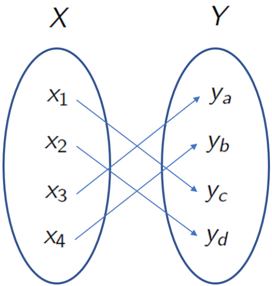
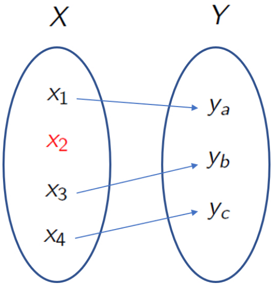
Functions: the intuition

Intuitively, a function is a rule that associates to **each** element of a set X, **only and only one** element in another set Y.

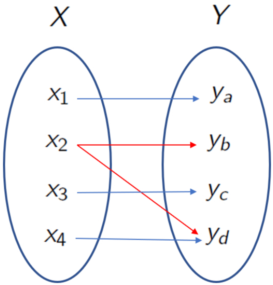


This is a function between X and Y

Functions: the intuition, cont'd



This is **not** a function between X and Y because $x_2 \in X$ is not mapped into any element in Y



This is **not** a function between X and Y because x_2 is mapped into more than one element in Y

Functions: the definition

Definition

Let $D \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. A function is a rule that associates to each element of D one and only one element of $\mathbb{R}$. In symbols we write:

$$f : D \to \mathbb{R}$$

meaning that

$$\forall x \in D \ \Rightarrow \exists!\ y \in \mathbb{R} : \ y = f(x)$$

The set D is called the **domain** of the function.

- The variable x is called “independent variable”, it can take values in D
- The variable y is called the “dependent variable”, it can take values in $\mathbb{R}$.
- In economics x is called the exogenous variable and y is called the endogenous variable

The domain of a function

Definition

Let $f : D \to \mathbb{R}$ be a function. The domain of the function, $D \subseteq \mathbb{R}$, is the set of all values $x \in \mathbb{R}$ for which the expression $f(x)$ makes sense.

Three cases require computations:

1. $f(x)$ contains a division
2. $f(x)$ contains a root with even power
3. $f(x)$ contains a logarithm

or any combinations of the above conditions.

Domain of rational functions

Let $f(x) = \dfrac{P(x)}{Q(x)}$, and assume that $Q(x)$ makes sense for all $x \in \mathbb{R}$.
Then the function $f(x)$ is well defined if and only if $Q(x) \neq 0$. This means that

$$D = \{x \in \mathbb{R} : \; Q(x) \neq 0\}$$

Examples

- $f(x) = \dfrac{x+3}{x^2-1}$

$$D = \{x \in \mathbb{R} : \; x \neq \pm 1\}$$

- $f(x) = e^{\frac{x+5}{x-3}}$

$$D = \{x \in \mathbb{R} : \; x \neq 3\}$$

Domain of irrational functions

Let $f(x) = \sqrt[n]{G(x)}$, and assume that $G(x)$ makes sense for all $x \in \mathbb{R}$.
There are two possibilities:

- if n is even, then the function $f(x)$ is well defined if and only if $G(x) \geq 0$. This means that

$$D = \{x \in \mathbb{R} : \; G(x) \geq 0\}$$

- if n is odd, then the function $f(x)$ is well defined for all $x \in \mathbb{R}$

Examples

- $f(x) = \sqrt{x^2 - 5}$
  This is an irrational function with even index ($n = 2$). Then we have

$$D = \{x \in \mathbb{R} : \; x \leq -\sqrt{5} \text{ or } x \geq \sqrt{5}\}$$

- $f(x) = \sqrt[3]{x + 2}$
  This is an irrational function with odd index ($n = 3$). Then we have

$$D = \mathbb{R}$$

Domain of logarithmic functions

Let $f(x) = \log H(x)$, and assume that $H(x)$ makes sense for all $x \in \mathbb{R}$. Then the function $f(x)$ is well defined if and only if $H(x) > 0$. This means that

$$D = \{x \in \mathbb{R} : \; H(x) \geq 0\}$$

Examples

- $f(x) = \log(1 - x^2)$

$$D = \{x \in \mathbb{R} : \; -1 < x < 1\}$$

- $f(x) = \log(x^2 + 2)$
  Since $x^2 + 2 > 0$ for all $x \in \mathbb{R}$ we get that

$$D = \mathbb{R}$$

Example

These three condition must be combined together if a function contains fractions, roots and logarithms.

