Quantitative Methods – I

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Practice 6

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THEME #1

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Probability

The classical definition of probability

Given the event A the probability of A is:

$$P(A) = \frac{\text{number of favorable outcomes}}{\text{total of outcomes}}$$

Probability of the **intersection of events A and B**: P(A ∩ B) or P(A and B).

Probability of the **union of A and B**: P(A ∪ B) or P(A or B).

Events **mutually exclusive** or **disjoint**:
if they cannot occur at the same time.

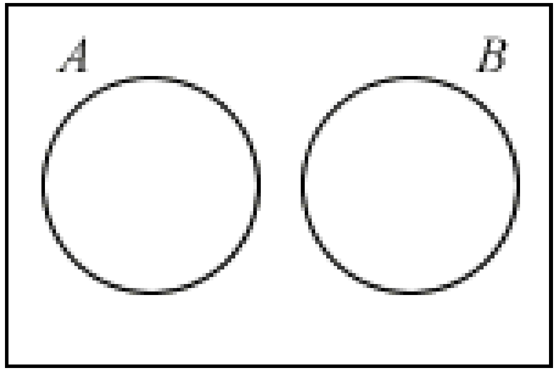
If A and B are disjoint event:

$$P(A \cap \mathrm{B}) = \mathrm{P(A\ and\ B)} = 0$$

Disjoint

$A \cap B = \varnothing$

$A \cup B =$
$n(A) + n(B)$



Union of Events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are disjoint $P(A \cap \mathrm{B}) = 0$, so $P(A \cup B) = P(A) + P(B)$

Conditional probability: the probability that event A occurs, given that event B has occurred P(A | B).

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Intersection of the 2 events

Conditioning Event

Intersection of Events: $P(A \cap B) = P(A|B)P(B)$

Independent Events: if the occurrence of event A does not change the probability of event B, then **events** are **independent**.

If A and B are indipendent: $P(A|B) = P(A)$

Complementary Event: The complement of event A, denoted by Ā, is the event that includes all the outcomes that are not in A.

P(A)+P(Ā)=1. Indeed, P(Ā)=1 – P(A)

The Axioms of Probability

First Axiom of Probability

$$0 \leq P(A) \leq 1$$

$P(A)=0 \rightarrow$ impossible event, $P(A)=1 \rightarrow$ certain event

Second Axiom of Probability

For an experiment with outcomes $S=\{E_1, E_2, E_3, \ldots,\}$, $\sum P(E_i) = P(E_1) + P(E_2) + P(E_3) + \ldots = 1$

Third Axiom of Probability

If A and B are mutually exclusive events in S, then

$$P(A \cup B)=P(A) + P(B)$$

A and B are mutually exclusive events if
$P(A \cap B) = 0$

*If A and B are **NOT** mutually exclusive events in S*

$$P(A \cup B)=P(A) + P(B) - P(A \cap B)$$

Question 1

A die is rolled, find the probability that an even number is obtained.

Solution
Let us first write the sample space S of the experiment. S = {1,2,3,4,5,6}.
Let E be the event "*an even number is obtained*": E = {2,4,6}
We now use the formula of the classical probability. P(E) = n(E) / n(S) = 3 / 6 = 1 / 2

Question 2

Two coins are tossed, find the probability that two heads are obtained.

Solution
The sample space S is given by. S = {(H,T),(H,H),(T,H),(T,T)}
Let E be the event "*two heads are obtained*": E = {(H,H)}
We use the formula of the classical probability. P(E) = n(E) / n(S) = 1 / 4

Question 3

Which of these numbers cannot be a probability?

a) -0.00001
b) 0.5
c) 1.001
d) 0
e) 1
f) 20%

Solution
A probability is always greater than or equal to 0 and less than or equal to 1.

Question 4

Two dice are rolled, find the probability that the sum is

a) equal to 1

b) equal to 4

c) less than 13

Solution

The sample space S of two dice is shown below.

S ={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

n(S)=36 -> Events of the 1th dice (6) × Events of the 2nd dice (6)

a) There are no outcomes which correspond to a sum equal to 1, hence P(E)=n(E)/n(S) = 0 / 36 = 0

b) Three possible outcomes give a sum equal to 4:

E = {(1,3),(2,2),(3,1)}, hence. P(E) = n(E) / n(S) = 3 / 36 = 1 / 12 = 0.083 = 8.3%

c) All possible outcomes, E = S, give a sum less than 13, hence. P(E) = n(E) / n(S) = 36 / 36 = 1

Question 5

A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

Solution

The sample space S of the experiment described in question 5 is as follows

S = { (1,H),(2,H),(3,H),(4,H),(5,H),(6,H),(1,T),(2,T),(3,T),(4,T),(5,T),(6,T)}

n(S)=12 -> Events of the coin (2) × Events of the dice (6)

Let E be the event "the die shows an odd number and the coin shows a head".

