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During the lecture the teacher explains and integrates what is written in the slides.

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Linear Spaces

A. Fabretti

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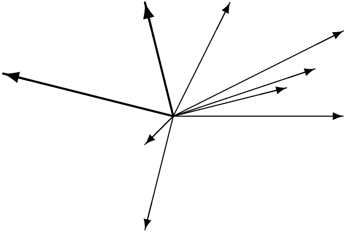
Introduction

A linear space (called also a vector space) is a set of objects called vectors that can be added or multiplied by a scalar. They are useful for modelling a wide variety of economic phenomena because n-tuples of numbers may be interpreted in many ways.

$\mathbb{R}^n$ is the "most common" vector space we deal to. For example, $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ can represent a consumption bundle or a portfolio allocation over n assets.



Examples of vectors

Draw the following vectors

- in $\mathbb{R}^2$:

$$(1,1) \quad \left(-\frac{1}{2},\frac{3}{2}\right) \quad (0,0) \quad (-2,4)$$

- in $\mathbb{R}^3$:

$$(-1,2,-1) \quad (0,1,0) \quad \left(2,\frac{1}{2},\frac{1}{2}\right)$$

- in $\mathbb{R}^4$:

$$(0,1,2,0) \quad (2,0,1,0)$$

Linear Space Definition

Definition

A **linear space** over a field $\mathcal{F}$ is a set X whose elements are called vectors and two operations are defined:

- **addition** $+ : X \times X \to X$
  for any pair $\mathbf{u}, \mathbf{v} \in X$, $\mathbf{u} + \mathbf{v} \in X$
- **multiplication by scalar** $\cdot : X \times \mathcal{F} \to X$
  for $\alpha \in \mathcal{F}$ and $\mathbf{u} \in X$, $\alpha\mathbf{u} \in X$

these two operations satisfies some axioms

Linear Space Definition: Axioms for addition

Definition

- *Associative*: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- *Commutative*: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- *identity element*: there exists an element $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- *Inverse element of addition*: for any $\mathbf{u} \in X$ there exists an element $-\mathbf{u} \in X$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, this element is called the addivitive inverse of $\mathbf{u}$ or its opposite.

Linear Space Definition: Axioms for multiplication by a scalar

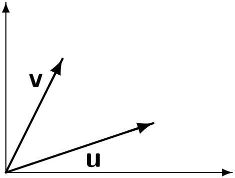
Definition

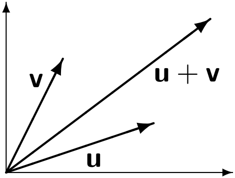
- *Associative*: $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$
- *Distribuitive 1*: $\alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
- *Distribuitive 2*: $(\alpha+\beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
- if 1 is the *Identity element* of $\mathcal{F}$, for any $\mathbf{u} \in X$, $1\mathbf{u} = \mathbf{u}$
- $0\mathbf{u} = \mathbf{0}$, for any $\mathbf{u} \in X$.

Addition

We add vectors as we add numbers and matrices, simply adding corresponding coordinates. Thus, having $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$, their sum is

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

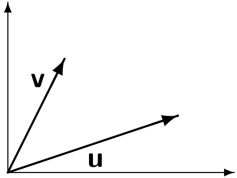


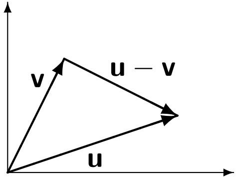


Subtraction

We subtract vectors as we subtract numbers and matrices, simply subtracting corresponding coordinates. Thus, having $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$, their sum is

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2)$$

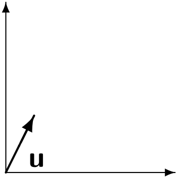


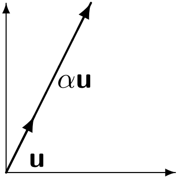


Scalar multiplication

If we multiply a vector $\mathbf{u}$ by a scalar α, we get a vector pointing in the same direction of $\mathbf{u}$ but with length α times the length of $\mathbf{u}$. Thus, having $\mathbf{u} = (u_1, u_2)$ we get

$$\alpha\mathbf{u} = (\alpha u_1, \alpha u_2)$$





A linear space on dimension n

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ two vectors in a linear space of dimension n and α a scalar

- *addition* $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$;
- *subtraction* $\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, ..., x_n - y_n)$;
- *multiplication by a scalar* $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$;

Exercises

- **Exercize 1** Let $\mathbf{x} = (1, 2, -1)$, $\mathbf{y} = (0, 1, 1)$ and $\mathbf{z} = (-2, 1, 3)$. Compute the following vectors

$$\mathbf{x} + 3\mathbf{y} - \mathbf{z} \qquad \mathbf{z} - 2\mathbf{x} \qquad 3(\mathbf{x} + \mathbf{y}) - 4\mathbf{z}$$

