1 Open set, closed set, bounded sets, internal points, accumulation points, isolated points, boundary points and exterior points.

Definition 1. Consider the set of real numbers $\mathbb{R}$ and let $E \subset \mathbb{R}$ be a subset of $\mathbb{R}$. The following definitions are standard:

- A **neighborhood** of radius $\varepsilon \in \mathbb{R}$, $\epsilon > 0$, of a point x_0 of $\mathbb{R}$ is the set defined by:

  $$N_\varepsilon(p) = (x_0 - \varepsilon, x_0 + \varepsilon).$$

  A right-neighborhood of radius $\varepsilon \in \mathbb{R}$, $\epsilon > 0$, of a point x_0 of $\mathbb{R}$ is the set defined by:

  $$N_\varepsilon(p)^+ = (x_0, x_0 + \varepsilon).$$

  A left-neighborhood of radius $\varepsilon \in \mathbb{R}$, $\epsilon > 0$, of a point x_0 of $\mathbb{R}$ is the set defined by:

  $$N_\varepsilon(p)^- = (x_0 - \varepsilon, x_0).$$

- A point $p \in \mathbb{R}$ is a **limit point** (also said **accumulation point**) of E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

  In formula

  $$(p \text{ accumulation point of } E \subseteq \mathbb{R}) \Leftrightarrow \left(\forall \varepsilon > 0 \Rightarrow \left(N_\varepsilon(p) \bigcap E\right) \setminus \{p\} \neq \varnothing\right)$$

  The set made by all the accumulation point of E is called the derivative set and it is indicated with E'.

- If $p \in E$ and p is not a limit point of E then p is called an **isolated point** of E.

- E is **closed** if every limit point of E is a point of E. For example the set $E = (0, 1) \subset \mathbb{R}$ is not closed because the points 0 and 1 are limit points of E but they are not in E. Nevertheless the set $E = [0, 1] \subset \mathbb{R}$ is colsed.

- A point p of E is an **interior point** of E if there is a neighborhood $N_\varepsilon(p)$ of p such that $N \subset E$.

  In formula

  $$(p \text{ interior point of } E \subseteq \mathbb{R}) \Leftrightarrow (\exists \varepsilon > 0 : N_\varepsilon(p) \subset E)$$

- E is **open** if every point of E is an interior point of E. For example the set $E = (0, 1) \subset \mathbb{R}$ is open.

- E is **bounded** if there is a real number M and a point $q \in \mathbb{R}$ such that $|p-q| < M$ for all $p \in E$.
- A point p is a **boundary** point for E if every neighborhood of p contains at least one point of E and at least one point in E^c.

  The set made by all the boundary points of a set E is indicated with ∂E and it is called the border of E. For example

$$\partial(0,1) = \partial[0,1] = \partial(0,1] = \partial[0,1) = \{0,1\}.$$

- A point $p \in \mathbb{R}$ is called an **exterior point** of E if

$$\exists \varepsilon > 0 : N_\varepsilon(p) \subset E^c.$$

  Recall that E^c is defined as the complement set $E^c = \mathbb{R} \setminus E = x \in \mathbb{R} | x \not| nE$.

- The **closure** of E is defined as the set

$$\bar{E} = E \bigcup E'$$

  that is the union of E with the set of all its limit points.

Remark. Trivially the closure of a set is always a closed set since it contains all its limit points by definition.

Example. Consider the set $E = (0,1)$. It is clear that

- E is open.
- The accumulations points are all the points of $[0,1]$
- All $x \in (0,1)$ are interior points.
- All $x \notin [0,1]$ are exterior points, i.e all x such that $x < 0$ or $x > 1$.
- The numbers 0 and 1 are the boundary points.
- The set E has no isolated points.

Theorem 1.1. *If $S_1, ..., S_n$ are open subsets of $\mathbb{R}$ then*

$$\bigcap_{i=1}^{n} S_i$$

is open.

Let $I \subseteq \mathbb{N}$ be an arbitrary set of indexes. Let S_i be open for all $i \in I$. Then

$$\bigcup_{i \in I} S_i$$

is open.

Proof.

1) Let $x \in \bigcap_{i \in I}^{n} S_i$, hence $x \in S_i$ for all $i = 1, ..., n$. Therefore given the openness of all S_i, for all $i = 1, ..., n$, there exists $\rho_i > 0$ such that $N_{\rho_i}(x) \subset S_i$. Consider $\rho = \min_i \rho_i$, hence for all i we have $N_\rho(x) \subseteq N_{\rho_i}(x) \subset S_i$ hence $N_\rho(x) \subset \bigcap_{i=1}^{n} S_i$.