Example

$$f(x) = \frac{x}{\log(x+2)}$$

We have that:

- $\log(x+2) \neq 0$ for the existence of the fraction
- $x+2 > 0$ for the existence of the logarithm

Hence we have

$$\begin{cases} \log(x+2) & \neq 0 \\ x+2 & > 0 \end{cases}$$

which implies that

$$D = \{x \in \mathbb{R} : \; x > -2 \text{ and } x \neq -1\}$$

The range of a function

Intuitively, the range of a function is the set of all points of $\mathbb{R}$ that can be obtained by applying the function f to the points of D. That is, the set of all possible “dependent variables”.

We can also say that the range os the set of all *images* of points of the domain through the function f

Definition

Let $f : D \to \mathbb{R}$ be a function. The range of f is the set:

$$R_f = \{y \in \mathbb{R} \mid \exists x \in D \ : \ y = f(x)\}$$

Examples

- $y = f(x) = x, \quad D = \mathbb{R}, \quad R_f = \mathbb{R}$, because by applying $f(x)$ to each $x \in D$ we obtain any point in $\mathbb{R}$.
- $y = f(x) = x^2, \quad D = \mathbb{R}, \quad R_f = \{x \in \mathbb{R} \mid x \geq 0\}$, because by applying $f(x)$ to each $x \in D$ we obtain only zero or a positive number.
- $y = f(x) = \frac{1}{x}, \quad D = \mathbb{R}\backslash\{0\}, \quad R_f = \mathbb{R}\backslash\{0\}$, because by applying $f(x)$ to each $x \in D$ we obtain any point of $\mathbb{R}$ except zero.

Odd/even functions

Definition

Let $f : D \to \mathbb{R}$ be a function. f is ***even*** if

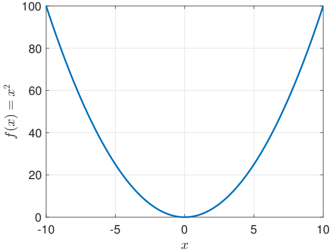
1. for any $x \in D$ then also $-x \in D$
2. $f(-x) = f(x)$

Notice that both conditions must hold.

Because of $f(x) = f(-x)$ for all $x \in \mathbb{R}$, then the plot of the graph of the function is symmetric with respect to the axis $x = 0$.

Example of an even function

$$f(x) = x^2$$



Indeed: The domain of the functions is $D = \mathbb{R}$. For all $x \in D$

- $-x \in D$
- $f(-x) = (-x)^2 = x^2 = f(x)$

Odd/even functions

Definition

Let $f : D \to \mathbb{R}$ be a function. f is ***odd*** if

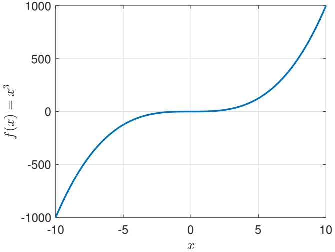
1. for any $x \in D$ then also $-x \in D$
2. $f(-x) = -f(x)$

Notice that both conditions must hold.

Because of $f(0) = f(-0) = -f(0)$ then $f(0) = 0$. The plot of the graph of the function must cross the origin.

Example of an even function

$$f(x) = x^3$$

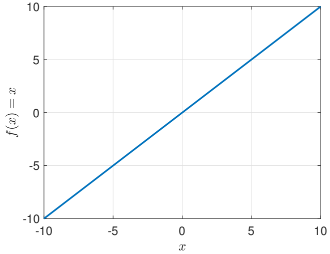


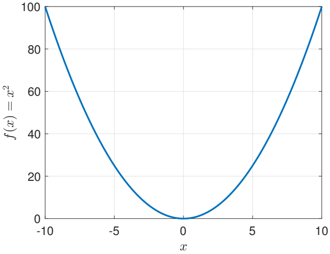
Indeed: The domain of the functions is $D = \mathbb{R}$. For all $x \in D$