Event E may be described as follows E={(1,H),(3,H),(5,H)}

The probability P(E) is given by P(E) = n(E) / n(S) = 3 / 12 = 1 / 4

Question 6

A card is drawn at random from a deck of cards. Find the probability of getting the 3 of diamond.

Solution
An examination of the sample space shows that there is one "3 of diamond"
so that $n(E) = 1$ and $n(S) = 52$.
Hence the probability of event E occurring is given by $P(E) = 1 / 52$

Question 7

A card is drawn at random from a deck of cards. Find the probability of getting a queen.

Solution
An examination of the sample space shows that there are 4 "Queens"
so that $n(E) = 4$ and $n(S) = 52$.
Hence the probability of event E occurring is given by $P(E) = 4 / 52 = 1 / 13$

Question 8

A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?

Solution
We now use the empirical formula of the probability
$P(E)$ = Frequency for white color / Total frequencies in the above table = $10 / 20 = 1 / 2 = 50\%$

Question 9

The blood groups of 200 people is distributed as follows: 50 have type A blood, 65 have B blood type, 70 have 0 blood type and 15 have type AB blood. If a person from this group is selected at random, what is the probability that this person has 0 blood type?

Solution

We use the empirical formula of the probability

P(E) = Frequency for 0 blood / Total frequencies = 70 / 200 = 0.35

Question 10

At the production of a certain item, two types of defects, A and B, can occur.

We know that $P(A) = 0.1$, $P(B) = 0.2$ and $P(A \cap B)=0.05$.

Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

Solution

a) Probability of Union: $P(AUB) = P(A)+P(B)-P(A \cap B)=0.1+0.2-0.05=0.25$
b) $P(A)-P(A \cap B)=0.1-0.05=0.05$
c) Complementary of the Union of Event: $P(\overline{A \cup B}) = 1 - 0.25=0.75$
d) $P(AUB) - P(A \cap B) = 0.2$

Summary of probabilities

	Event	Probability
	A	$P(A) \in [0,1]$
Complementary Event	not A	$P(A^{\complement}) = 1 - P(A)$
Union of Events	A or B	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cup B) = P(A) + P(B) \qquad \text{if A and B are mutually exclusive}$
Intersection of Events	A and B	$P(A \cap B) = P(A\|B)P(B) = P(B\|A)P(A)$ $P(A \cap B) = P(A)P(B) \qquad \text{if A and B are independent}$
Conditional probability	A given B	$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B\|A)P(A)}{P(B)}$

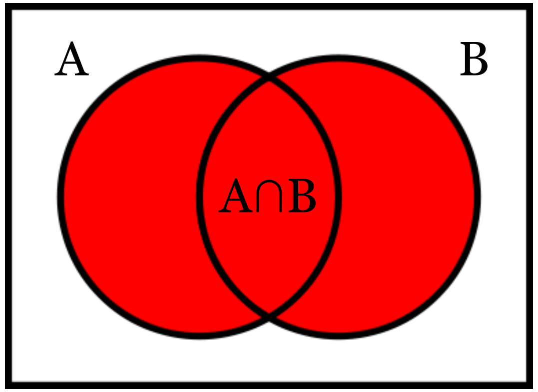
Question 1

At the production of a certain item, two types of defects, A and B, can occur.
We know that $P(A) = 0.1$, $P(B) = 0.2$ and $P(A \cap B)=0.05$.
Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

Solution

a) To calculate the probability that a produced unit has "at least one of the defect" we have to calculate the probability that a produced unit has "the defect A <u>OR</u> the defect B".
This is the probability of the union of the two events A or B, $P(A \cup B)$.
A and B are not disjoint (because $P(A \cap B) \neq 0$).



So, $P(A \cup B) = P(A)+P(B)-P(A \cap B) = 0.1 + 0.2 - 0.05 = 0.25$

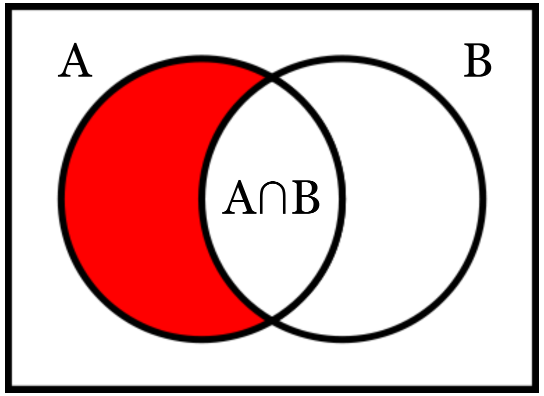
Question 1

At the production of a certain item, two types of defects, A and B, can occur.
We know that $P(A) = 0.1$, $P(B) = 0.2$ and $P(A \cap B) = 0.05$.
Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

Solution

b) We have to calculate the probability that a produced unit has "the defect A but not the defect B".
We can calculate this probability removing from the probability that a produced unit has the defect A, $P(A)$, the probability that this produced unit has also the defect B (the part of the event A in common with B), represented by the intersection of the two events, $P(A \cap B)$.