2) Let $x \in \bigcup_{i \in I} S_i$. Therefore there exists $i^* \in I$ such that $x \in S_{i^*}$. Since S_{i*} is open there exists a ρ_{i^*} such that $N_{\rho_{i^*}} \subset S_{i^*} \subseteq \bigcup_{i \in I}^{n} S_i$. □

Consider the sequence of open sets $S_n = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) \subset \mathbb{R}$. Hence

$$S_1 = (0, 2) \supset S_2 = \left(\frac{1}{2}, \frac{3}{2}\right) \supset S_3 = \left(\frac{2}{3}, \frac{4}{3}\right) \supset \cdots$$

Now note that

- $\bigcap_{n=1}^{\infty} S_n = \{1\}$ is not open (in particular, since the set of limit points of $\{1\}$ is empty, then $\{1\}$ contains all its limit point so it is closed).
- $\bigcup_{n=1}^{\infty} S_n = (0, 2)$ is open.

It is immediate to se that E is closed if and only if $E^c = \mathbb{R} \backslash E$ is open. Hence

Theorem 1.2. *If $S_1, ..., S_n$ are closed subsets of $\mathbb{R}$ then*

$$\bigcup_{i=1}^{n} S_i$$

is closed.

Let $I \subseteq \mathbb{N}$ be an arbitrary set of indexes. Let S_i be closed for all $i \in I$. Then

$$\bigcap_{i \in I} S_i$$

is closed.

Proof. It follows from the fact that if S is closed then S^c is open and

$$\left(\bigcup_i S_i\right)^c = \bigcap_i S_i^c. \quad \square$$

Consider the sequence of closed sets $S_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \subset \mathbb{R}$. Now note that

- $\bigcup_{n=3}^{\infty} S_n = (0,1)$ is not closed (in particular, is open).
- $\bigcap_{n=3}^{\infty} S_n = \left[\frac{1}{3}, \frac{2}{3}\right]$ is closed.

2 Maximum, minimum, supremum and infimum of a subset of $\mathbb{R}$

In what follows, as before, we indicate with E a generic subset of $\mathbb{R}$.

Definition 2. We say that $M \in E$ is a **maximum** for E if

$$\forall x \in E \Rightarrow x \leq M,$$

similarly we say that $m \in E$ is a **minimum** for E if

$$\forall x \in E \Rightarrow m \leq x,$$

Warning: it is not absolutely guaranteed that a set E has a minimum and a maximum! This is mainly due to the fact that, in the definition of maximum and minimum, we require that both of them **belong** to E. As an example consider $E = (0,1)$. It is clear that $\forall x \in (0,1)$ we have $x > 0$ and $x < 1$. Nevertheless 0 is not a minimum (neither 1 is a maximum) since it does not belong to $(0,1)$. In fact the open set $(0,1)$ does not have neither a minimum nor a maximum. This is quite easy to see: let $x \in \mathbb{R}$, if $x \leq 0$ or $x \geq 1$, then x cannot be a minimum nor a maximum since $x \notin (0,1)$. On the contrary if $x \in (0,1)$ then x is an interior point and so we can find elements of $(0,1)$ that are both below and above x, so x is neither a minimum nor a maximum.

Example. Consider the case

$$E = \left\{\frac{1}{n} \;\middle|\; n \in \mathbb{N},\ n > 0\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$$

Note that $1 \in E$ and $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$, so $M = 1$. Does the set E have a minimum point? The answer is trivially no. Suppose that we have found a minimum point, and let $\frac{1}{n^\star}$ be such point. Trivially

$$\frac{1}{n^\star + 1} < \frac{1}{n^\star},$$

but $\frac{1}{n^\star+1} \in E$ so $\frac{1}{n^\star}$ is not a minimum point.

Example. Consider

$$E = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N},\ n > 0 \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots \right\}$$

then $\nexists$ the minimum of E, nevertheless

$$E' = E \bigcup \{0\} = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots \right\}$$

is such that 0 is the minimum point of E'. Similarly

$$F = \left\{ -\frac{1}{n} \,\middle|\, n \in \mathbb{N},\ n > 0 \right\} = \left\{ -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, \cdots \right\}$$

Then $\nexists$ the maximum of F, nevertheless

$$F' = F \bigcup \{0\} = \left\{ 0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, \cdots \right\}$$

is such that 0 is the maximum point of F'.

