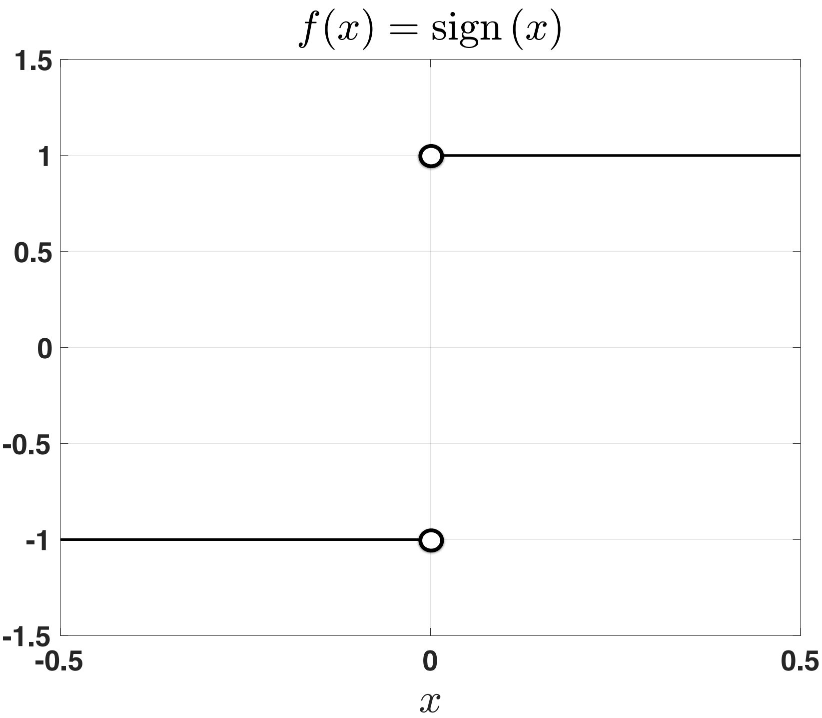
$$\lim_{x \to 0^+} \operatorname{sign}(x) = +1$$

If $x < 0$, then $\operatorname{sign}(x) = -1$ so that:

$$\lim_{x \to 0^-} \operatorname{sign}(x) = -1$$

Since the right and left limits are different, the limit does not exist:

$$\nexists \lim_{x \to 0} \operatorname{sign}(x) .$$



$f(x) = \mathrm{sign}\,(x)$
1.5
1
0.5
0
-0.5
-1
-1.5
-0.5
0
0.5
x

“Finite limit at infinity”: $\lim_{x\to+\infty} f(x) = \ell$

Definition

Let $f : D \to \mathbb{R}$ be a function and let D be unbounded on the right, e.g. $D = (a, +\infty)$ or $D = [a, +\infty)$. We say that

$$\lim_{x\to+\infty} f(x) = \ell$$

if

$$\forall \epsilon > 0 \quad \exists K > 0 : \quad \forall x \in D : x > K \;\Rightarrow\; |f(x) - \ell| < \epsilon$$

In this case we say that the line with equation $y = \ell$ is an horizontal asymptote at $+\infty$

Important: Notice that this case is very similar to limits of sequences. Here K plays the role as n^*

“Finite limit at infinity”: $\lim_{x\to-\infty} f(x) = \ell$

Definition

Let $f : D \to \mathbb{R}$ be a function and let D be unbounded on the left, e.g. $D = (-\infty, b)$ or $D = (-\infty, b]$. We say that

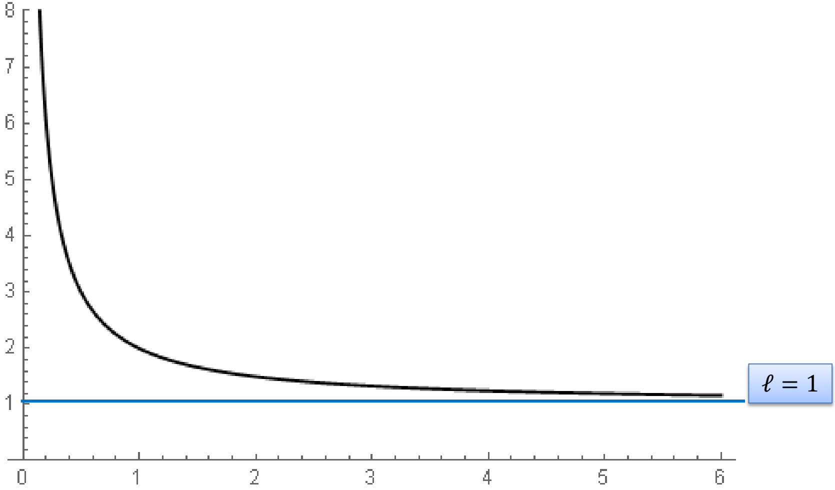
$$\lim_{x\to-\infty} f(x) = \ell$$

if

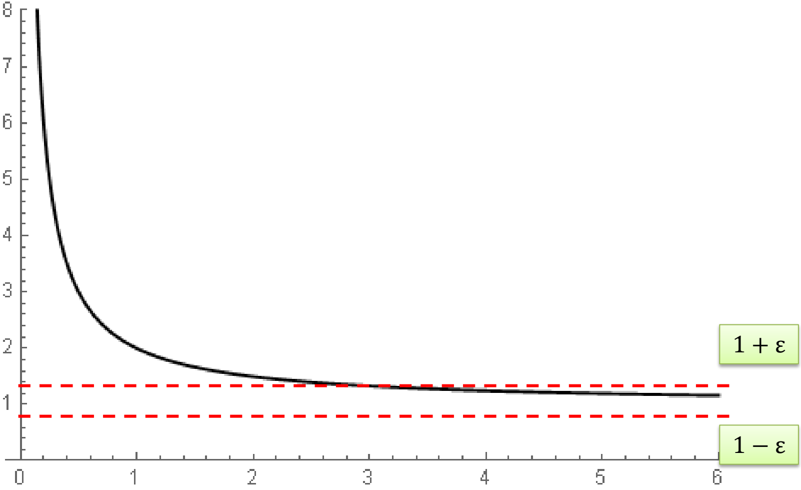
$$\forall \epsilon > 0 \quad \exists K > 0 : \quad \forall x \in D : x < -K \ \Rightarrow\ |f(x) - \ell| < \epsilon$$

In this case we say that the line with equation $y = \ell$ is an horizontal asymptote at $-\infty$