Distance in euclidean spaces

Consider an euclidean space X of dimension n and two points $P = (x_1, x_2, ..., x_n)$ and $Q = (y_1, y_2, ..., y_n)$.
We can define the distance between P and Q, which is the length of the segment PQ:

$$||PQ|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$$

Note that $||PQ||$ is also the length of the vector obtained by subtracting the vectors $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$:

$$||\mathbf{x} - \mathbf{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}.$$

Length

If we take $\mathbf{y} = \mathbf{0}$ we have the distance of $\mathbf{x}$ from the origin which is the **length** of vector $\mathbf{x} = (x_1, x_2, ..., x_n)$

$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

If α is a scalar the length of the vector $\alpha\mathbf{x}$ will be $|\alpha|$ times the length of $\mathbf{x}$:

Theorem

$$||\alpha\mathbf{x}|| = |\alpha|||\mathbf{x}||$$

for all $\alpha \in \mathbb{R}$ *and* $\mathbf{x} \in \mathbb{R}^n$.

Versor

Given a vector $\mathbf{v} = (v_1, v_2, ..., v_n)$ we can be interested in having a vector pointing in the same direction and having length equal 1. Such a vector is called **unit vector** or **versor** and can be obtained simply dividing $\mathbf{v}$ by its length $||\mathbf{x}||$. Denoting this vector by $\mathbf{w}$ we have

$$\mathbf{w} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

Example

Given the vector $\mathbf{u} = (1, 3, -2)$ in $\mathbb{R}^n$ find its corresponding versor $\mathbf{v}$.

First compute $||\mathbf{u}||$:

$$||\mathbf{u}|| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$$

It follows

$$\mathbf{v} = \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}} \right)$$

Inner Product

Definition

Let $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{v} = (v_1, ..., v_n)$ be two vectors in $\mathbb{R}^n$. The **euclidean inner product** of $\mathbf{u}$ and $\mathbf{v}$, written $\mathbf{u} \cdot \mathbf{v}$ or $< \mathbf{u}, \mathbf{v} >$ is the number

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + ... + u_n v_n$$

Example: If $\mathbf{u} = (2, -2, 3)$ and $\mathbf{v} = (1, -1, 2)$:

$$\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + -2 \cdot (-1) + 3 \cdot 2 = 10.$$

Inner Product (2)

Theorem

Let $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be arbitrary vectors in* $\mathbb{R}^n$ *and let* α *be a scalar. Then,*

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,
- $\mathbf{u} \cdot (\alpha \mathbf{v}) = \alpha(\mathbf{u} \cdot \mathbf{v}) = (\alpha \mathbf{u}) \cdot \mathbf{v}$,
- $\mathbf{u} \cdot \mathbf{u} \geq 0$,
- $\mathbf{u} \cdot \mathbf{u} = 0$ *implies* $\mathbf{u} = \mathbf{0}$,
- $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}$.

Inner Product (3)

The euclidean inner product is closely related to the length of a vector. Since

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \ldots + u_n^2 \qquad \text{and} \qquad ||\mathbf{u}|| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

we have that

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

and the distance between two vectors **u** and **v** can be written

$$||\mathbf{u} - \mathbf{v}|| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}.$$

Inner Product (4)

Any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ determine a plane, in this plane we can measure the angle between $\mathbf{u}$ and $\mathbf{v}$, suppose it is θ. The inner product gives a relation between the length of $\mathbf{u}$ and $\mathbf{v}$ and the angle θ between them.

Theorem

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$. Let θ be the angle between them. Then,

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \ ||\mathbf{v}|| \cos\theta.$$

Orthogonality

Theorem

the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$ *is*

- *acute, if* $\mathbf{u} \cdot \mathbf{v} > 0$,
- *obtuse, if* $\mathbf{u} \cdot \mathbf{v} < 0$,
- *right,* $\mathbf{u} \cdot \mathbf{v} = 0$.

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** if and only if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

Triangular Inequality

Theorem

For any two vectors $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$*,*

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||.$$

The theorem says that any side of a triangle is shorter than the sum of the lengths of the other two sides.

Norm

Any assignment of a real number to a vector as the euclidean length $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ is a **norm** if satisfies this three properties

- $||\mathbf{u}|| \geq 0$ and $||\mathbf{u}|| = 0$ only if $\mathbf{u} = \mathbf{0}$,
- $||r\mathbf{u}|| = |r| \; ||\mathbf{u}||$,
- $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.

Exercises

Exercise 1 Find the length of the following vectors

$$(1,2,3) \qquad (-1,0,3,4) \qquad \left(\frac{1}{2},-2,0,1\right)$$

Exercise 2 Find the distance from $P=(5,2)$ and $Q=(3,3)$

Exercise 3 Use vector notation to prove that the diagonals of a rhombus are orthogonal to each other.