Theorem 2.1. Maximum and minimum are unique (when they exist) *Let $E \subset \mathbb{R}$ be a subset of $\mathbb{R}$. The maximum of E, provided that it exists, is unique. The same holds for the minimum.*

Proof. Let M be a maximum of E. Suppose that M' is another maximum. Since M is a maximum and since $M' \in E$ then it must be $M' \le M$. Nevertheless since, by hypothesis, also M' is a maximum and since, by the definition of maximum, $M \in E$ then it must be $M \le M'$. The only way in which $M' \le M$ and $M \le M'$ are possible is $M = M'$. □

A more general definition is given introducing the concept of upper and lower bounds.

Definition 3. Let $E \subset \mathbb{R}$. A number $M \in \mathbb{R}$ is called an upper bound for E if

$$\forall x \in E \Rightarrow x \le M.$$

Similarly, a $m \in \mathbb{R}$ is lower bound for E if

$$\forall x \in E \Rightarrow m \le x.$$

Note that, since the set $\mathbb{R}$ is provided with the "special" numbers $\pm\infty$ a lower and an upper bound for any set always exist. In particular if E is bounded (see definition before) then both the upper and the lower bound are $\neq \infty$.

Example. Let $E = (0, 1)$. Then

- Any $x \in \mathbb{R}$ with $x \ge 1$ is an upper bound.

- Any $x \in \mathbb{R}$ with $x \leq 0$ is a lower bound.

Example. Let $E = (0, \infty)$. Then

- Only $+\infty$ is an upper bound.
- Any $x \in \mathbb{R}$ with $x \leq 0$ is a lower bound.

Definition 4. Let $E \subset \mathbb{R}$. Consider these two subsets

$$\begin{aligned} U_E &= \{x \in \mathbb{R} | x \text{ is an upper bound for } E\}, \\ L_E &= \{x \in \mathbb{R} | x \text{ is a lower bound for } E\}. \end{aligned} \tag{2.1}$$

The minimum of U_E (which always exists) is called **supremum** of E and it is indicated with $\sup E$. The maximum of L_E (which always exists) is called **infimum** of E and it is indicated with $\inf E$.

Remark. Note that, differently from maximum and minimum, the supremum and the infimum always exist, they could be however $+\infty$ or $-\infty$.

Remark. Note that, if E has a maximum element M then $\sup E = M$. Similarly, if E has a minimum element m then $\inf E = m$.

Example. Let $E = (0, 1)$. We already know that E has neither a minimum nor a maximum element. However since

$$U_{(0,1)} = \{x \in \mathbb{R} \mid x \geq 1\}$$

then $\sup(0, 1) = 1$ and since

$$L_{(0,1)} = \{x \in \mathbb{R} \mid x \leq 0\}$$

then $\inf(0, 1) = 0$.

Remark. Thanks to Theorem 2.1 supremum and infimum of a set are unique.

Example. Consider the set

$$E = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N},\ n > 0 \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots \right\}$$

then

$$U_E = \{u \in \mathbb{R} \mid u \geq 1\}, \quad L_E = \{\ell \in \mathbb{R} \mid \ell \leq 0\}$$

and so, although E **has no minimum element**, we have

$$\inf(E) = \max(L_E) = 0,$$

while, straightforwardly, the maximum element and the supremum coincide

$$\sup(E) = \min(U_E) = 1.$$

Examples. The following table gives some example of max/min and inf/sup of sets.

Set	max	min	sup	inf
$\{n \in \mathbb{N} \mid n \le 10\}$	10	0	10	0
$\{q \in \mathbb{Q} \mid q > -1\}$	$\nexists$	$\nexists$	$+\infty$	-1
$\{q \in \mathbb{Q} \mid q \ge -1\}$	$\nexists$	-1	$+\infty$	-1
$\{q \in \mathbb{Q} \mid q < 0\}$	$\nexists$	$\nexists$	0	$-\infty$
$\{q \in \mathbb{Q} \mid q \le 0\}$	0	$\nexists$	0	$-\infty$
$\left\{1 - \frac{1}{n} \mid n \in \mathbb{N},\ n > 0\right\}$	$\nexists$	0	1	0
$\left\{1 - \frac{1}{n^3} \mid n \in \mathbb{N},\ n > 0\right\}$	$\nexists$	0	1	0
$\{q \in \mathbb{Q} \mid 0 \le q < 2\}$	$\nexists$	0	2	0
$\{q \in \mathbb{Q} \mid 0 < q \le 2\}$	2	$\nexists$	2	0
$\{q \in \mathbb{Q} \mid 0 \le q \le 2\}$	2	0	2	0

Exercise. *Let*

$$A = (0, 1) \cup 2$$

determine its accumulation points, isolated points, boundary points, interior points, exterior points, maximum, minimum, infimum, supremum.