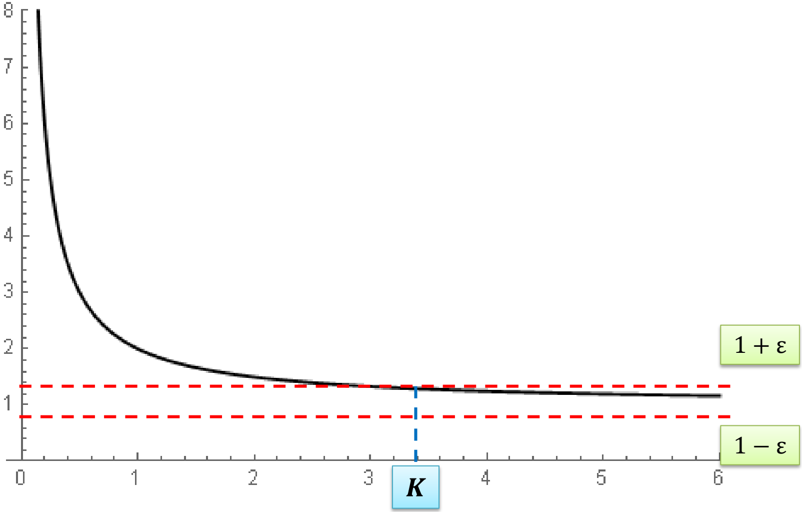
$$f(x) = \frac{x+1}{x}$$



$$f(x) = \frac{x+1}{x}$$



$$f(x) = \frac{x+1}{x}$$



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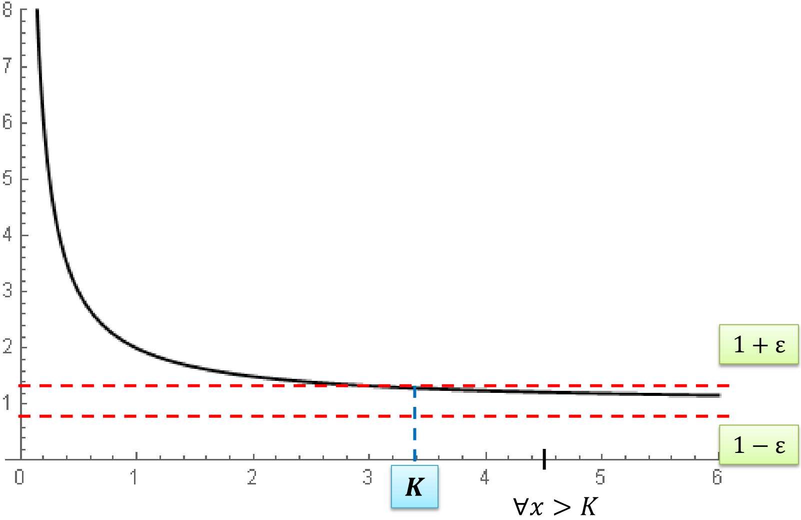
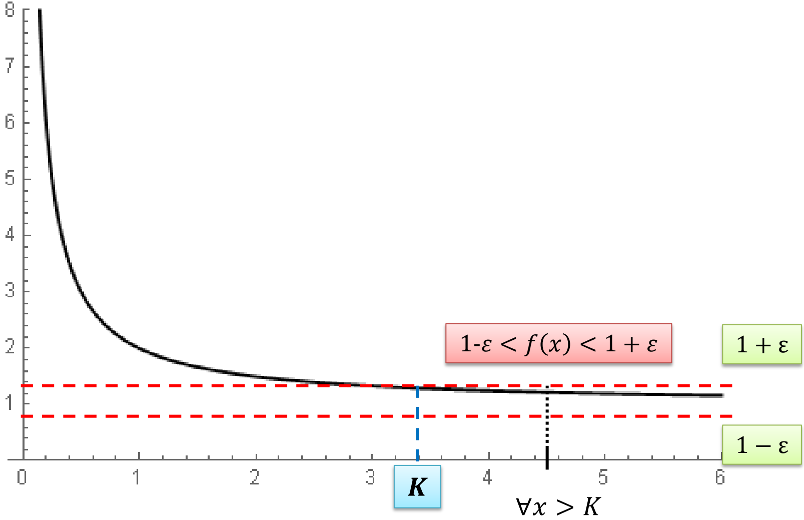


$$f(x) = \frac{x+1}{x}$$



"Infinite limit at a point"

It may happen that a function becomes larger and larger when x approaches a point x_0.

$$f(x) = \frac{1}{(x+5)^2}$$

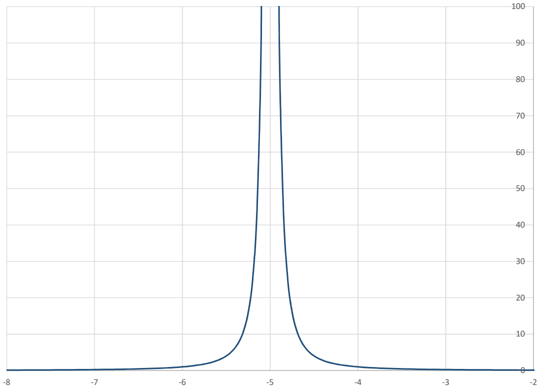
The domain is $D = \mathbb{R} \setminus \{-5\}$. What happens when x approaches -5?

x approaching -5 from right	
-4.9	100
-4.95	400
-4.99	1000
-4.999	100000
-4.9999	100000000

x approaching -5 from left	
-5.1	100
-5.05	400
-5.005	40000
-5.0005	4000000
-5.0001	100000000

$$f(x) = \frac{1}{(x+5)^2} \Rightarrow D = (-\infty, -5) \cup (-5, +\infty)$$

$$f(x) = \frac{1}{(x+5)^2} \Rightarrow D = (-\infty, -5) \cup (-5, +\infty)$$



"Infinite limit at a point"

We now consider the case $\lim_{x \to x_0} f(x) = \pm\infty$

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let x_0 be a limit point of D. We say that

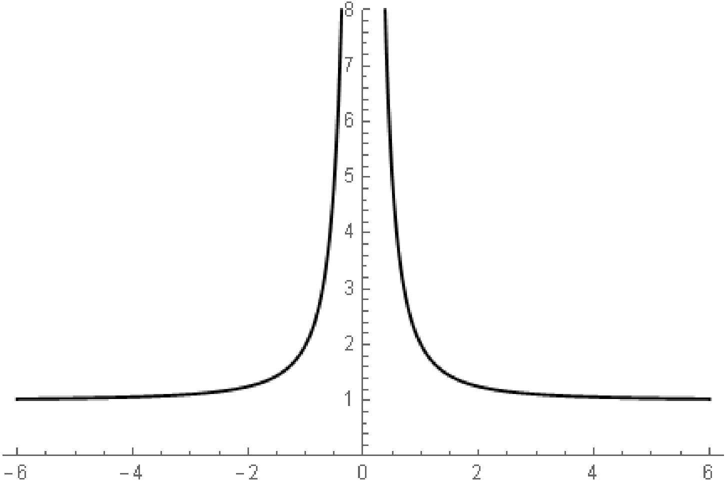
$$\lim_{x \to x_0} f(x) = +\infty$$

if

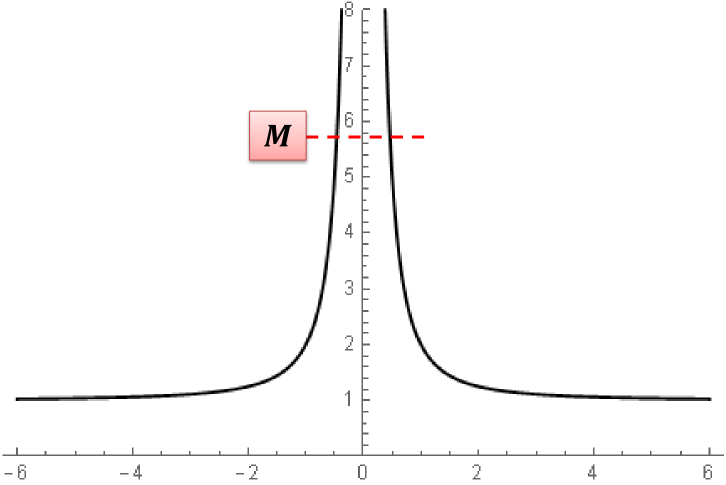
$$\forall M > 0 \quad \exists \delta > 0 : \quad \forall x \in D : |x - x_0| < \delta, \; x \neq x_0 \; \Rightarrow \; f(x) > M$$

In this case we say that the line with equation $x = x_0$ is a vertical asymptote

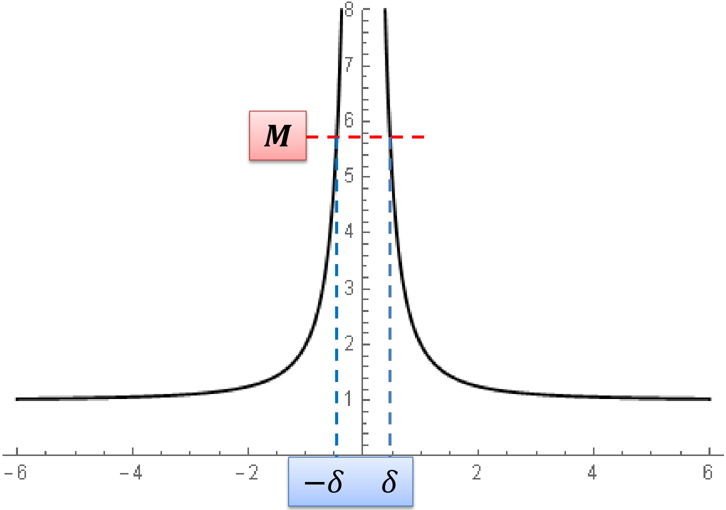
$$f(x) = \frac{x^2 + 1}{x^2}$$



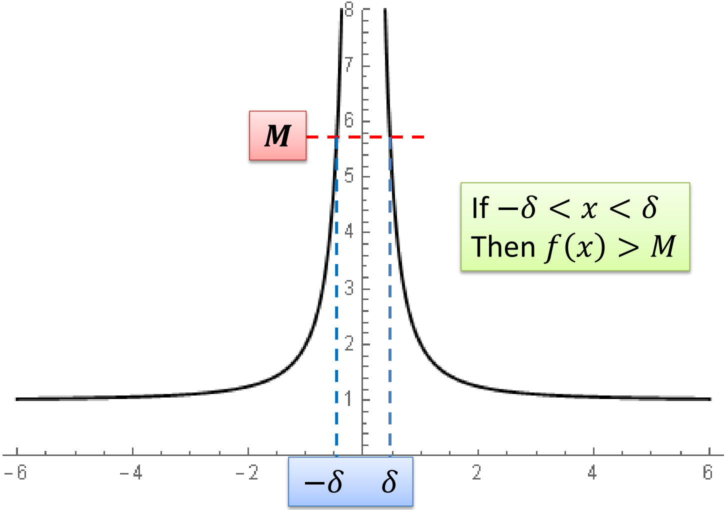
$$f(x) = \frac{x^2 + 1}{x^2}$$



$$f(x) = \frac{x^2+1}{x^2}$$



$$f(x) = \frac{x^2 + 1}{x^2}$$



“Infinite limit at a point”

