Mathematics II

Third Practice: Solutions

1. Compute (when possible) the inverse of the following matrices:

$$a)\ \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \qquad b)\ \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \qquad c)\ \begin{pmatrix} 1 & 3 \\ -2 & -1 \end{pmatrix}$$

$$d)\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix} \qquad e)\ \begin{pmatrix} 1 & 2 & 2 \\ 1 & -3 & -1 \\ 1 & 0 & -1 \end{pmatrix} \qquad f)\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$g)\ \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 0 & -5 \\ -4 & 3 & 4 & 1 \\ 4 & 3 & -4 & 0 \end{pmatrix}$$

Solution:

$$a)\ \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix} \qquad b)\ \textbf{Not invertible} \qquad c)\ \begin{pmatrix} -1/5 & -3/5 \\ 2/5 & 1/5 \end{pmatrix}$$

$$d)\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \qquad e)\ \begin{pmatrix} \frac{1}{3} & \frac{2}{9} & \frac{4}{9} \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{9} & -\frac{5}{9} \end{pmatrix} \qquad f)\ \textbf{Not invertible}$$

$$g)\ \textbf{Not invertible}$$

2. Compute the rank of the following matrices:

$$a) \quad \begin{pmatrix} 1 & 3 & 1 \\ -1 & 1 & 2 \\ 3 & 1 & -3 \end{pmatrix} \qquad\qquad b) \quad \begin{pmatrix} 0 & 1 & 2 & -4 \\ 1 & -1 & 1 & 1 \\ -2 & 3 & 0 & -6 \end{pmatrix}$$

$$c) \quad \begin{pmatrix} 3 & 1 & 0 & -1 & 2 \\ 1 & 5 & 3 & 2 & 1 \\ -1 & 0 & 0 & -1 & 4 \end{pmatrix} \qquad d) \quad \begin{pmatrix} 0 & 2 & -4 & 3 \\ 0 & 1 & -2 & 1 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -2 & 1 & 0 & -2 \end{pmatrix}$$

Solution: a) 2 ; b) 2 ; c) 3; d) 3.

3. Determine for which value of k the following vectors in $\mathbb{R}^4$ are linearly independent:

$$\vec{v} = (2, -1, 2, -1); \quad \vec{w} = (1, 1, 1, 1); \quad \vec{z} = (3, k, 3, -k).$$

Solution: $k \neq 0$.

4. Which of the following sets of vectors are linearly independent? Which of them span the space $\mathbb{R}^n$ to which they belong? Which of them form a basis?

$$\begin{aligned} &a)\ \mathbb{R}^2 : \ \{(0,0), (1,0), (0,1)\} \\ &b)\ \mathbb{R}^3 : \ \{(1,1,1), (1,0,0)\} \\ &c)\ \mathbb{R}^2 : \ \{(1,2), (3,-1)\} \\ &d)\ \mathbb{R}^4 : \ \{(2,0,1,1), (3,1,0,2), (2,0,1,4)\} \end{aligned}$$

Solution: a) Ind (No), Span (Yes), Basis (No); b) Ind (Yes), Span (No), Basis (No); c) Ind (Yes), Span (Yes), Basis (Yes); ; d) Ind (Yes), Span (No), Basis (No).

5. Calculate (if any) all solutions for each of the following systems:

$$(a) \ : \begin{cases} x + y + z = 1 \\ 3x + 3y - 2z - 2 = 0 \end{cases} , \quad (b) \ : \begin{cases} x + y + z = 1 \\ 3x + 3y - 2z - 2 = 0 \\ z = 1 \end{cases}$$

$$(c) \ : \begin{cases} x + y = 1 \\ 3x + y = 0 \\ 5x + 3y = 0 \end{cases} , \quad (d) \ : \begin{cases} x + y = 1 \\ 3x + y = 0 \\ 5x + 3y = 2 \end{cases}$$

Solution: a) $\left(t, \frac{4}{5} - t, \frac{1}{5}\right)$; b) no solution; c) no solution; d) $\left(-\frac{1}{2}, \frac{3}{2}\right)$.

6. Calculate the solutions of the following systems

$$a) \ : \begin{cases} kx + y - z = 0 \\ 2y + kz = k \\ x - z = 0 \end{cases} \qquad b) \ : \begin{cases} x + y + z + w = 0 \\ y - z = 1 \\ x + 2y + kw = 2 \end{cases}$$

$\forall k \in \mathbb{R}$.

Solution: a) no solution for $k = 2$, otherwise a single solution $x = \frac{k}{2-k}$, $y = \frac{k-k^2}{2-k}$, $z = \frac{k}{2-k}$; b) no solution for $k = 1$, otherwise ∞^1 solutions $x = \frac{2-k}{1-k} - 2t$, $y = t$, $z = -1 + t$, $w = \frac{1}{k-1}$.