Solution.

Accumulation points. Take $x \in \mathbb{R}$. Then if $x < 0$ clearly it is not an accumulation point since

$$\left(x - \frac{|x|}{2}, x + \frac{|x|}{2}\right) \bigcap A = \varnothing$$

This is immediate since $x + \frac{|x|}{2} < 0$ for a $x < 0$. The point $x = 0$ is an accumulation point since the neighborhood $(-\varepsilon, \varepsilon)$ has always a non-empty intersection with $(0, 1)$, that is

$$(-\varepsilon, \varepsilon) \bigcap A \setminus \{0\} \neq \varnothing.$$

Any $x \in (0,1)$ is trivially an accumulation point. Also $x = 1$ is an accumulation point since the neighborhood $(1-\varepsilon, 1+\varepsilon)$ has always a non-empty intersection with $(0,1)$, that is

$$(1-\varepsilon, 1+\varepsilon) \bigcap A \setminus \{1\} \neq \varnothing.$$

All other x with $x > 1$ cannot be accumulation points. Summing up the derivative set is given by

$$A' = [0,1].$$

Isolated points. Since $2 \in A$ and 2 is not an accumulation point then 2 is an isolated point. Besides it is the unique isolated point of A.

Boundary points. Clearly 0 and 1 are boundary points since every neighborhood of 0 or 1 contains points of A and of A^c. Also 2 is a boundary points since every neighborhood of 2 contains 2, which is a point of A, and points of A^c, so

$$\partial A = \{0,1,2\}.$$

Interior points. The interior points of A are the points of the set $(0,1)$. In fact, if $x \in (0,1)$ then I can always find a $N_\varepsilon(x)$ such that $N_\varepsilon(x) \subset A$ (try to find it). Vice versa all other points of $\mathbb{R}$ cannot be interior points, neither 2 can be since, for example, the neighborhood $\left(2-\frac{1}{2}, 2+\frac{1}{2}\right)$ is not contained in A.

Exterior points. All x such that $x < 0$, $1 < x < 2$, $x > 2$.

Maximum. Clearly we have that $\forall x \in A$ then $x \leq 2$, besides since $2 \in A$ we have that the set A has a maximum and this maximum is 2.

Minimum. A has no minimum element.

Supremum. Since A has 2 as a maximum then $\sup A = 2$.

Infimum. Consider that

$$L_A = \{x \in \mathbb{R} | x \leq 0\},$$

hence

$$\inf A = \max L_A = 0.$$

Exercize. Let B_r represents the open interval $(-1, r)$ and let J be the set of positive real numbers. Describe, with proof, the set $\cap_{r\in J} B_r$.

Solution. We claim that the intersection of this family consists of all $-1 < x \le 0$. First, if $x \le -1$ then $x \notin B_r$ for any r according to the definition of B_r, and thus x is clearly not in their intersection. Furthermore, if $-1 < x \le 0$ then $x \in B_r$ for every positive real number r, since B_r consists of all real numbers between -1 and r, which certainly includes any x in the range $-1 < x \le 0$. Hence these values of x belong to the intersection $\cap_{r\in J} B_r$. Finally, given any $x > 0$, choose $r = 0.5x$. Then r is a smaller positive real number, so $x \notin B_r$ for this particular r. Since x is absent from at least one such set, it does not belong to their intersection.

3 Functions

Definition 5. A function $f : A \to B$ is a correspondence that associate to each element of a set A (called **domain**) **one and only** element of a set B (called **co-domain**).

The function is typically indicated with the letter f, g, h, ... and it is specified once all elements of A are assigned to one and only one element of B.

Definition 6. The image of a function $f : A \to B$ is the set indicated with I_f and defined as

$$I_f = \{b \in B | \exists a \in A : f(a) = b\}.$$

Examples.

1. *Consider the sets $A = \{0, 5, \pi\}$ and $B = \{0, 1, 2, 3, 4\}$ and the correspondence $f : A \to B$ defined as*

$$\begin{array}{ccc} 0 & \longrightarrow & 3 \\ 5 & \longrightarrow & 3 \\ \pi & \longrightarrow & 4 \end{array}$$

   which is typically shortened in $f(0) = 3$, $f(5) = 3$ and $f(\pi) = 4$. So we have $I_f = \{3, 4\} \subset B$.