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let x_0 be a limit point of D. We say that

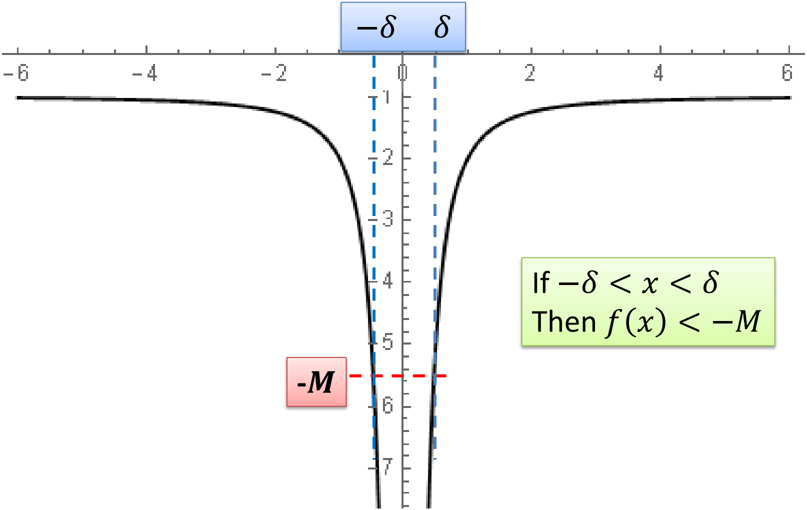
$$\lim_{x \to x_0} f(x) = -\infty$$

if

$$\forall M > 0 \quad \exists \delta > 0 : \quad \forall x \in D : |x - x_0| < \delta,\ x \neq x_0 \ \Rightarrow\ f(x) < -M$$

In this case we say that the line with equation $x = x_0$ is a vertical asymptote

$$f(x) = -\frac{x^2+1}{x^2}$$



"Infinite limit at a point": right and left limits

The concept of right and left limits can be extended to the case of "infinite limit at a point".

Theorem

Let $f : D \subseteq \mathbb{R} \Rightarrow \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0} f(x) = +\infty$$

if and only if*:* $\lim_{x \to x_0^+} f(x) = +\infty$ **and** $\lim_{x \to x_0^-} f(x) = +\infty$. *We also say that*

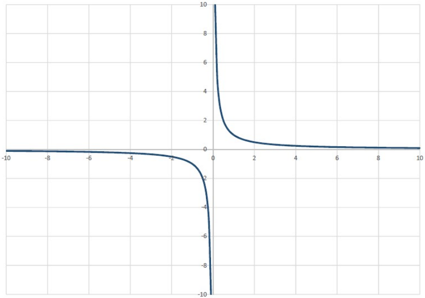
$$\lim_{x \to x_0} f(x) = -\infty$$

if and only if*:* $\lim_{x \to x_0^+} f(x) = -\infty$ **and** $\lim_{x \to x_0^-} f(x) = -\infty$.

Remark: If the limit does not exists (i.e. left and right limits are different), but at least one between the right and left limits are $\pm\infty$, **we still say that the function has a vertical asymptote at $x = x_0$.**

“Infinite limit at a point”: right and left limits

$$f(x) = \frac{1}{x} \Rightarrow D = (-\infty, 0) \cup (0, +\infty)$$



We have: $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$. Therefore, the limit does not exists, however (we will see this later!) we say that the function has a vertical asymptote at $x = 0$.

Infinite limit at infinity

This case covers four sub-cases:

1. $\lim_{x\to+\infty} f(x) = +\infty$
2. $\lim_{x\to-\infty} f(x) = -\infty$
3. $\lim_{x\to+\infty} f(x) = -\infty$
4. $\lim_{x\to-\infty} f(x) = +\infty$

Definitions are very intuitive. For instance $\lim_{x\to+\infty} f(x) = +\infty$

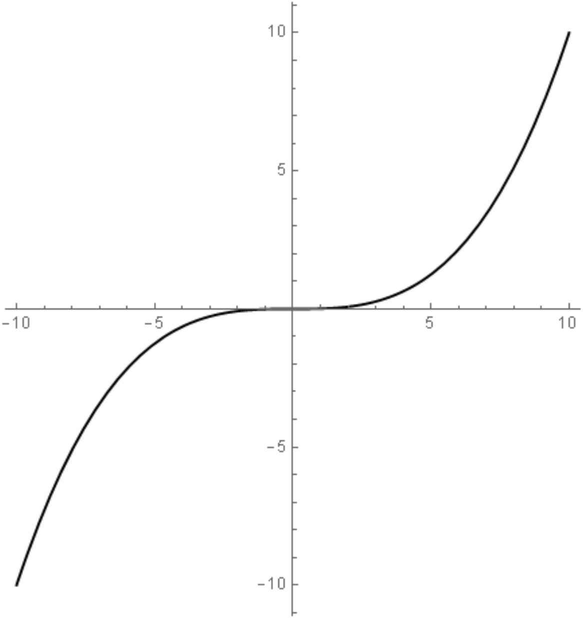
Definition

Let $f : D \to \mathbb{R}$ be a function and let D be unbounded from above. We say that: $\lim_{x\to+\infty} f(x) = +\infty$ if

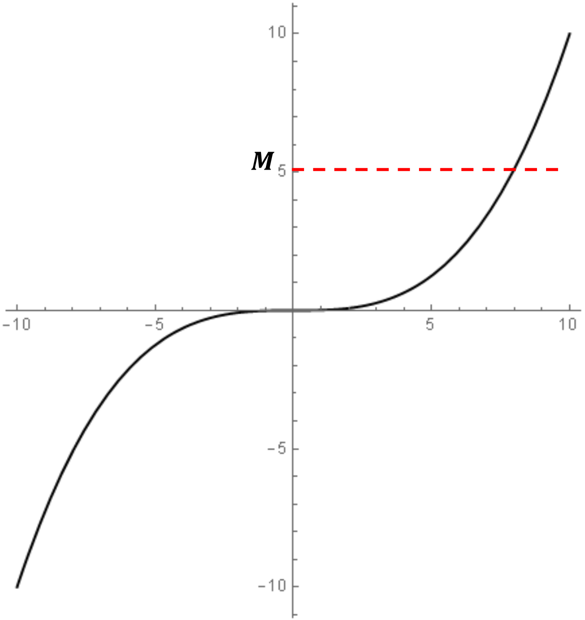
$$\forall M > 0 \quad \exists K > 0 : \quad \forall x \in D : x > K \;\Rightarrow\; f(x) > M$$

Exercise: Write the definitions of limits for the other three cases.