Exercise 4 Prove that in $\mathbb{R}^2$, $||(u_1,u_2)|| = max\{|u_1|,|u_2|\}$ is a norm.

Lines

Lines are among the fundamental objects in an Euclidean space. a line is determined by a point x_0 and a direction $\mathbf{v}$. To determine any point on the line we add scalar multiples of $\mathbf{v}$ to $\mathbf{x_0}$, we obtain the **parametric representation** of the line

$$\mathbf{x}(t) = \mathbf{x_0} + t\mathbf{v}.$$

Example

The parametric representation of a line going through the point $(3,2)$ in the direction $(1,2)$:

$$\mathbf{x}(t) = (x_1(t), x_2(t)) = (3,2) + t(1,2) = (3 + 1 \cdot t, 2 + 2 \cdot t)$$

it is equivalent to write

$$\begin{cases} x_1(t) = & 3 + t \\ x_2(t) = & 2 + 2t \end{cases}$$

Lines (2)

Another way to determine a line is to identify two points on the line. Suppose $\mathbf{x}$ and $\mathbf{y}$ lie on a line r, then r can be seen as the line going from $\mathbf{x}$ in the direction $\mathbf{y} - \mathbf{x}$. Hence

$$\mathbf{r}(t) = \mathbf{x} + t(\mathbf{y} - \mathbf{x}) = (1 - t)\mathbf{x} + t\mathbf{y}$$

A combination of $\mathbf{x}$ and $\mathbf{y}$ with coefficients summing to 1.

Non parametric equation

There is another way to identify a lines in the plane $\mathbb{R}^2$. Consider a point $P = (x_0, y_0)$ and a vector $\mathbf{u} = (u_1, u_2)$. The parametric equation is

$$\begin{cases} x(t) = & x_0 + t\ u_1 \\ y(t) = & y_0 + t\ u_2 \end{cases}$$

Expliciting t in the first and substituting in the second we get

$$-u_2\ x + u_1\ y = u_1 y_0 - u_2 x_0.$$

Note that the vector $\mathbf{n} = (-u_2, u_1)$ is orthogonal to $\mathbf{u}$ hence, denoting $\mathbf{p} = (x, y)$ and $\mathbf{p_0} = (x_0, y_0)$ we can write the non parametric equation

$$\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{p_0}$$

Exercises

Exercise 1 Find the parametric equation of the line through $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (-1, 2, 1)$.

Exercise 2 Is the point $\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$ on the line $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$? If it is, find the value of t.

Exercise 3 Transform the following parameterized equation into cartesian equation

$$\begin{cases} x_1(t) = & 4 + 2t \\ x_2(t) = & 2 - t \end{cases}$$

Planes

A plane is determined by two vectors $\mathbf{v}$ and $\mathbf{w}$ which points in different directions (linearly independent). For any scalar s and t, the vector $s\mathbf{v} + t\mathbf{w}$ is called a **linear combination** of $\mathbf{v}$ and $\mathbf{w}$. It is clear that all the linear combinations of $\mathbf{v}$ and $\mathbf{w}$ lie on the plane determined by the two vectors. Hence the plane has parametric equation

$$\mathbf{x} = s\mathbf{v} + t\mathbf{w}.$$

If the plane does not pass through the origin but trough the point $\mathbf{p} \neq 0$ and $\mathbf{v}$ and $\mathbf{w}$ are linear independent direction vector from $\mathbf{p}$ then the plane has equation

$$\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w} \qquad s, t \in \mathbb{R}.$$

Planes (2)

Three non collinear points determine a plane. Given the points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, we can take $\mathbf{q}-\mathbf{p}$ and $\mathbf{r}-\mathbf{p}$ as displacement vectors from $\mathbf{p}$, so we get the parameterized equation

$$\mathbf{x}=\mathbf{p}+s(\mathbf{q}-\mathbf{p})+t(\mathbf{r}-\mathbf{p})=(1-s-t)\mathbf{p}+s\mathbf{q}+t\mathbf{r}.$$

We can say that a plane is the set of those linear combinations of three fixed vectors whose coefficients sum to 1:

$$\mathbf{x}=t_1\mathbf{p}+t_2\mathbf{q}+t_3\mathbf{r}; \qquad t_1+t_2+t_3=1.$$

Non parametric equation

In $\mathbb{R}^3$ a plane can be determined also by a point $\mathbf{p} = (p_1, p_2, p_3)$ and a vector $\mathbf{n} = (a, b, c)$ orthogonal to the plane, called a **normal vector**. If $\mathbf{x} = (x_1, x_2, x_3)$ is an arbitrary point, it will belong to the plane if $\mathbf{x} - \mathbf{p}$ is a vector in the plane and if it is orthogonal to $\mathbf{n}$, hence their inner product must be zero:

$$\begin{aligned} 0 &= (\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = (x_1 - p_1, x_2 - p_2, x_3 - p_3) \cdot (a, b, c) \\ &= a(x_1 - p_1) + b(x_2 - p_2) + c(x_3 - p_3) \end{aligned}$$