2. *Consider the function*

$$\begin{array}{rcl} f : \mathbb{R} \smallsetminus \{0\} & \longrightarrow & \mathbb{R} \\ x & \longrightarrow & \dfrac{1}{x} \end{array} \tag{3.1}$$

   shortened in $f(x) = \frac{1}{x}$. The domain of the function is $\mathbb{R} \smallsetminus \{0\}$ while the image is given by

$$I_f = \left\{ y \in \mathbb{R} | \exists x \in \mathbb{R} \smallsetminus \{0\} : y = \frac{1}{x} \right\} = \mathbb{R} \smallsetminus \{0\}$$

3. *Consider the function*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow x^2 \end{aligned} \tag{3.2}$$

In this case $I_f = \mathbb{R}^+ = \{y \in \mathbb{R} | y \geq 0\}$.

4. *Consider the function*

$$\begin{aligned} f : \mathbb{R} \smallsetminus \{0\} &\longrightarrow \mathbb{R} \\ x &\longrightarrow x^2 - 1 \end{aligned} \tag{3.3}$$

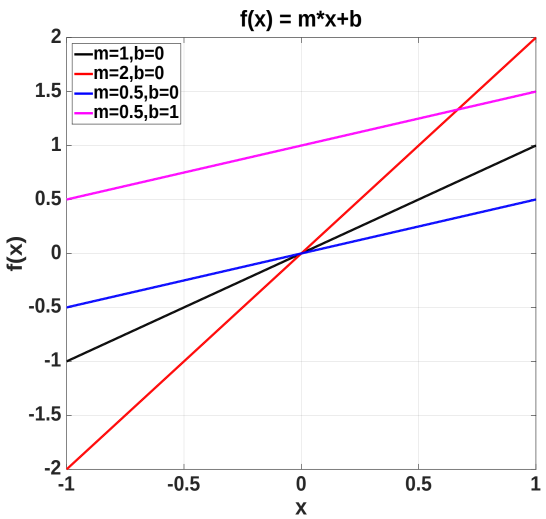
In this case $I_f = \{y \in \mathbb{R} | y \geq -1\}$.

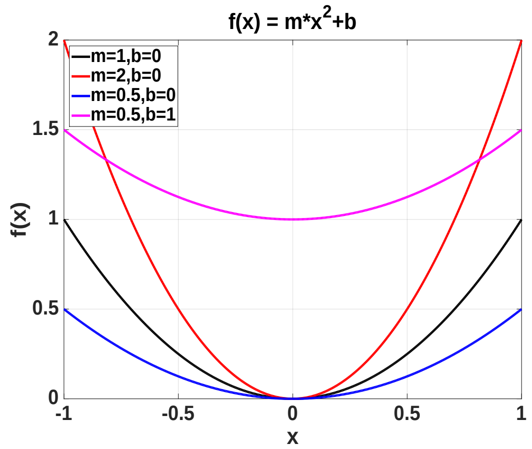
Economic example. *The most simple function form* $\mathbb{R} \to \mathbb{R}$ *is the linear function*

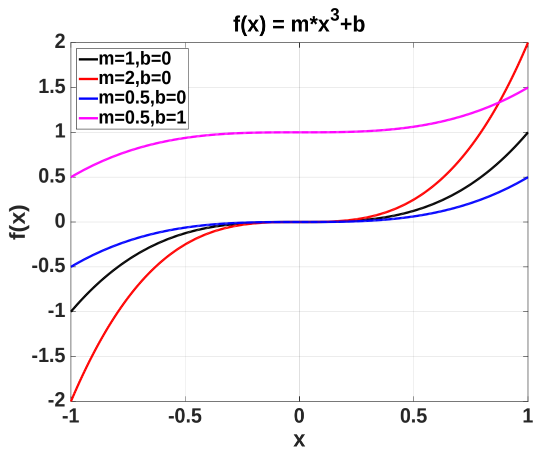
$$f(x) = m\,x + b$$

where m *and* b *are given* ***fixed*** *parameters.*

To have an idea of its plot it is enough to choose a value for m *and* b *and then, in the function so obtained, plug some value for* x *and extract the corresponding value for* $f(x)$*:*




f(x) = m*x^2+b
m=1,b=0
m=2,b=0
m=0.5,b=0
m=0.5,b=1
f(x)
x
2
1.5
1
0.5
0
-1
-0.5
0
0.5
1


f(x) = m*x^3+b
m=1,b=0
m=2,b=0
m=0.5,b=0
m=0.5,b=1
f(x)
x
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
-0.5
0
0.5
1