- $-x \in D$
- $f(-x) = (-x)^3 = -x^3 = -f(x)$

Increasing and decreasing functions: the intuition

To get an intuition of when a function is increasing or decreasing, simply look at its graph:





- The function $f(x) = x$ is increasing in its entire domain
- The function $f(x) = x^2$ in decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$

Increasing and decreasing functions: the definition

Definition

Let $f : D \to \mathbb{R}$ a function and let $I = (a, b) \subseteq D$ an open interval in D. The function f is strictly increasing in I if:

$$\forall x_1, x_2 \in I \ : \ x_1 < x_2 \ \Rightarrow \ f(x_1) < f(x_2)$$

The function f is increasing if:

$$\forall x_1, x_2 \in I \ : \ x_1 < x_2 \ \Rightarrow \ f(x_1) \leq f(x_2)$$

Increasing and decreasing functions: the definition

Definition

Let $f : D \to \mathbb{R}$ a function and let $I = (a, b) \subset D$ an open interval in D. The function f is strictly decreasing in I if:

$$\forall x_1, x_2 \in I \ : \ x_1 < x_2 \ \Rightarrow \ f(x_1) > f(x_2)$$

The function f is decreasing if:

$$\forall x_1, x_2 \in I \ : \ x_1 < x_2 \ \Rightarrow \ f(x_1) \geq f(x_2)$$

Increasing and decreasing functions: examples

$$f(x) = 2x + 3$$

Show that the function is strictly increasing in $\mathbb{R}$.

Note that $D = \mathbb{R}$. Consider two points $x_1, x_2 \in \mathbb{R}$, with $x_1 < x_2$. We will show that $f(x_1) < f(x_2)$ (Notice that $f(x_1) = 2x_1 + 3$ and $f(x_2) = 2x_2 + 3$)

To do this, we apply properties of real numbers:

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow 2x_1 + 3 < 2x_2 + 3$$

Thus, f is strictly increasing in $\mathbb{R}$.

Increasing and decreasing functions: examples, cont'd

$$f(x) = x^2$$

Show that the function is decreasing in $(-\infty, 0)$.

Note that $D = \mathbb{R}$. Consider two points $x_1, x_2 \in (-\infty, 0)$, with $x_1 < x_2$. We will show that $f(x_1) > f(x_2)$ (Notice that $f(x_1) = x_1^2$ and $f(x_2) = x_2^2$)

To do this, we apply properties of real numbers:

Since $x_1 < x_2$ and they are both negative, we get that $|x_1| > |x_2|$, then it holds that

$$x_1 < x_2 \Rightarrow (x_1)^2 > (x_2)^2$$

Thus, f is strictly decreasing in $(-\infty, 0)$.

A more general definition of function

Definition

Given two sets $X \subseteq \mathbb{R}$, $Y \subseteq \mathbb{R}$ a function f is a rule that associates to each element $x \in X$ one and only one element $y \in Y$. We write:

$$f : X \to Y$$

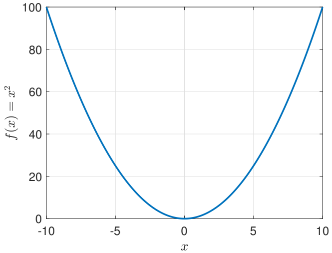
meaning that

$$\forall x \in X \;\Rightarrow \exists!\, y \in Y : \; y = f(x)$$

Notice that for the function f to be well defined we must have that $X \subseteq D$. The set D is the largest set of real numbers for which the expression $f(x)$ makes sense.

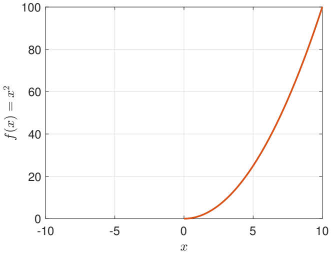
A more general definition of function, example

Let's consider the same function $f(x) = x^2$ defined in two different domains:



$$f(x) : \mathbb{R} \to \mathbb{R}$$

$$f(x) = x^2$$



$$f(x) : [0, +\infty) \to \mathbb{R}$$

$$f(x) = x^2$$

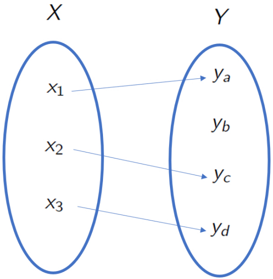
Even if the formula $f(x) = x^2$ is the same in both cases, the two functions are completely different.