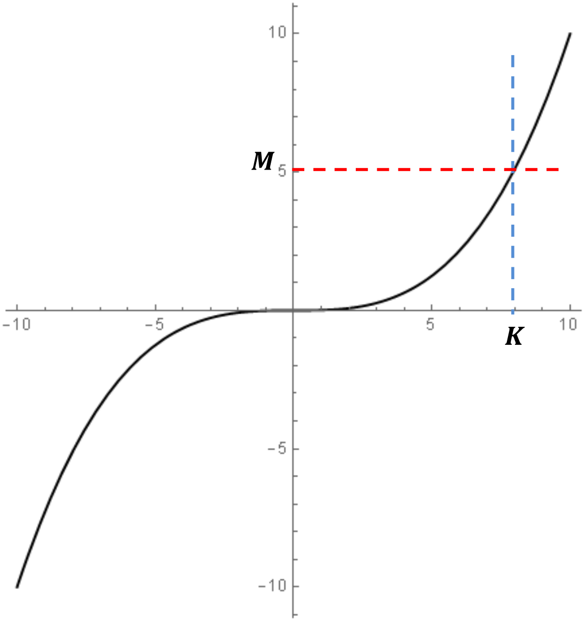
$$f(x) = \frac{x^3}{10}$$



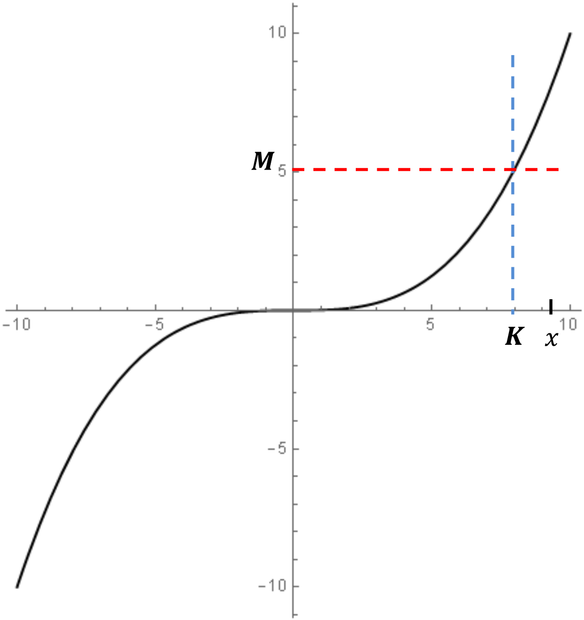
$$f(x) = \frac{x^3}{10}$$



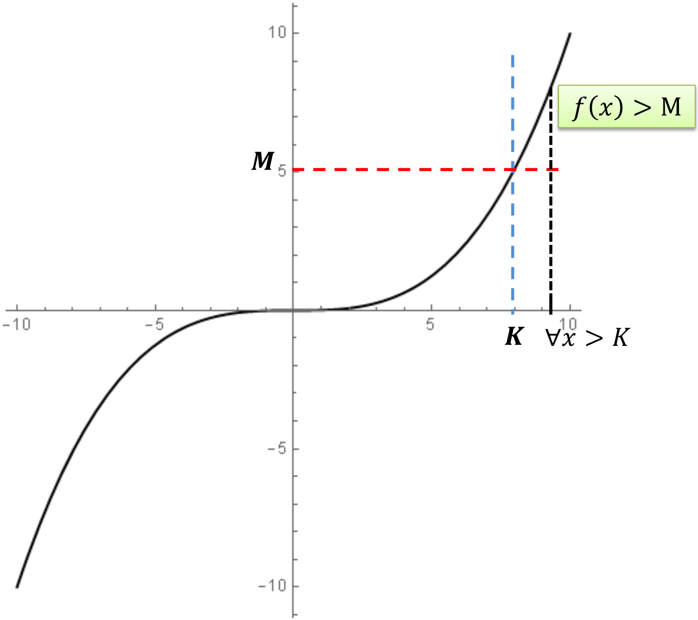
$$f(x) = \frac{x^3}{10}$$



$$f(x) = \frac{x^3}{10}$$



$$f(x) = \frac{x^3}{10}$$



Computations with limits

Let f and g be functions such that

$$\lim_{x\to x_0} f(x) = l, \qquad \lim_{x\to x_0} g(x) = m$$

with l and m both finite. Then

$$\lim_{x\to x_0} f(x) + g(x) = l + m, \qquad \lim_{x\to x_0} f(x) - g(x) = l - m$$

$$\lim_{x\to x_0} f(x) \cdot g(x) = l \cdot m$$

$$\text{if} \quad m \neq 0, \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{l}{m}, \qquad \text{if} \quad m = 0 \quad \text{and} \quad l \neq 0, \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = \infty$$

$$\lim_{x\to x_0} [f(x)]^p = l^p, \qquad \lim_{x\to x_0} a^{f(x)} = a^l$$

$$\text{if} \quad l > 0, \quad \lim_{x\to x_0} [f(x)]^{g(x)} = l^m$$

Undetermined forms

Suppose that

$$\lim_{x \to x_0} f(x) = +\infty, \qquad \lim_{x \to x_0} g(x) = -\infty.$$

What is limit of the sum $f(x) + g(x)$?

This is an undetermined form.
Undetermined forms

$$+\infty - \infty, \quad , 0 \cdot \infty, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0^0, \quad \infty^0, \quad 1^\infty.$$

Intuitive and Notable limits

$$\lim_{x\to+\infty} a^x = \begin{cases} +\infty, & \text{if } a > 1 \\ 0, & \text{if } 0 < a < 1. \end{cases} \qquad \lim_{x\to-\infty} a^x = \begin{cases} 0, & \text{if } a > 1 \\ +\infty, & \text{if } 0 < a < 1. \end{cases}$$

$$\lim_{x\to+\infty} \log_a(x) = \begin{cases} +\infty, & \text{if } a > 1 \\ -\infty, & \text{if } 0 < a < 1. \end{cases} \qquad \lim_{x\to 0^+} \log_a(x) = \begin{cases} -\infty, & \text{if } a > 1 \\ +\infty, & \text{if } 0 < a < 1. \end{cases}$$

$$\lim_{x\to+\infty} \frac{\log x}{x^p} = 0, \qquad p > 0.$$

$$\lim_{x\to+\infty} \frac{x^p}{a^x} = 0, \qquad p > 0, a > 1.$$

$$\lim_{x\to 0^+} x^p \log x = 0.$$

Intuitive and Notable limits

- $f(x) = x^n$, $n \in \mathbb{N}$, $\quad D = \mathbb{R}$

$$\lim_{x \to +\infty} x^n = +\infty, \quad \lim_{x \to -\infty} x^n = \begin{cases} -\infty & n \text{ odd} \\ +\infty & n \text{ even} \end{cases}$$

- $f(x) = x^{-n} = \frac{1}{x^n}$, $n \in \mathbb{N}$, $\quad D = \mathbb{R} \setminus \{0\}$

$$\lim_{x \to +\infty} \frac{1}{x^n} = 0, \quad \lim_{x \to -\infty} \frac{1}{x^n} = 0$$

$$\lim_{x \to 0^-} \frac{1}{x^n} = \begin{cases} -\infty & n \text{ odd} \\ +\infty & n \text{ even} \end{cases}, \quad \lim_{x \to 0^+} \frac{1}{x^n} = +\infty$$

Limits of powers, exponentials and logarithms

- $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x},\ n \in \mathbb{N}, \quad D = \begin{cases} \mathbb{R} & \text{if } n \text{ is odd} \\ [0, +\infty) & \text{if } n \text{ is even} \end{cases}$
  - n odd
  $$\lim_{x \to +\infty} \sqrt[n]{x} = +\infty, \qquad \lim_{x \to -\infty} \sqrt[n]{x} = -\infty$$
  - n even
  $$\lim_{x \to +\infty} \sqrt[n]{x} = +\infty, \qquad \lim_{x \to 0^+} \sqrt[n]{x} = 0$$
- $f(x) = x^{-\frac{1}{n}} = \frac{1}{x^{\frac{1}{n}}},\ n \in \mathbb{N}, \quad D = \begin{cases} \mathbb{R} \backslash \{0\} & \text{if } n \text{ is odd} \\ (0, +\infty) & \text{if } n \text{ is even} \end{cases}$
  - n odd
  $$\lim_{x \to +\infty} x^{-\frac{1}{n}} = 0, \qquad \lim_{x \to -\infty} x^{-\frac{1}{n}} = 0$$
  $$\lim_{x \to 0^-} x^{-\frac{1}{n}} = -\infty, \qquad \lim_{x \to 0^+} x^{-\frac{1}{n}} = +\infty$$
  - n even
  $$\lim_{x \to 0^+} x^{-\frac{1}{n}} = +\infty, \qquad \lim_{x \to +\infty} x^{-\frac{1}{n}} = 0$$

Intuitive and Notable limits

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e, \qquad \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = 1, \qquad \lim_{x\to 0}\frac{e^x-1}{x} = 1, \qquad a>0$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$$

Intuitive and Notable limits

If $\lim_{x\to\dots} g(x) = +\infty$ (or $-\infty$)

$$\lim_{x\to\dots}\left(1+\frac{1}{g(x)}\right)^{g(x)} = e$$

If $\lim_{x\to\dots} h(x) = 0$

$$\lim_{x\to\dots}\frac{\log(1+h(x))}{h(x)} = 1, \qquad \lim_{x\to\dots}\frac{e^{h(x)}-1}{h(x)} = 1$$

$$\lim_{x\to\dots}\frac{\sin(h(x))}{h(x)} = 1, \qquad \lim_{x\to\dots}\frac{1-\cos(h(x))}{h^2(x)} = \frac{1}{2}$$

Why dots? Because what really matters is the behaviour of $h(x)$ and $g(x)$!