Suppose now for example that p is the price of a good and that the demand for that good is a linear function of p

$$D(p) = -\alpha p + \beta$$

with $\alpha > 0$ and $\beta > 0$. Since α is positive then higher prices correspond to lower demand. Similarly suppose that the supply is given by

$$S(p) = \gamma p + \delta$$

with $\gamma > 0$ and $\delta > 0$. Since γ is positive then higher prices correspond to higher supply. Which is the economic interpretation of α and γ? Suppose that an economic shock changes the price form p to $p + \Delta$ with $\Delta > 0$, which are the changes in demand and supply because of this shock? Let's focus first on the demand

$$D(p+\Delta) - D(p) = -\alpha\,(p+\Delta) + \beta - (-\alpha p + \beta) = -\alpha p - \alpha\Delta + \beta + \alpha p - \beta = -\alpha\Delta,$$

so the rate of change of the demand, defined as the absolute change in demand per unit of price is

$$\left|\frac{D(p+\Delta) - D(p)}{\Delta}\right| = \alpha,$$

similarly the rate of change of the supply is

$$\left|\frac{S(p+\Delta) - S(p)}{\Delta}\right| = \gamma.$$

Which is the equilibrium price? The equilibrium price is defined as that particular p^* such that

$$D(p^*) = S(p^*)$$

so that

$$-\alpha p^* + \beta = \gamma p^* + \delta \Leftrightarrow p^* = \frac{\beta - \delta}{\gamma + \alpha}.$$

Hence a strictly positive equilibrium price exits if and only if $\beta > \delta$, besides the larger the γ or the α (i.e. the larger the rate of change of supply and the rate of change of demand, respectively) the smaller the equilibrium price.

4 The Absolute Value and The Triangular Inequality

As a special function consider

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

called "the absolute value of x" and defined for all $x \in \mathbb{R}$.

Theorem 4.1. Triangular Inequality. *For all a and b in $\mathbb{R}$ it holds that*

$$|a+b| \leq |a| + |b|.$$

Proof.

$$-|a| \leq a \leq |a|.$$

$$-|b| \leq b \leq |b|.$$

by summing we get

$$-|a| - |b| \leq a + b \leq |a| + |b|.$$

Let $c = |a| + |b|$, then:

$$-c \leq a + b \leq c,$$

which means

$$|a+b| \leq |c| = |a| + |b|. \quad \square$$

5 Surjective and Injective functions.

Definition 7. A function is called injective if

$$\forall a, a' \in A : a \neq a' \Rightarrow f(a) \neq f(a'),$$

which is logically equivalent to

$$\forall a, a' \in A : f(a) = f(a') \Rightarrow a = a'.$$

The functions in the examples above are , respectively: 1) not injective 2) injective 3) not injective 4) not injective.

Definition 8. A function $f : A \to B$ is called surjective if

$$\forall b \in B : \exists a \in A : f(a) = b.$$

Note that if we restrict the definition of a function this way

$$f : A \to I_f$$

all functions are surjective, the problem arises if we think the function as having a co-domain in which the image of the function is strictly included.

Definition 9. A function which is surjective and injective is called bijective.

For example the function

$$f(x) = \frac{1}{x}$$

is a bijection from $\mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+ \setminus \{0\}$ but not a bijection from $\mathbb{R}^+ \setminus \{0\} \to \mathbb{R}$ since, for example, the element $y = 0$ of $\mathbb{R}$ is not the image of any $x \in \mathbb{R}$ through f.

Definition 10. Let $f : A \to B$ and let $g : D \to C$. Suppose that $I_f \subseteq D$. Then it is well defined the composite function

$$g \circ f : A \to C$$

defined as

$$\forall a \in A : (g \circ f)(a) = g(f(a)).$$

Remark. Note that it is essential that the domain of the second function g contains the image of f, otherwise it may be the case that $f(a)$ falls outside of the domain of definition of g and then the writing $f(g(a))$ is meaningless. For example consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $x \to -|x|$ and the function $g : \mathbb{R} \to \mathbb{R}$ defined as $x \to \sqrt{x}$, the composition of $(g \circ f)$ would require to compute $\sqrt{-|x|}$ which is not defined (at least not in $\mathbb{R}$).