A function is not only determined by the form of $f(x)$. Sets X and Y matter!

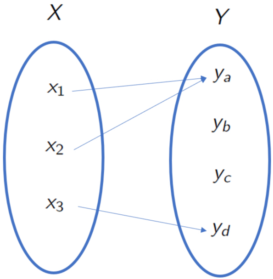
Important: When X and Y are not specified it is implicitly assumed that $X = D$ and $Y = \mathbb{R}$.

Injective functions: the intuition

Intuitively, a function $f : X \to Y$ is injective if the images of distinct points in X correspond to distinct points in Y



This function is injective because distinct elements of X are mapped into distinct elements of Y



This function is **not** injective because x_1 and x_2 are mapped into the same element of Y

Injective functions: the definition

Definition

Let $f : X \to Y$ be a function, with $X \subseteq D$, $Y \subseteq \mathbb{R}$. f is said to be injective in X if

$$\forall x_1, x_2 \in X, \text{ if } x_1 \neq x_2, \Rightarrow \ f(x_1) \neq f(x_2)$$

Equivalently: f is said to be injective in X if

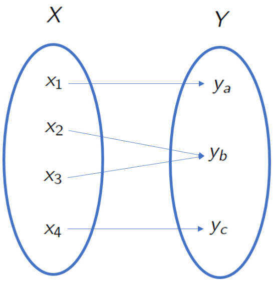
$$\text{for all } \ x_1, x_2 \in X \text{ such that } \ f(x_1) = f(x_2) \text{ then } x_1 = x_2$$

Examples

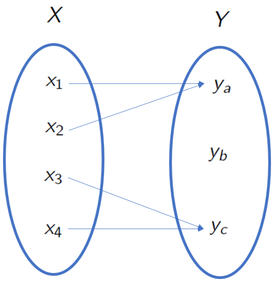
- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not injective.
  Indeed, $f(2) = 4$ and $f(-2) = 4$. Thus, f maps $2, -2 \in \mathbb{R}$ to the same point $y = 4$.
- The function $f : [0, +\infty) \to \mathbb{R}$, $f(x) = x^2$ is injective!
  There is at most one positive number x such that $x^2 = y$, for all $y \in \mathbb{R}$.
- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is injective.
  Indeed, f maps each $x \in \mathbb{R}$ to distinct points $y \in \mathbb{R}$.

Surjective functions: the intuition

Intuitively, a function $f : X \to Y$ is surjective if all the elements of the co-domain are "reached" by the function



This function is surjective because all elements of Y are "reached" by f. Note however that f is not injective



This function is **not** surjective because there are no elements in X that are mapped into y_b. Note also that f is not injective

Surjective functions: the definition

Definition

A function $f : X \to Y$ is said to be surjective if

$$\forall y \in Y, \quad \exists x \in X : \; y = f(x)$$

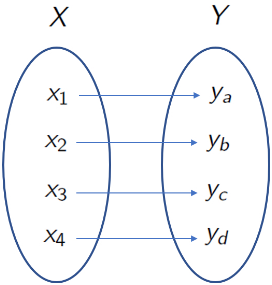
Examples

- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not surjective.
  Indeed, $y = x^2$ is always a non-negative number, and therefore negative real numbers (which belong to the co-domain) are **not** "reached" by the function.
- The function $f : \mathbb{R} \to [0, +\infty)$, $f(x) = x^2$ is surjective!
  Every point $y \in [0, +\infty)$ can be "reached" by the function.
- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is surjective.
  Every point of $\mathbb{R}$ can be "reached" by the function.

Bijective functions

Definition

A function $f : X \to Y$ is said to be bijective if it is both injective and surjective.



This function is injective because it associates to distinct elements in X distinct elements in Y and at the same time it is surjective because all elements of Y are "reached" by the function.