We get that the **point-normal equation** of the plane is

$$ax + by + cz = d.$$

Examples

Example 1 The equation of the plane through the point $(1, 2, 3)$ and with normal vector $(4, 5, 6)$ is

$$4(x-1)+5(y-2)+6(z-3)=0$$

hence

$$4x+5y+6z=32.$$

Example 2 Find the point-normal equation of the plane $\mathcal{P}$ which contains the points

$$\mathbf{p}=(2,1,1) \quad \mathbf{q}=(1,0,-3) \quad \mathbf{r}=(0,1,7).$$

Solution: $3x-7y+z=0$.

Hyperplanes

A line in $\mathbb{R}^2$ can be written as

$$ax_1 + bx_2 = c$$

a plane in $\mathbb{R}^3$ can be written as

$$ax_1 + bx_2 + cx_3 = d$$

Similarly an **hyperplane** in $\mathbb{R}^n$ can be written in point-normal form as

$$a_1x_1 + a_2x_2 + ... + a_nx_n = d.$$

An hyperplane in $\mathbb{R}^n$ has dimension $n - 1$.

Exercises

Exercise 1 Determine if the following pairs of planes intersect

1. $x + 2y - 3z = 6$ and $x + 3y - 2z = 6$
2. $x + 2y - 3z = 6$ and $-2x - 4y + 6z = 10$.

Exercise 2 Derive parametric and non parametric equations for the plane through each of the following triplets of points

1. $(6, 0, 0)$ $(0, -6, 0)$ $(0, 0, 3)$
2. $(0, 3, 2)$ $(3, 3, 1)$ $(2, 5, 0)$.

Introduction

Given a vector $\mathbf{v}$ when we consider the set of vectors obtained multiplying $\mathbf{v}$ by a scalar we obtain a straight line through the origin, we denote this set by $\mathcal{L}(\mathbf{v})$

$$\mathcal{L}(\mathbf{v}) = \{r\mathbf{v} \ : r \in \mathbb{R}\}$$

and call it the line spanned (or generated) by $\mathbf{v}$.
If we consider two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, we can take all possible their linear combinations and obtaining a set spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\mathcal{L}(\mathbf{v}_1, \mathbf{v_2}) = \{r_1\mathbf{v_1} + r_2\mathbf{v_2} \ : r_1, r_2 \in \mathbb{R}\}$$

Linear Dependence

If $\mathbf{v_1}$ is a multiple of $\mathbf{v_2}$, $\mathcal{L}(\mathbf{v}_1, \mathbf{v_2})$ is the line $\mathcal{L}(\mathbf{v_2})$. However if $\mathbf{v_1}$ is not a multiple of $\mathbf{v_2}$ then the set spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$, $\mathcal{L}(\mathbf{v}_1, \mathbf{v_2})$, is a plane.
If $\mathbf{v_1}$ is a multiple of $\mathbf{v_2}$ or vice versa we say that $\mathbf{v_1}$ and $\mathbf{v_2}$ are **linearly dependent**, otherwise we say tha $\mathbf{v_1}$ and $\mathbf{v_2}$ are **linearly independent**.

Linear Dependence

Definition

Vectors $\mathbf{v_1}, \mathbf{v}_2, \ldots, \mathbf{v_k}$ in $\mathbb{R}^n$ are **linearly dependent** if and only if there exist scalars $c_1, c_2, \ldots, c_k$ not all zero such that

$$c_1\mathbf{v_1} + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v_k} = \mathbf{0}$$

Vectors $\mathbf{v_1}, \mathbf{v}_2, \ldots, \mathbf{v_k}$ in $\mathbb{R}^n$ are **linearly independent** if and only if $c_1\mathbf{v_1} + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v_k} = \mathbf{0}$ implies that $c_1 = c_2 = \ldots = c_k = 0$.

Example

The vectors

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{v_3} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

are linear dependent since

$$2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Matrix Rank

The **rank** of a matrix A is the dimension of the space spanned by its vectors (columns or rows). It is denoted by $\mathbf{Rank}(\mathbf{A})$.

It is equivalent to say:
the rank is the number of linear independent vectors.

Examples

Ex 1 Find the rank of matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & -2 \end{pmatrix}$$

Ex 2 Study the rank of matrix A in function of a?

$$A = \begin{pmatrix} a & 1 & 4 \\ 2 & 1 & a^2 \\ 1 & 0 & -3 \end{pmatrix}$$