Definition 11. A function $f : A \to B$ is called invertible if there exists a second function called inverse, indicated with f^{-1}, defined from B to A such that

$$f^{-1} \circ f = \iota,$$

where is ι is the identity function defined as

$$\begin{array}{c} \iota : A \to A \\ a \to a \end{array},$$

simply put

$$f^{-1}(f(a)) = a.$$

Theorem 5.1. Suppose that $f : A \to B$ is bijective. Then f is invertible.

Proof. We have to define the inverse of f. So let $b \in B$. Since f is surjective then $\exists a \in A$ such that $f(a) = b$. Besides since f is injective this a is unique, in fact if I consider any $a' \neq a$ I would have $f(a') \neq f(a) = b$. So we can say that $\exists! a \in A$ so that $f(a) = b$. Hence I define, for all $b \in B$,

$$g(b) \stackrel{def}{=} a, \text{ with } a \text{ the unique element of } A \text{ such that } f(a) = b.$$

this definition is well-posed since the a is unique (remember that a function must associate to each point of the domain one and only one point of the co-domain). Note that I have defined the g in all the points of B. Now it is obvious that, for all a in A

$$(g \circ f)(a) = g(f(a)) = a,$$

which means $g = f^{-1}$. □

Examples.

- Let $f : \mathbb{Z} \to \mathbb{Z}$ the function defined as

$$f : \quad n \to n+1$$

  which is typically written as $f(n) = n + 1$. The function is trivially a bijection hence it exists the inverse function $g : \mathbb{N} \to \mathbb{N}$ which is $g(n) = n - 1$.

Exercizes.

1. Let $A = \{0, 1, 2, 3\}$ and $f : A \to \mathbb{Z}$ defined as $f(x) = 2x - 3$. Determine the image. $(\{-3, -1, 1, 3\})$.
2. Say if the following law defines a function from $\mathbb{R}$ to $\mathbb{R}$:

$$f(x) = \begin{cases} x+3 & \text{if } x \geq 1 \\ -x^2 + x & \text{if } x \leq 1 \end{cases} \tag{5.1}$$

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(x) = \begin{cases} 2x-1 & \text{if } x \geq 2 \\ 1 - 3x & \text{if } x < 2 \end{cases} \tag{5.2}$$

   Determine x such that $f(x) = 7$.
4. Let $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x^2 - 1$.
   - Determine the minimum value of the image of f. $(y = -1)$

- Try to draw the graph of the function f.
- Determine x such that $f(x) = 8$. ($x = \pm 3$)
- Determine x such that $f(x) = 18$. (there are no $x \in \mathbb{Z}$ with this property)

Exercize. Given $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x}$ determine for which x is possible to define the composite function $h = g \circ f$ and write explicitly the form of h. Determine then for which x is possible to define the composite function $u = f \circ g$ and write the u explicitly.

Solution. Since $\sqrt{x}$ is define only for $x \geq 0$ we have to impose that

$$\frac{1}{x} \geq 0$$

which corresponds to $x > 0$ ($x = 0$ has no reciprocal). So the inverse function h is defined from $(0, \infty)$ to $(0, \infty)$ and it is given by

$$h(x) = g(f(x)) = \frac{1}{\sqrt{x}}.$$

Similarly $f \circ g$ is defined only in $(0, \infty)$ because, even if the square root function is defined in 0 (so $g(0)$ can be defined) f is not defined in 0 hence

$$u(x) = f(g(x)) = \frac{1}{\sqrt{x}} = h(x).$$

6 Increasing and decreasing functions

Definition 12. The function

$$f : A \subseteq \mathbb{R} \to \mathbb{R}$$

is said to be increasing on A if

$$\forall x_1, x_2 \in A : \quad x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$

and it is said that it is strictly increasing if

$$\forall x_1, x_2 \in A : \quad x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

Similarly it is said to be decreasing on A if

$$\forall x_1, x_2 \in A : \quad x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

and it is said that it is strictly decreasing if

$$\forall x_1, x_2 \in A : \quad x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

Examples.

- Any linear functions of the form

$$f(x) = m\,x + b$$

is strictly increasing if $m > 0$ and strictly decreasing if $m < 0$. If fact take $x_1 \le x_2$ then

$$f(x_2) - f(x_1) = m(x_2 - x_1)$$

so if $x_1 < x_2$ and $m > 0$ we have $m(x_2 - x_1) > 0$ and then $f(x_1) < f(x_2)$. Vice versa if $x_1 < x_2$ and $m < 0$ we have $m(x_2 - x_1) < 0$ and then $f(x_1) > f(x_2)$.