The inverse function

Why are bijective functions important? Bijective functions are **invertible**

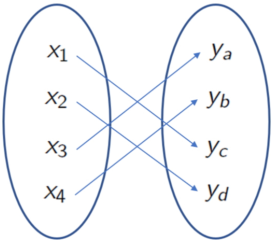
Theorem

Let $f : X \to Y$ be a function, with $X \subseteq D$ and $Y \subseteq \mathbb{R}$.

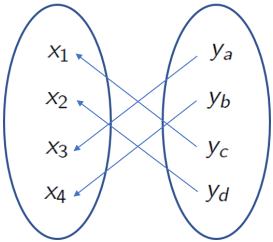
*f is invertible **if and only if** f is bijective.*

Moreover the inverse function $f^{-1} : Y \to X$ exists and it is unique.

$f : X \to Y$



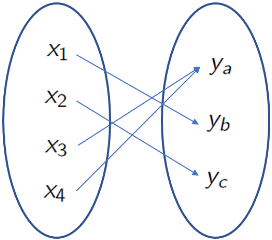
$f^{-1} : Y \to X$



The inverse function: existence

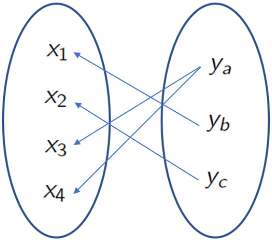
If the function f is not bijective, the inverse function does not exist

$f : X \to Y$



Note that this function is not injective

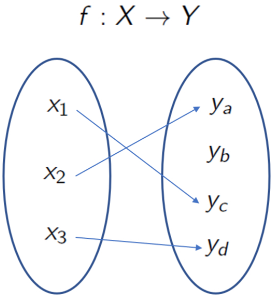
$f^{-1} : Y \to X$



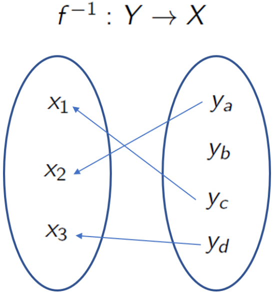
$f^{-1} : Y \to X$ **cannot** be a function because it associates to y_a more than one element in X

The inverse function: existence, cont'd

If the function f is not bijective, the inverse function does not exist



Note that this function is not surjective



$f^{-1} : Y \to X$ **cannot** be a function because there are elements in Y that are not mapped into any element in X

The inverse function

How is the inverse function defined?

Definition

Let $f : X \to Y$ be an invertible function, with $X \subseteq D$ and $Y \subseteq \mathbb{R}$.
Then the inverse function $f^{-1} : Y \to X$ is the function that verifies

$$f(f^{-1}(y)) = y \text{ and } f^{-1}(f(x)) = x$$

Sufficient conditions for invertibility

Is there a **sufficient condition** that guarantees invertibility of a function?

Theorem

Sufficient conditions for a function to be invertible Let $f : X \to Y$ be a function, with $X \subseteq D$ and $Y \subseteq \mathbb{R}$.

If f is strictly monotonic in X (i.e. strictly increasing or strictly decreasing in X) and Y coincides with the set of all images of real numbers in X, then f is invertible.

Two conditions must hold:

1. f must be strictly monotonic in X
2. Y must coincides with the set of all images of elements in X

Notice that the theorem goes in one direction only!!

The computation of the inverse function

Problem: given an invertible function $f : X \to Y$, how do we determine the inverse of f?

Solution: given $y \in Y$, we want to find $x \in X$ such that $y = f(x)$.

For doing this we just solve the equation $y = f(x)$ with respect to x!

The inverse function: examples

- $f : \mathbb{R} \to [0, +\infty), \quad f(x) = x^2$

  The function f defined in this way is **not** injective, and therefore it is **not** invertible.

- $f : [0, +\infty) \to [0, +\infty), \quad f(x) = x^2$

  First, observe that f is injective and surjective. Therefore, the inverse exists and it is unique. To determine it, we solve the equation $y = x^2$ with respect to x. We have:

  $$y = x^2 \;\Leftrightarrow\; x = \sqrt{y}$$

  So the inverse is the function $f^{-1} : [0, +\infty) \to [0, +\infty)$, $f^{-1}(y) = \sqrt{y}$.