So the probability that a produced unit has "the defect A but not the defect B" is: $P(A)-P(A \cap B)=0.1-0.05=0.05$

Question 1

At the production of a certain item, two types of defects, A and B, can occur.
We know that P(A) = 0.1, P(B) = 0.2 and P(A∩B)=0.05.
Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

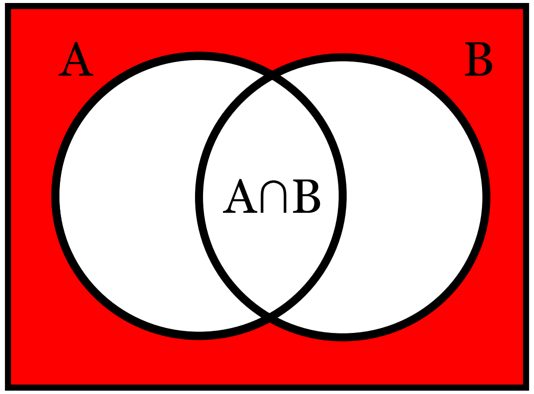
Solution

c) To calculate the event "none of the defects", we have to calculate the complementary event of both events, A and B.
This event is the Complementary event of the union of the events A or B: P($\overline{A \cup B}$).

P(AUB) = 0.25

From the property of the complementary events:

$$P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.25 = 0.75$$



Question 1

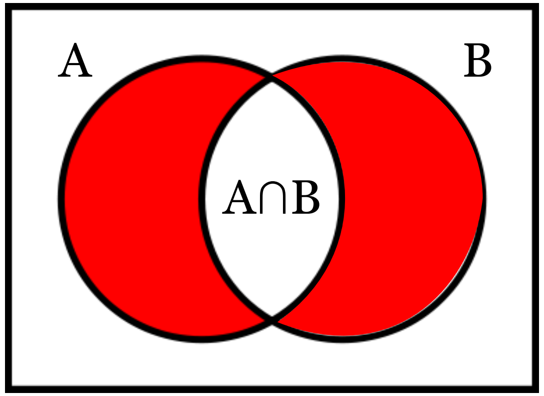
At the production of a certain item, two types of defects, A and B, can occur.
We know that P(A) = 0.1, P(B) = 0.2 and P(A∩B)=0.05.
Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

Solution

d) To calculate the probability of the event that a produced unit has "precisely one of the defects A and B" we have to calculate first of all the probability union of the events A and B.
But in this probability we have also the probability that a product has both the defects, A and B. For this reason, from the probability of the union of the two events we have to remove the intersection of the events.
P(AUB) = 0.25 and P(A∩B)=0.05

So, P("precisely one of the defect A and B")=P(AUB) - P(A∩B) = 0.2



Question 2

A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test.
What percent of those who passed the first test also passed the second test?

Solution
First, we have to consider that:
P(A)=P(Second Test), the probability that a student passed the second test,
P(B)=P(First Test), the probability that a student passed the first test,
P(A ∩ B)=P(Second ∩ First), the probability that a student passed both the tests.

So, the probability that a student who passed the first test also passed the second test is given by the conditional probability, $P(\text{Second}|\text{First})$, and this probability is:

$$P(\text{Second}|\text{First}) = \frac{P(\text{Second} \cap \text{First})}{P(\text{First})}$$

where P(First) is the "conditioning event" (select a students that passed the second test given that has passed the first test).

P(First)=0.42
P(First ∩ Second)=0.25

Hence, $P(\text{Second}|\text{First}) = \frac{P(\text{Second} \cap \text{First})}{P(\text{First})} = 0.25\ /\ 0.42 = 0.5952 = 59.6\%$

Question 3

At a middle school, 18% of all students play football and basketball and 32% of all students play football. What is the probability that a student plays basketball given that the student plays also football?

Solution

We have to calculate the Conditional Probability: P(Basketball | Football)

We have to consider that:

P(F), the probability that a student plays football,

P(B), the probability that a student pays basketball,

P(F ∩ B), the probability that a student plays both the sports.

So, the conditional probability is:

$$P(B|F) = \frac{P(F \cap B)}{P(F)}$$

P(F ∩ B)=0.18

P(F)=0.32

$$P(B|F) = \frac{P(F \cap B)}{P(F)} = 0.18 \; / \; 0.32 = 0.5625 = 56.3\%$$

Question 4

The two-way table below gives the thousands of commuters in Massachusetts in 2018 by transportation method and one-way length of commute.