Exercises

Compute the following limit:

$$\lim_{x \to +\infty} \frac{x-2}{x^2-3x+2}$$

This is an indeterminate form $\frac{\infty}{\infty}$. We use the usual trick:

$$\frac{x-2}{x^2-3x+2} = \frac{x\left(1-\frac{2}{x}\right)}{x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)} = \frac{1}{x}\frac{1-\frac{2}{x}}{1-\frac{3}{x}+\frac{2}{x^2}} \to 0$$

Now, compute the following limit:

$$\lim_{x \to 2} \frac{x-2}{x^2-3x+2}$$

This time, we have an indeterminate form $\frac{0}{0}$. To solve these kinds of limit, observe that the equation $x^2-3x+2=0$ has solutions $x_1=2$, $x_2=1$. We can thus write:

$$\lim_{x \to 2} \frac{x-2}{x^2-3x+2} = \lim_{x \to 2} \frac{x-2}{(x-2)(x-1)} = \lim_{x \to 2} \frac{1}{x-1} = 1$$

Exercises

Compute the following limit:

$$\lim_{x \to +2} \frac{x^2 - 4}{x^2 - 3x + 2}$$

It is of the form $\frac{0}{0}$. Factorize both the numerator and the denominator:

$$\lim_{x \to +2} \frac{x^2 - 4}{x^2 - 3x + 2} = \lim_{x \to +2} \frac{(x-2)(x+2)}{(x-2)(x-1)} = \lim_{x \to +2} \frac{x+2}{x-1} = 4$$

Exercises

Compute the following limit:

$$\lim_{x\to+\infty} \sqrt{x-2} - \sqrt{x}$$

It is of the form $+\infty - \infty$. Multiply and divide by $\sqrt{x-2} + \sqrt{x}$:

$$\begin{aligned}
\lim_{x\to+\infty} \sqrt{x-2} - \sqrt{x} &= \lim_{x\to+\infty} \frac{(\sqrt{x-2} - \sqrt{x})(\sqrt{x-2} + \sqrt{x})}{\sqrt{x-2} + \sqrt{x}} = \\
&= \lim_{x\to+\infty} \frac{x-2-x}{\sqrt{x-2} + \sqrt{x}} = \\
&= \lim_{x\to+\infty} \frac{-2}{\sqrt{x-2} + \sqrt{x}} = 0
\end{aligned}$$

Exercises

Compute the following limit:

$$\lim_{x\to 0}\frac{\sqrt{1+2x}-1}{3x}$$

It is of the form $\frac{0}{0}$. Multiply and divide by $\sqrt{1+2x}+1$:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+2x}-1}{3x} &= \lim_{x\to 0}\frac{(\sqrt{1+2x}-1)(\sqrt{1+2x}+1)}{3x(\sqrt{1+2x}+1)} = \\
&= \lim_{x\to 0}\frac{1+2x-1}{3x(\sqrt{1+2x}+1)} = \\
&= \lim_{x\to 0}\frac{2x}{3x(\sqrt{1+2x}+1)} = \\
&= \lim_{x\to 0}\frac{2}{3(\sqrt{1+2x}+1)} = \frac{1}{3}
\end{aligned}$$

Exercises

Compute the following limit:

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{3x}=\lim_{x\to+\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{3}=e^{3}$$

Compute the following limit:

$$\lim_{x\to+\infty}\left(1+\frac{3}{x}\right)^{x}$$

Let $g(x)=\frac{x}{3}$ and notice that $\lim_{x\to+\infty} g(x)=+\infty$ then we have:

$$\lim_{x\to+\infty}\left(1+\frac{3}{x}\right)^{x}=\lim_{x\to+\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{x}=\lim_{x\to+\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}\cdot 3}$$

$$=\lim_{x\to+\infty}\left(\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}}\right)^{3}=e^{3}$$

Exercises

Compute the following limit:

$$\lim_{x\to+\infty}\left(1+\frac{7}{3x}\right)^{x-1}$$

Let $g(x)=\frac{3x}{7}$ and notice that $\lim_{x\to+\infty} g(x)=+\infty$. Then

$$\lim_{x\to+\infty}\left(1+\frac{7}{3x}\right)^{x-1}=\lim_{x\to+\infty}\left(1+\frac{1}{\frac{3x}{7}}\right)^{x-1}$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{\frac{3x}{7}}\right)^{\frac{3x}{7}\frac{7}{3}}\left(1+\frac{1}{\frac{3x}{7}}\right)^{-1}=e^{\frac{7}{3}}$$

Exercises

Compute the following limit:

$$\lim_{x \to 0} \frac{\log(1 + 3x^2)}{x^2}$$

Looks similar to the notable limit:

$$\lim \frac{\log(1 + g(x))}{g(x)} = 1, \quad \text{with} \quad g(x) \to 0$$

Let $g(x) = 3x^2$ and notice that if $x \to 0$, then also $g(x) = 3x^2 \to 0$.

Then we have

$$\lim_{x \to 0} \frac{\log(1 + 3x^2)}{x^2} = \lim_{x \to 0} \frac{\log(1 + 3x^2)}{3x^2} \cdot 3 = 1 \cdot 3 = 3$$

Exercises

Compute the following limits:

$$\lim_{x\to 0} \frac{\sin(3x)}{2x} \qquad \text{Hint: set } g(x) = 3x$$

$$\lim_{x\to 0} \frac{2^{2x^3} - 1}{x^3} \qquad \text{Hint: set } g(x) = 2x^3$$

$$\lim_{x\to 0} \frac{\sin x}{x^3} \qquad \text{Hint: observe that } \frac{\sin x}{x^3} = \frac{1}{x^2}\frac{\sin x}{x}$$

$$\lim_{x\to 0} \frac{1 - \cos x}{x} \qquad \text{Hint: observe that } \frac{1 - \cos x}{x} = x\frac{1 - \cos x}{x^2}$$

Asymptotes

Vertical Asymptotes

- Compute the **Domain**
- Look for vertical asymptotes at **finite** limit points of the Domain
- Compute $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$
- Is any of these $+\infty$ or $-\infty$?
    - YES $\Rightarrow x = x_0$ is a vertical asymptote
    - NO $\Rightarrow$ there is NO vertical asymptote

Asymptotes

Horizontal Asymptotes

- Compute the **Domain**
- Is $+\infty$ and/or $-\infty$ an extreme point of the domain?
- if YES:
  - Compute $\lim_{x \to +\infty} f(x)$. If the limit is a finite number ℓ_1, then $y = \ell_1$ is a horizontal asymptote at $+\infty$.
  - Compute $\lim_{x \to -\infty} f(x)$. If the limit is a finite number ℓ_2, then $y = \ell_2$ is a horizontal asymptote at $-\infty$.
- if the domain is NOT unbounded or any of the limits ($\lim_{x \to +\infty} f(x)$ and/or $\lim_{x \to -\infty} f(x)$) are ∞, then there is no horizontal asymptote (eventually only in one of the two sides).