The inverse function: examples

- $f : (-\infty, 0] \to [0, +\infty), \quad f(x) = x^2$

  f is injective and surjective. Therefore, the inverse exists and it is unique. To determine it, we solve the equation $y = x^2$ with respect to x. Note that now we are looking for negative x. We have:

  $$y = x^2 \iff x = -\sqrt{y}$$

  So the inverse is the function $f^{-1} : [0, +\infty) \to (-\infty, 0]$, $f^{-1}(y) = -\sqrt{y}$.

Linear functions

$$f:\mathbb{R}\rightarrow\mathbb{R} \text{ such that } \mathbf{f(x)=mx+q}, \quad m,q\in\mathbb{R}$$

- Graphically, this is the equation of a straight line.
- $m=\frac{y_2-y_1}{x_2-x_1}=\frac{f(x_2)-f(x_1)}{x_2-x_1}$, for any $x_1\neq x_2$, is the slope of the line
  - If $m>0$ the function is increasing
  - If $m<0$ the function is decreasing
  - If $m=0$ we have a flat (horizontal) line
- The absolute value of m indicates how fast the line increases or decreases

Linear functions

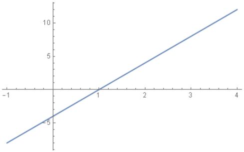


Figure: $m > 0$

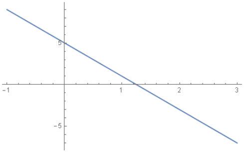


Figure: $m < 0$

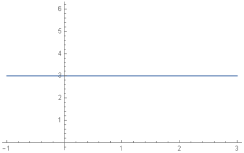


Figure: $m = 0$

How to compute the equation of a straight line

1. Point-slope equation.
   Let $P_1 = (x_1, y_1)$ be a point on the line and m the slope.
   Then the equation of the line through P_1 with slope m is

   $$y - y_1 = m(x - x_1)$$

2. Two points.
   Let $P_1 = (x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on the line.
   To compute the equation of the line we first compute the slope:
   $m = \frac{y_2 - y_1}{x_2 - x_1}$
   and next we use the formula

   $$y - y_1 = m(x - x_1)$$

Quadratic functions

$$f : \mathbb{R} \to \mathbb{R} \text{ such that } f(x) = ax^2 + bx + c, \quad a, b, c \in \mathbb{R},\ a \neq 0$$

- Graphically, this is the equation of a parabola.
- The parabola is convex if $a > 0$
- The parabola is concave if $a < 0$
- the vertex of the parabola is the point with coordinates $V = \left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$, with $\Delta = b^2 - 4ac$

Position of a parabola with respect to the x-axis

Let $\Delta = b^2 - 4ac$. Suppose that $a > 0$

1. If $\Delta > 0$ The parabola intercepts the x-axis at two points, which are the solutions of
$$ax^2 + bx + c = 0$$
2. If $\Delta = 0$ The parabola intercepts the x-axis at one point, which is the unique solution of
$$ax^2 + bx + c = 0$$
3. If $\Delta < 0$ The parabola stays always above the x-axis: the equation $ax^2 + bx + c = 0$ does not have any solution

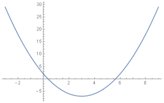


Figure: $a > 0, \quad \Delta > 0$

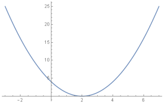


Figure: $a > 0, \quad \Delta = 0$

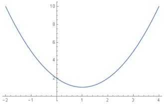


Figure: $a > 0, \quad \Delta < 0$

Position of a parabola with respect to the x-axis

Let $\Delta = b^2 - 4ac$. Suppose that $a < 0$

1. If $\Delta > 0$ The parabola intercepts the x-axis at two points, which are the solutions of
$$ax^2 + bx + c = 0$$
2. If $\Delta = 0$ The parabola intercepts the x-axis at one point, which is the unique solution of
$$ax^2 + bx + c = 0$$
3. If $\Delta < 0$ The parabola stays always below the x-axis: the equation $ax^2 + bx + c = 0$ does not have any solution

