MATHEMATICS 2

Exam

5/02/2019, A.Y. 2017/2018

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1) *(8 p.ts)* Solve the following integral

$$\int x\sqrt{1+x^2}dx$$

$$\int x\sqrt{1+x^2}dx = \frac{1}{2}\int 2x\sqrt{1+x^2}dx = \frac{1}{2}(1+x^2)^{\frac{3}{2}}\cdot\frac{2}{3}+c = \frac{1}{3}\sqrt{(1+x^2)}^3+c$$

2) *(12 p.ts)* Find eigenvalues and eigenvectors of the following matrix and determine if it is diagonalizable? If so, identify the invertible matrix T that transforms A into a diagonal matrix, and show how T realizes this transformation.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

Let us look for Real eigenvalues

$$|A - \lambda| = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & -2 & 1-\lambda \end{vmatrix} = -\lambda(\lambda-2)^2$$

the characteristic polynomial has three real zeroes $\lambda_1 = 0$ (simple eigenvalue) and $\lambda_2 = \lambda_3 = 2$ (double eigenvalue). Let us check if the algebraic and geometric multiplicities are equal. For the eigenvalue λ_1 the condition is trivially verified. Regarding $\lambda_2 = 2$, whose algebraic multiplicity is $m_a = 2$, notice that

$$A - 2I_3 = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & -2 & -1 \end{pmatrix}$$

is such that $rank(A - 2I_3) = 1$ (first and third row are multiple by factor -1, and the second row is a null row), hence the geometric multiplicity of $\lambda_2 = 2$ is $m_g = 3 - rank(A - 2I_3) = 2$. This implies the diagonalizability of the given matrix A. Let us calculate a base of three independent eigenvectors (1 correspondent to the eigenvalue $\lambda_1 = 0$ and two correspondent to the double eigenvalue $\lambda_2 = 2$). If $\lambda = 0$ we obtain a first eigenvector by solving the homogeneous system

$$(A - 0 \cdot I_3)\mathbf{X} = \mathbf{0}, \quad \text{where} \quad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

that we can write as

$$\begin{cases} x + 2y + z = 0 \\ 2y = 0 \\ x - 2y + z = 0 \end{cases}$$

this system has solutions $\mathbf{v_1} = \begin{pmatrix} -\alpha \\ 0 \\ \alpha \end{pmatrix}$, with $\alpha \in \mathbf{R}$ and $\alpha \neq 0$. If we choose $\alpha = 1$ we obtain $\mathbf{v_1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

If $\lambda = 2$, we obtain the remaining independent eigenvectors by solving the homogeneous system

$$(A - 2 \cdot I_3)\mathbf{X} = \mathbf{0}, \quad \text{where} \quad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

that we can write as

$$\begin{cases} -x+2y+z=0 \\ 0=0 \\ x-2y-z=0 \end{cases}.$$

This system has solutions $\mathbf{v_2} = \begin{pmatrix} 2\beta+\gamma \\ \beta \\ \gamma \end{pmatrix}$, with $\beta, \gamma \in \mathbf{R}$ and $\beta^2+\gamma^2 \neq 0$. If we choose $(\beta,\gamma)=(1,0)$ we obtain $\mathbf{v_2} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

If we choose $(\beta,\gamma)=(0,1)$ we obtain $\mathbf{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.If we display v_1, v_2 and v_3 in the following matrix

$$T = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

by setting $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, diagonal matrix with eigenvalues in the diagonal, it is easy to prove that $T^{-1}AT = D$, where

$$T^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}.$$

3) *(10 p.ts)* Find max/min of the following function

$$f(x,y) = x^2 - y^2$$

subject to the following constraint

$$x^2 - y = 0$$

The Lagrangian funtion is

$$\mathcal{L}(x,y,\lambda) = x^2 - y^2 - \lambda(x^2 - y),$$

with gradient

$$\nabla(x,y,\lambda) = (2x - 2\lambda x, -2y + \lambda, -x^2 + y),$$

whose components cancel in the three following cases

$$(0,0) \text{ for } \lambda = 0$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \text{ both for } \lambda = 1.$$

Second order conditions imply calculation of the Bordered Hessian $H(x,y,\lambda)$:

$$H(x,y,\lambda) = \begin{pmatrix} 0 & 2x & -1 \\ 2x & 2-2\lambda & 0 \\ -1 & 0 & -2 \end{pmatrix} \quad \text{hence } |H(x,y,\lambda)| = 2\lambda - 2 + 8x^2.$$

Let us analyze second order conditions case by case :
If $(x,y,\lambda) = (0,0,0)$, $|H(0,0,0)| = -2 < 0$, and this implies that $(0,0)$ is a local minimum.

If $(x,y,\lambda) = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right)$, $|H\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right)| = 4 > 0$, and this implies that $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ is a local maximum.

If $(x,y,\lambda) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right)$, $|H\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right)| = 4 > 0$, and this implies that $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ is a local maximum.