Limits of functions: exercises

Determine the asymptotes of the following function:

$$f(x) = \frac{5 - x^2}{x + 3}$$

First, we determine the domain: $D = \mathbb{R} \setminus \{-3\}$. Thus, we compute the limit in -3, $+\infty$ and $-\infty$. We have:

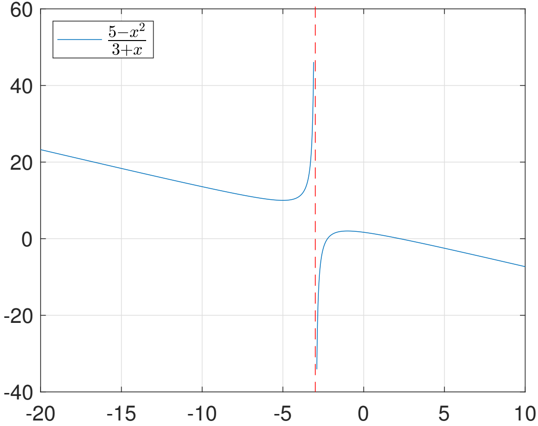
$$\lim_{x \to -3^+} \frac{5 - x^2}{x + 3} = \frac{5 - (-3^+)^2}{0^+} = -\frac{4}{0^+} = -\infty$$

$$\lim_{x \to -3^-} \frac{5 - x^2}{x + 3} = \frac{5 - (-3^-)^2}{0^-} = -\frac{4}{0^-} = +\infty$$

$$\lim_{x \to +\infty} \frac{5 - x^2}{x + 3} = \lim_{x \to +\infty} \frac{x^2 \left(\frac{5}{x^2} - 1\right)}{x \left(1 + \frac{3}{x}\right)} = \lim_{x \to +\infty} x \left(\frac{\frac{5}{x^2} - 1}{1 + \frac{3}{x}}\right) = +\infty \times (-1) = -\infty$$

$$\lim_{x \to -\infty} \frac{5 - x^2}{x + 3} = \lim_{x \to -\infty} \frac{x^2 \left(\frac{5}{x^2} - 1\right)}{x \left(1 + \frac{3}{x}\right)} = \lim_{x \to -\infty} x \left(\frac{\frac{5}{x^2} - 1}{1 + \frac{3}{x}}\right) = -\infty \times (-1) = +\infty$$

Limits of functions: exercises



The function has a vertical asymptote in $x = -3$. The function has no horizontal asymptotes because $\lim_{x \to \pm\infty} f(x) = \mp\infty$.

Limits of functions: exercises

Determine the asymptotes of the following function:

$$f(x) = \frac{\sqrt{x^2+5}}{x+1}$$

The domain is $D = \mathbb{R} \backslash \{-1\}$. We compute the limit in -1, $+\infty$, $-\infty$.

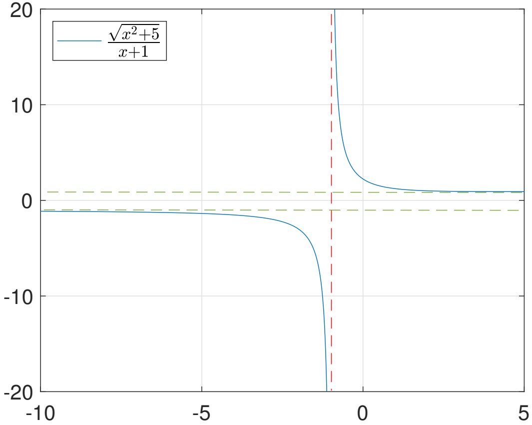
$$\lim_{x \to -1^+} \frac{\sqrt{x^2+5}}{x+1} = \frac{\sqrt{(-1^+)^2+5}}{0^+} = \frac{\sqrt{6}}{0^+} = +\infty$$

$$\lim_{x \to -1^-} \frac{\sqrt{x^2+5}}{x+1} = \frac{\sqrt{(-1^-)^2+5}}{0^-} = \frac{\sqrt{6}}{0^-} = -\infty$$

$$\lim_{x \to +\infty} \frac{\sqrt{x^2+5}}{x+1} = \lim_{x \to +\infty} \frac{\sqrt{x^2\left(1+\frac{5}{x^2}\right)}}{x(1+\frac{1}{x})} = \lim_{x \to +\infty} \frac{|x|}{x} \frac{\sqrt{1+\frac{5}{x^2}}}{(1+\frac{1}{x})} = 1$$

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+5}}{x+1} = \lim_{x \to -\infty} \frac{\sqrt{x^2\left(1+\frac{5}{x^2}\right)}}{x(1+\frac{1}{x})} = \lim_{x \to -\infty} \frac{|x|}{x} \frac{\sqrt{1+\frac{5}{x^2}}}{(1+\frac{1}{x})} = -1$$

Limits of functions: exercises



The function has a vertical asymptote in $x = -1$. The function has two horizontal asymptotes in $y = 1$ and $y = -1$.

Limits of functions: exercises

Determine the asymptotes of the following function:

$$f(x) = \frac{x^2+1}{x^2-1}$$

The domain is $D = \mathbb{R}\backslash\{-1, 1\}$. We compute the limit in -1, 1, $+\infty$, $-\infty$.

$$\lim_{x\to -1^+} \frac{x^2+1}{x^2-1} = \frac{(-1^+)^2+1}{0^-} = \frac{2}{0^-} = -\infty$$

$$\lim_{x\to -1^-} \frac{x^2+1}{x^2-1} = \frac{(-1^-)^2+1}{0^+} = \frac{2}{0^+} = +\infty$$

$$\lim_{x\to 1^+} \frac{x^2+1}{x^2-1} = \frac{(1^+)^2+1}{0^+} = \frac{2}{0^+} = +\infty$$

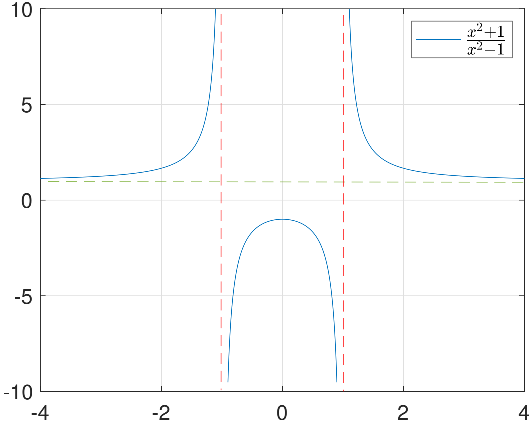
$$\lim_{x\to 1^-} \frac{x^2+1}{x^2-1} = \frac{(1^-)^2+1}{0^-} = \frac{2}{0^-} = -\infty$$

Limits of functions: exercises

$$\lim_{x\to+\infty}\frac{x^2+1}{x^2-1}=\lim_{x\to+\infty}\frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2(1-\frac{1}{x^2})}=\lim_{x\to+\infty}\frac{\left(1+\frac{1}{x^2}\right)}{(1-\frac{1}{x^2})}=1$$

$$\lim_{x\to-\infty}\frac{x^2+1}{x^2-1}=\lim_{x\to-\infty}\frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2(1-\frac{1}{x^2})}=\lim_{x\to-\infty}\frac{\left(1+\frac{1}{x^2}\right)}{(1-\frac{1}{x^2})}=1$$

Limits of functions: exercises



The function has two vertical asymptotes in $x = -1$ and $x = 1$. The function has one horizontal asymptotes in $y = 1$.

Continuity: the intuition

Intuitively, a function is continuous if its graph is a “single unbroken” curve

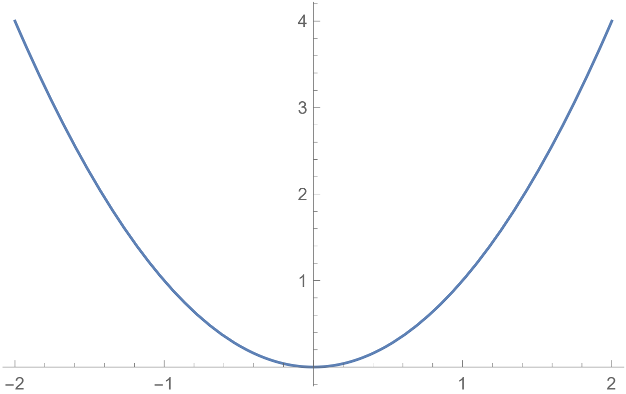
Examples

- $f(x) = x^2$, $f(x) = \sin(x)$, $f(x) = |x|$ are continuous functions
- $f(x) = \text{sign}(x)$, $f(x) = \frac{1}{x}$, $f(x) = \frac{\sin(x)}{x}$, $f(x) = \begin{cases} x & x \leq -4 \\ -4 & -4 < x < 2 \\ 3 - x & x \geq 2 \end{cases}$