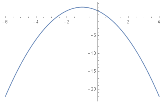


Figure: $a < 0, \quad \Delta > 0$

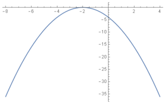


Figure: $a < 0, \quad \Delta = 0$

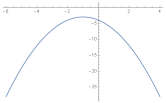


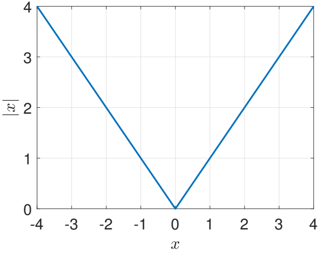
Figure: $a < 0, \quad \Delta < 0$

The function “Absolute Value”

Definition

The function “absolute value” of x, is given by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



- $D = \mathbb{R}$, $R_f = [0, +\infty)$

Power functions

For all $n \in \mathbb{N}$ we define the function:

$$f(x) = x^n$$

which is nothing but the multiplication of x by itself n times

- This function is defined for all $x \in \mathbb{R}$, $D = \mathbb{R}$
- If n is even, the range is $R_f = [0, +\infty)$
- If n is odd, the range is $R_f = \mathbb{R}$

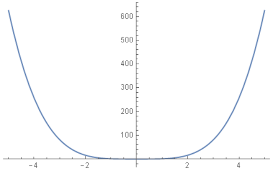


Figure: $f(x) = x^4$

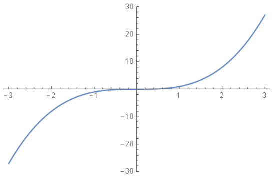


Figure: $f(x) = x^3$

A few characteristics of Power function with exponent $n \in \mathbb{N}$

If *n* is even, the function is not globally invertible. However if we consider only

$$f(x) : [0, +\infty) \to [0, +\infty)$$

the function is invertible and

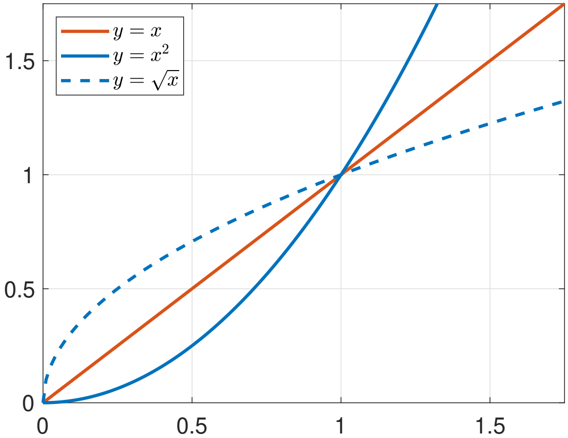
$$f^{-1}(y) = y^{\frac{1}{n}} = \sqrt[n]{y}$$

If *n* is odd, the function is globally invertible and

$$f^{-1}(y) = y^{\frac{1}{n}} = \sqrt[n]{y}$$

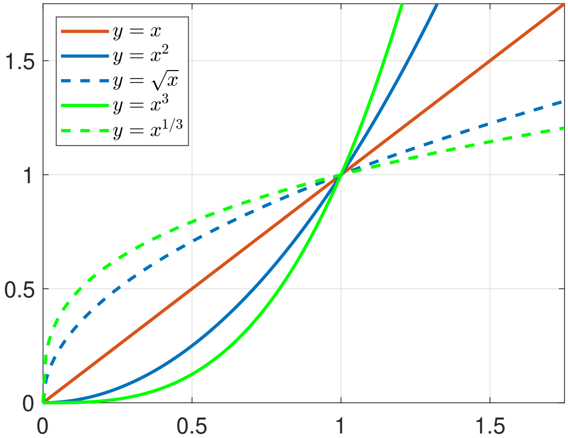
The inverse function: graphical representation

The graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ over the line $y = x$.



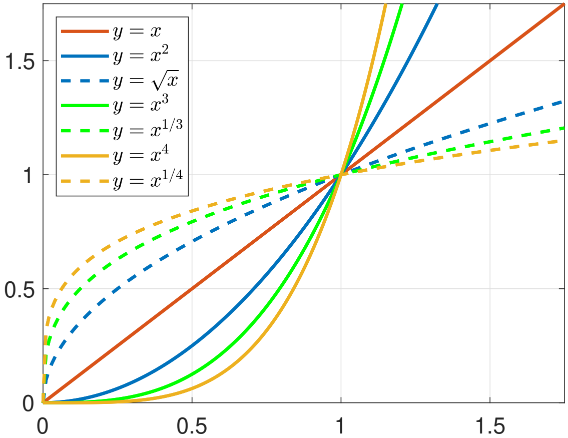
The inverse function: graphical representation

The graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ over the line $y = x$.