	Less than 15 minutes	15-29 Minutes	30-44 Minutes	45-59 Minutes	>60 Minutes	**Total**
Private vehicle	636	908	590	257	256	**2647**
Public Transportation	9	54	96	62	108	**329**
Other	115	70	23	7	7	**222**
Total	**760**	**1032**	**709**	**326**	**371**	**3198**

a. Given that the commuter used public transportation, find the probability that the commuter had a commute of 60 or more minutes.

b. Given that the commuter used other method of transportation, find the probability that the commuter had a commute of less than 15 minutes.

c. Given that the commuter had a commute of 35 minutes, find the probability that the commuter used a private vehicle.

Solution

a. 108/329

b. 115/222

c. 590/709

Question 5 A random sample of 250 adults was taken, and they were asked whether they prefer watching sports or opera on television. The following table gives the two-way classification of these adults.

	Prefer Watching Sports	Prefer Watching Opera	
Male	104	18	122
Female	55	73	128
	159	91	250

If one adult is selected at random from this group, find the probability that this adult

a. prefers watching opera

b. is a male

c. prefers watching sports given that the adult is a female

d. is a male given that he prefers watching sports

e. is a female and prefers watching opera

f. prefers watching sports or is a male

a. P(Opera)=91/250=0.364

b. P(Male)=122/250=0.488

c. P(Sport|Female)=(Sport and Female)/Female=
=55/128=0.4297

d. P(Male|Sport)=104/159=0.654

e. P(Female and Opera)=73/250=0.292

f. P(Sport or Male) = P(Sport) + P(Male) – P(Sport and Male) = 159/250 + 122/250 – 104/250 = 177/250 = 0.7080

THEME #2

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Bayes' Theorem

The Bayes' Theorem

Bayes's theorem is stated mathematically as the following equation:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

where A and B are events and $P(B) \neq 0$.

P(A|B) is a conditional probability: the likelihood of event A given B.

P(B|A) is also a conditional probability: the likelihood of event B given A.

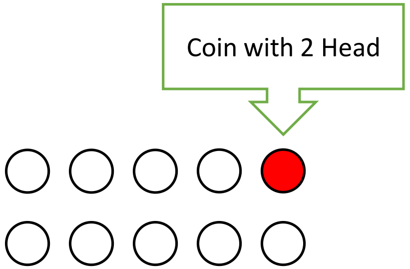
P(A) and P(B) are the probabilities of observing A and B respectively; they are called marginal probability.

Question 15.

I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads.

a. If I take one of the coins at random from my pocket, what is the probability that it is the coin with two heads?
b. If I toss the coin one further time and it comes up tails, what is the probability that it is one of the nine ordinary coins?
c. If I toss the coin and it comes up heads, what is the probability that it is the coin with two heads ?

Solution



Denote by D the event that the coin is the one with two heads.

(a) P(D) is the probability to select the coin with two heads from all the 10 coins.

For this reason $P(D) = \frac{1}{10}$

(b) If it comes up tails, it can't be the coin with two heads.
Therefore it must be one of the other nine.

For this reason the probability that it is one of the nine ordinary coins is 1 (certain event).

Question 15.

I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads.

a. If I take one of the coins at random from my pocket, what is the probability that it is the coin with two heads?
b. If I toss the coin one further time and it comes up tails, what is the probability that it is one of the nine ordinary coins?
c. If I toss the coin and it comes up heads, what is the probability that it is the coin with two heads ?

Solution

(c) Denote by H the event that we get a head when we toss the coin.

Then we have to find P(D|H) the probability that, having a head, is from the coin with two head.
By Bayes theorem, we have

$$P(D|H) = \frac{P(H|D) \cdot P(D)}{P(H)}$$

The probability P(H|D) is 1, because selecting the coin with two head, the result "HEAD" is a certain event.

The probability P(D) from the point (a) is 1/10.

We have to calculate the probability of H.

Question 15.

I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads.

a. If I take one of the coins at random from my pocket, what is the probability that it is the coin with two heads?
b. If I toss the coin one further time and it comes up tails, what is the probability that it is one of the nine ordinary coins?
c. If I toss the coin and it comes up heads, what is the probability that it is the coin with two heads ?

Solution

We have to calculate the probability of H.

Think of the Sample Space of all the possible tosses.

The bag contains 9 fair coins and 1 double-headed coin,

for this reason in the bag we have 11 heads (9 from the fair coin and 2 from the double-head coin) and 9 tails (all from the 9 fair coin), so that the probability of choosing a head is:

$$P(H) = \frac{\text{n.of head}}{\text{n.of head and tail}} = \frac{11}{(11+9)} = \frac{11}{20}$$

We now obtain the answer: $P(D|H) = \frac{P(H|D) \cdot P(D)}{P(H)} = \frac{1 \cdot \frac{1}{10}}{\frac{11}{20}} = \frac{2}{11} = 18.18\%$