  are not continuous functions

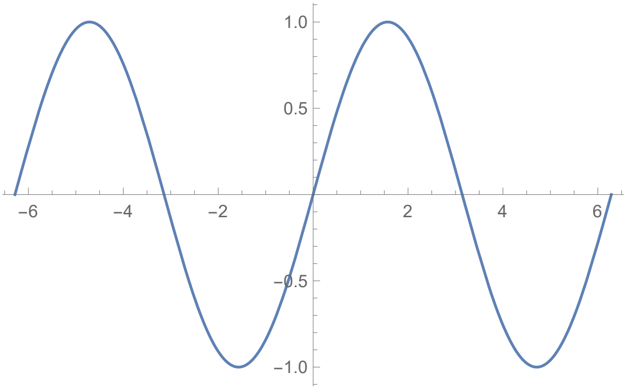
Continuity: the intuition

$$f(x) = x^2$$



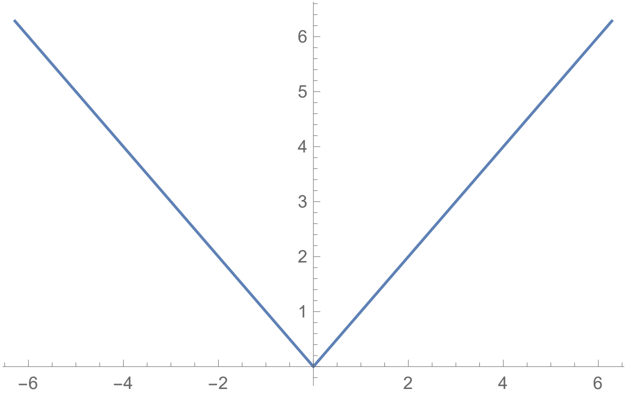
Continuity: the intuition

$$f(x) = \sin(x)$$



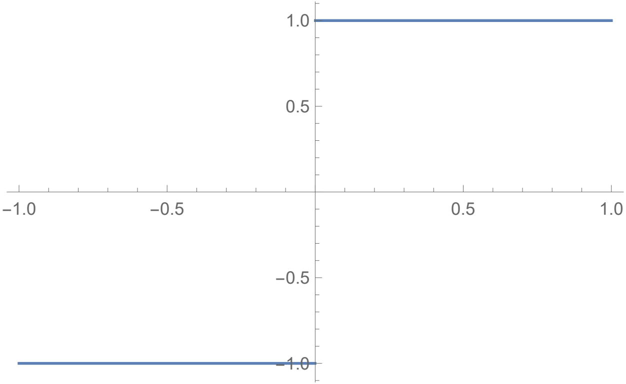
Continuity: the intuition

$$f(x) = |x|$$



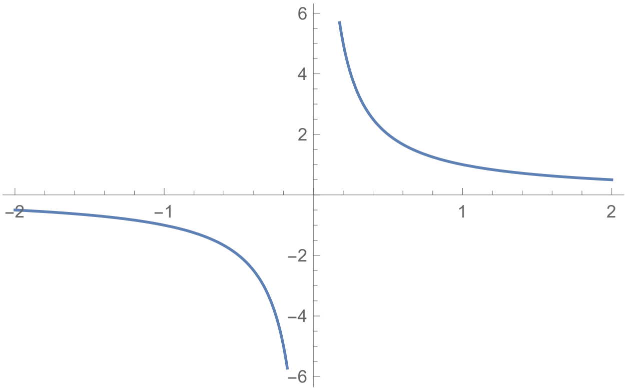
Continuity: the intuition

$$f(x) = \mathrm{sign}(x)$$



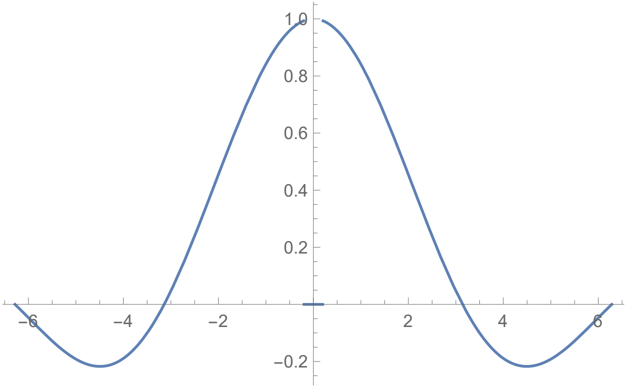
Continuity: the intuition

$$f(x) = \frac{1}{x}$$



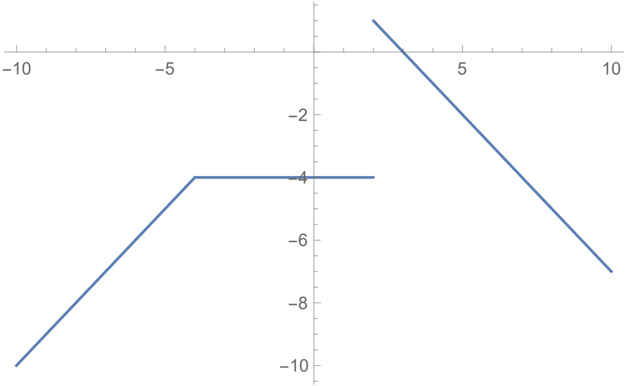
Continuity: the intuition

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Continuity: the intuition

$$f(x) = \begin{cases} x & x \leq -4 \\ -4 & -4 < x < 2 \\ 3 - x & x \geq 2 \end{cases}$$



Continuity: the definition

Definition (Continuity at a point)

Let $f : D \to \mathbb{R}$ be a function and let $x_0 \in D$. We say that f is continuous in x_0 if $\lim_{x \to x_0} f(x)$ exists finite and:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Definition (Continuity in an interval)

Let $f : I \subseteq D \to \mathbb{R}$ be a function. We say that f is continuous in I if f ia continuous in any point $x \in I$

Continuity: conditions for continuity

Conditions for continuity

Suppose we have a function $f : D \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$.

Then, $f(x)$ is continuous in x_0 if **ALL** these three conditions hold:

1. $x_0 \in D$
2. $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L \qquad L \neq \pm\infty$
3. $L = f(x_0)$

If at least one of the 3 conditions fails, we have a **discontinuity in** x_0

Classification of discontinuities

Classification of discontinuities

Let $f : D \to \mathbb{R}$ be a function.

- If:
$$\exists L_1 = \lim_{x \to x_0^+} f(x) \quad \text{and} \quad \exists L_2 = \lim_{x \to x_0^-} f(x)$$
but $L_1 \neq L_2$, the function f has a **jump discontinuity** in x_0.
- If:
$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$$
but either $L \neq f(x_0)$ or $x_0 \notin D$, the function f has a **removable discontinuity** in x_0.
- If at least one of the two limits:
$$\lim_{x \to x_0^+} f(x) \quad \text{or} \quad \lim_{x \to x_0^-} f(x)$$
is infinite or does not exist, the function f has an **essential discontinuity** in x_0.

Theorem on composition of continuous functions

Theorem

Suppose that f and g are continuous in x_0.

- *Then $f+g$, $f-g$, $f \cdot g$ are continuous in x_0.*
- *If in addition $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous in x_0.*
- *If in addition $f(x_0) > 0$, then f^g is continuous in x_0.*
- *If g is continuous in x_0 and f is continuous in $g(x_0) = y_0$, then $f \circ g$ is continuous in x_0*
- *If f is surjective, continuous and strictly increasing (or strictly decreasing) then f^{-1} is also continuous and strictly increasing (or strictly decreasing).*

Maximum and minimum of a function

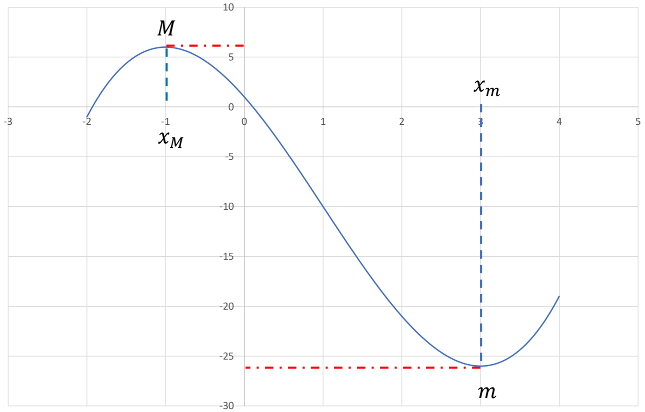
Intuitively, the maximum and minimum of a function f in a subset I of its domain D are the maximum and minimum values the function can reach in I.