The inverse function: graphical representation

The graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ over the line $y = x$.



Power functions

Consider the function:

$$f(x) = x^r, \quad r \in \mathbb{R}$$

This is a power function with real exponent (which generalizes the case of a power function with natural exponent)
A few examples

1. $f(x) = x^{-1} = \dfrac{1}{x}$
2. $f(x) = x^{\frac{1}{2}} = \sqrt{x}$
3. $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$
4. $f(x) = x^{1.3}$

Notice that an extra care must be applied in computing the domain power functions with real exponent. In particular they are well defined when $x > 0$, but they may be undefined for $x = 0$ or $x < 0$. For instance function 1 is not defined when $x = 0$, functions 3 and 4 are not defined when $x < 0$.

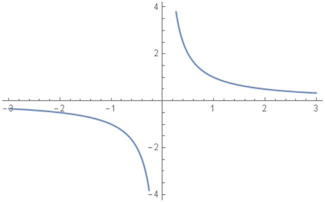


Figure: $f(x) = \frac{1}{x}$

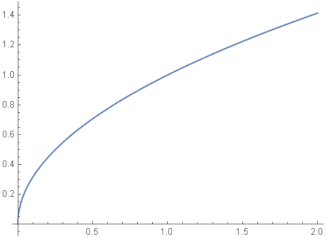


Figure: $f(x) = \sqrt{x}$

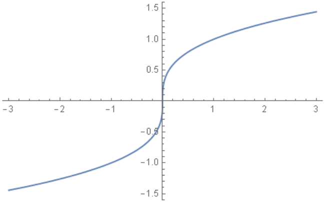


Figure: $f(x) = \sqrt[3]{x}$

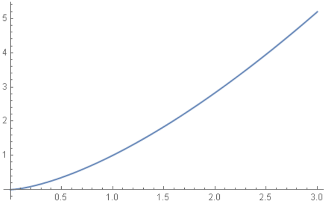


Figure: $f(x) = x^{1.3}$

The exponential function

$$f(x) = a^x, \qquad a > 0$$

Main characteristics:

- $D = \mathbb{R}$
- $R_f = (0, +\infty)$ meaning that $a^x > 0$ for all $x \in \mathbb{R}$
- $f(0) = a^0 = 1$
- if $a > 0$ the function is monotonic strictly increasing
- if $0 < a < 1$ the function is monotonic strictly decreasing
- if $a = 1$ we get the flat line

The exponential function

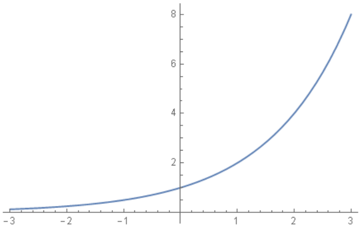


Figure: $f(x) = a^x$, $a > 1$

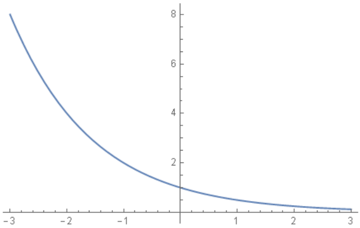


Figure: $f(x) = a^x$, $0 < a < 1$

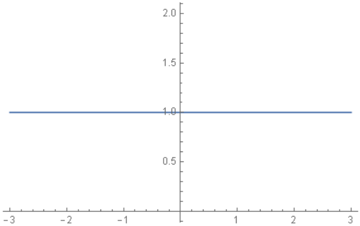


Figure: $f(x) = 1^x$

The logarithmic function

$$f(x) = \log_a(x), \qquad a > 0,\ a \neq 1$$

This is the inverse of the exponential function.

- $D = (0, +\infty)$,
- $R_f = \mathbb{R}$
- $f(1) = \log_a(1) = 0$ (this is a consequence of the fact that $a^0 = 1$)
- if $a > 0$ the function is monotonic strictly increasing
- if $0 < a < 1$ the function is monotonic strictly decreasing