MATHEMATICS 2

Exam

21/06/2018, A.Y. 2017/2018

Lastname .... Name ....

1) *(8 p.ts)* Solve the following integral

$$\int_1^e \frac{\ln x}{\sqrt{x}} dx$$

Applying integration by parts

$$\int_1^e \frac{\ln x}{\sqrt{x}} dx = \Big[2\sqrt{x}\ln x\Big]_1^e - 2\int_1^e \sqrt{x}\frac{1}{x} dx = 2\sqrt{e} - 2\int_1^e \frac{1}{\sqrt{x}} dx$$

$$= 2\sqrt{e} - 4\Big[\sqrt{x}\Big]_1^e = 2\sqrt{e} - 4\sqrt{e} + 4 = 4 - 2\sqrt{e} = 2(2-\sqrt{e})$$

2) *(12 p.ts)* Is the following matrix diagonalizable? If so, identify the invertible matrix T that transforms A into a diagonal matrix, and show how T realizes this transformation.

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Calculate eigenvalues

$$|A| = \begin{vmatrix} -1-\lambda & -2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = \lambda(1+\lambda)(2-\lambda) + 2 + 2 - 2(2-\lambda) - (1+\lambda) - 2\lambda$$

$$\begin{aligned} &= 2\lambda - \lambda^2 + 2\lambda^2 - \lambda^3 + 4 - 4 + 2\lambda - 1 - \lambda - 2\lambda \\ -\lambda^3 + \lambda^2 + \lambda - 1 &= -\lambda^2(\lambda-1) + (\lambda-1) = -(\lambda-1)(\lambda^2-1) \\ &= -(\lambda-1)^2(\lambda+1) \end{aligned}$$

hence the eigenvalues are $\lambda_1 = 1$ of algebraic multiplicity $m_a = 2$, and $\lambda_2 = -1$ of algebraic multiplicity $m_a = 1$. The given matrix is diagonalizable if for each eigenvalue, algebraic multiplicity is equal to geometric multiplicity. If $\lambda_1 = 1$ we have $rk(A - 1 \cdot I_3) = 1$, hence its geometric multiplicity is $m_g = 3 - 1 = 2$. If $\lambda_2 = -1$, we have $rk(A + 1 \cdot I_3) = 2$, hence its geometric multiplicity is $m_g = 3 - 2 = 1$. As $m_a = m_g$ in both cases, the given matrix is diagonalizable. Let us calculate the correspondent eigenvectors.
If $\lambda_1 = 1$, we find eigenvectors by solving

$$\begin{cases} -2x - 2y - 2z = 0 \\ x + y + z = 0 \\ -x - y - z = 0 \end{cases}$$

This system gives the following solutions

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

We can choose $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ as independent eigenvectors.
If $\lambda_2 = -1$, we find eigenvectors by solving

$$\begin{cases} -2y - 2z = 0 \\ x + 3y + z = 0 \\ -x - y + z = 0 \end{cases}$$

This system gives the following solutions

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2\gamma \\ -\gamma \\ \gamma \end{pmatrix} = \gamma \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

We can choose $v_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ as eigenvector correspondent to $\lambda_2 = -1$.

Place v_1, v_2, v_3 as columns of the requested matrix T:

$$T = \begin{pmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

This matrix is invertible as v_1, v_2, v_3 are linearly independent. Its inverse is

$$T^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 3 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and this matrix realizes the diagonalizabilty of A, as

$$T^{-1}AT = \frac{1}{2} \begin{pmatrix} 1 & 3 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

3) *(10 p.ts)* Given the function in $\mathbb{R}^2$

$$f(x,y) = 2x^2y - xy^2 + 2xy - y^2$$

find maximum and/or minimum and say if they are local or global.

Let us calculate the points for which the gradient is null

$$\begin{cases} f_x = 4xy - y^2 + 2y = y(4x - y + 2) = 0 \\ f_y = 2x^2 - 2xy + 2x - 2y = 2[x(x-y) + (x-y)] = 2(x+1)(x-y) = 0 \end{cases}$$

hence all the possible combinations that solve the previous system are $(-1,0)$, $(0,0)$, $(-1,-2)$, $(-\frac{2}{3}, -\frac{2}{3})$. To evaluate the nature of these points let us calculate the Hessian matrix

$$H(x,y) = \begin{pmatrix} 4y & 4x - 2y + 2 \\ 4x - 2y + 2 & -2x - 2 \end{pmatrix}$$

let us analyze each single point:

- $(-1,0)$ is such that $|H(-1,0)| = -2 < 0$, hence it is a saddle point
- $(0,0)$ is such that $|H(0,0)| = -4 < 0$, hence it is a saddle point
- $(-1,-2)$ is such that $|H(-1,-2)| = -4 < 0$, hence it is a saddle point
- $(-\frac{2}{3}, -\frac{2}{3})$ is such that $|H(-\frac{2}{3}, -\frac{2}{3}))| = \frac{4}{3} > 0$,and moreover $f_{xx} = -\frac{8}{3} < 0$, hence the point is a local maximum.

As the function is not globally convex, the max is just local.