Definition

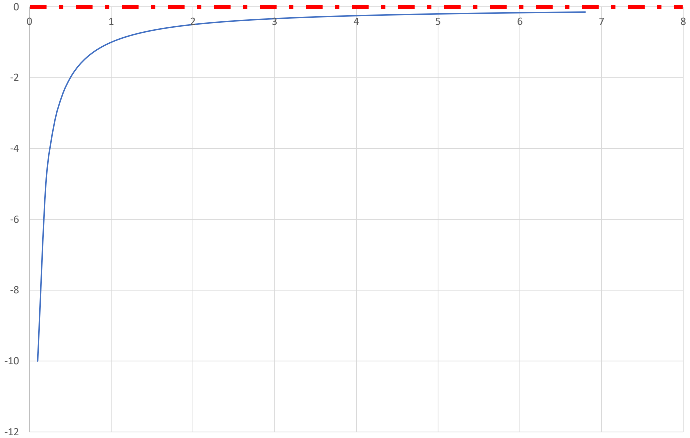
Let $f : I \subseteq D \to \mathbb{R}$. We say that f admits a maximum $M \in \mathbb{R}$ in the interval I if:

- there is a point $x_M \in I$ such that $f(x_M) = M$
- for any other point $x \in I$ it holds that $f(x) \leq M$

Example



Example



Maximum and minimum of a function

Definition

Let $f : I \subseteq D \to \mathbb{R}$. We say that f admits a minimum $m \in \mathbb{R}$ in the interval I if:

- there is a point $x_m \in I$ such that $f(x_m) = m$
- for any other point $x \in I$ it holds that $f(x) \geq m$

Weierstrass theorem

Maxima and minima are not guaranteed to exist. The Weierstrass theorem provides **sufficient** conditions under which a function has maxima and minima in a subset of its domain.

Theorem (Weierstrass Theorem - compact version)

Any function $f : D \to \mathbb{R}$ which is continuous function on a closed and bounded interval $[a, b]$ admits a maximum and a minimum in $[a, b]$.

Theorem (Weierstrass Theorem - extended version)

Let $f : D \to \mathbb{R}$ be a function. If:

- *f is continuous in an interval $[a, b]$*
- *$[a, b]$ is closed and bounded*

then there is a point $x_m \in [a, b]$ and a point $x_M \in [a, b]$ such that
$f(x_m) = m$ is the **minimum** *of f in $[a, b]$ and*
$f(x_M) = M$ is the **maximum** *of f in $[a, b]$.*

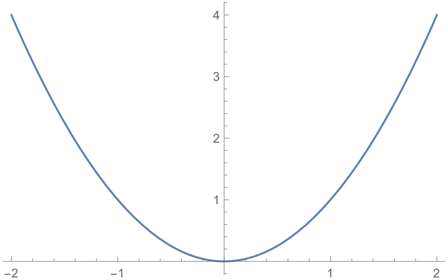
Weierstrass theorem

To apply Weierstrass Theorem three conditions must hold:

- f is continuous in $[a, b]$
- The interval $[a, b]$ is closed (i.e. no round brackets)
- The interval $[a, b]$ is bounded (i.e. $a \neq -\infty$ and $b \neq +\infty$)

Weierstrass theorem: examples

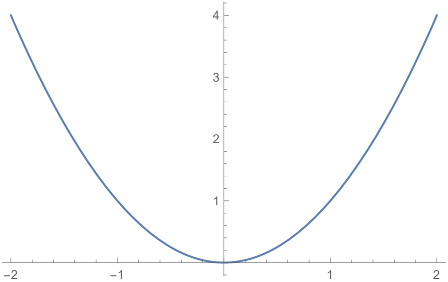
$$f(x) = x^2 \quad \text{in } [-2, 2]$$



The assumptions of the Weierstrass theorem are satisfied. Note that f has a minimum ($m = 0$) for $x = 0$ and a maximum ($M = 4$) for $x = -2$ and $x = 2$.

Weierstrass theorem: examples

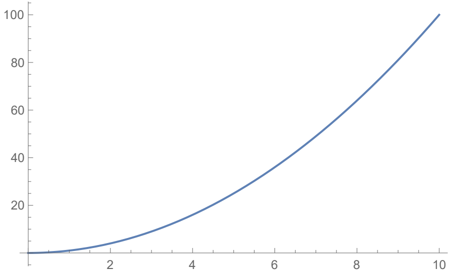
$$f(x) = x^2 \quad \text{in } (-2, 2)$$



The assumptions of the Weierstrass theorem are not satisfied because $(-2, 2)$ is not closed. Note that f has a minimum ($m = 0$) for $x = 0$ but has no maxima, why?

Weierstrass theorem: examples

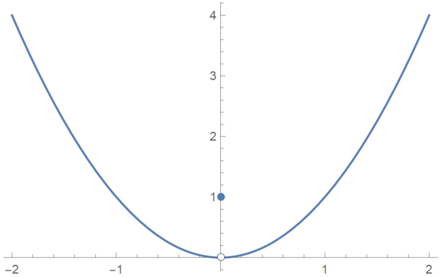
$$f(x) = x^2 \quad \text{in } [0, +\infty)$$



The assumptions of the Weierstrass theorem are not satisfied because $[0, +\infty)$ is unbounded. Note that f has a minimum $(m = 0)$ for $x = 0$ but has no maxima, why?

Weierstrass theorem: examples

$$f(x) = \begin{cases} x^2 & x \in [-2, 2] \setminus \{0\} \\ 1 & x = 0 \end{cases}$$



The assumptions of the Weierstrass theorem are not satisfied because f has a removable discontinuity in $x = 0$. Note that f has a maximum for $x = -2$ and $x = 2$ but has no minima, why?

Existence of zeros (or Intermediate zero theorem)

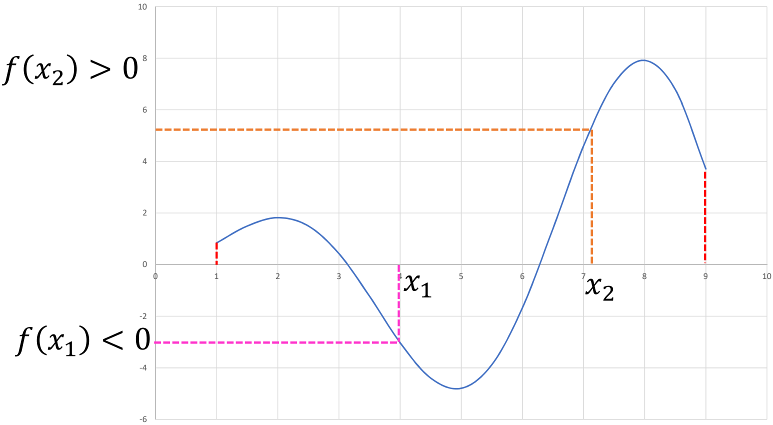
For any continuous function in a closed and bounded interval which in takes both positive and negative values in that interval, there exists at least one point in which the function is equal to zero.

Theorem (Existence of zeros)

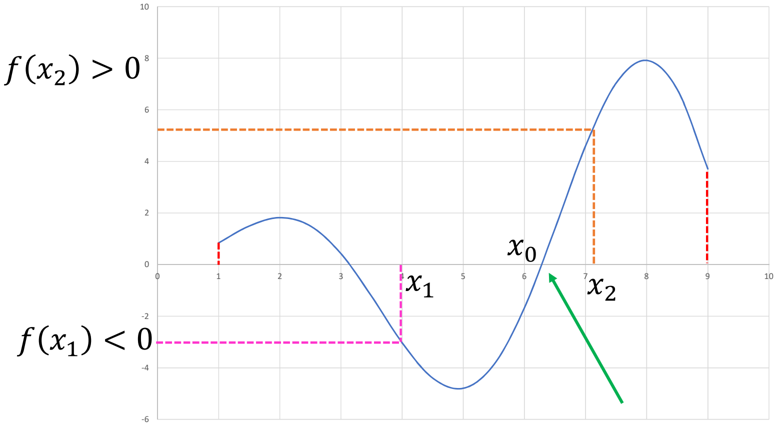
Let $f : D \to \mathbb{R}$ be a function. If:

- *f is continuous in $[a, b]$*
- *$[a, b]$ is a closed and bounded interval*
- *there are two points $x_1, x_2, \in [a, b]$ such that:*
  - $f(x_1) < 0$
  - $f(x_2) > 0$

then there exists a point $x_1 < x_0 < x_2$ (or $x_2 < x_0 < x_1$) such that $f(x_0) = 0$.



$f(x_2) > 0$
$f(x_1) < 0$
x_1
x_2
10
8
6
4
2
0
-2
-4
-6
0
1
2
3
4
5
6
7
8
9
10


$f(x_2) > 0$
$f(x_1) < 0$
x_0
x_1
x_2
10
8
6
4
2
0
-2
-4
-6
0
1
2
3
4
5
6
7
8
9
10