- The function $f(x) = x^2$ is strictly increasing in $[0, \infty]$ and strictly decreasing in $[-\infty, 0]$. Take any x_2 and x_1 such that $0 < x_1 < x_2$. Then $x_2^2 - x_1^2 = (x_1 + x_2)\,(x_2 - x_1)$ but $x_1 + x_2$ is positive since both x_1 and x_2 are positive and $x_2 - x_1 > 0$ since we assumed $x_1 < x_2$ then $x_2^2 - x_1^2 > 0$ and then the function is increasing. Similarly if $x_1 < x_2 < 0$ then again write

$$x_2^2 - x_1^2 = \underbrace{(x_1 + x_2)}_{<0}\,\underbrace{(x_2 - x_1)}_{>0} < 0,$$

hence the function is decreasing.

- The function $f(x) = x^3$ is defined on the entire real line and it is every where strictly increasing. This can be proved by noticing that, for any x_1 and x_2 real numbers, it holds that

$$x_2^3 - x_1^3 = (x_2 - x_1)(x_2^2 + x_2\,x_1 + x_1^2).$$

Besides the quantity $(x_2^2 + x_2\,x_1 + x_1^2)$ is always strictly positive unless $x_2 = x_1 = 0$, this can be seen by noticing that

$$(x_2^2 + x_2\,x_1 + x_1^2) = \frac{1}{2}\left(x_2^2 + x_1^2 + (x_1 + x_2)^2\right).$$

So in summary

$$x_2^3 - x_1^3 = \frac{1}{2}(x_2 - x_1)\left(x_2^2 + x_1^2 + (x_1 + x_2)^2\right)$$

hence if $x_1 < x_2$ then $x_1^3 < x_2^3$.

Theorem 6.1. Let $f : \mathbb{A} \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ be such that $f(A) \subseteq B$. Consider $h = g \circ f$ Then it holds that

- f increasing plus g increasing then $h = g \circ f$ increasing.
- f increasing plus g decreasing then $h = g \circ f$ decreasing.

- f decreasing plus g increasing then $h = g \circ f$ decreasing.
- f decreasing plus g decreasing then $h = g \circ f$ increasing.

Proof.

Let $x < y$ be points of A.

If f is increasing then $f(x) < f(y)$ and if g is increasing then $g(f(x)) < g(f(y))$ hence $g \circ f$ is increasing. On the contrary if g is decreasing $g(f(x)) > g(f(y))$ hence $g \circ f$ is decreasing.

If f is decreasing then $f(x) > f(y)$ and if g is increasing then $g(f(x)) > g(f(y))$ hence $g \circ h$ is decreasing. On the contrary if g is decreasing then $g(f(x)) < g(f(y))$ hence $g \circ h$ is increasing. □

Theorem 6.2. Let $f : \mathbb{A} \subseteq \mathbb{R} \to \mathbb{R}$ and $g : A \subseteq \mathbb{R} \to \mathbb{R}$ be two functions. Consider $h = g + f$ Then it holds that

- f increasing plus g increasing then $h = g + f$ increasing.
- f decreasing plus g decreasing then $h = g + f$ decreasing.

Proof. Immediate (try by yourselves). □

7 The Elementary Functions

Definition 13. *Given any real number $a > 0$, $a \neq 1$, we define, for all $n \in \mathbb{N}$, $n > 1$,*

$$a^n \stackrel{def}{=} \underbrace{a \cdot a \cdot a \cdots a}_{n-times}.$$

We further define $a^1 = a$ and

$$a^{-n} = \frac{1}{a^n}.$$

For any $m, n \in \mathbb{Z}$ we define

$$a^{n+m} = a^n\, a^m.$$

Hence we are forced to define

$$a^0 = 1,$$

since

$$a^0 = a^{n-n} = a^n\, a^{-n} = \frac{a^n}{a^n} = 1.$$

Definizione 7.1. For any $n > 1$, a function of the form

$$f(x) = x^n$$

is called a power function.

Properties. Let $f(x) = x^n$ be a power function. Then

- $f(x)$ is defined for all $x \in \mathbb{R}$.
- If n is even then $f(x) \geq 0$ for all $x \in \mathbb{R}$. If n is odd then $f(x)$ is positive in $\mathbb{R}^+$ and negative in $\mathbb{R}^-$.
- The image of $f(x)$ is $\mathbb{R}^+$ for n even and $\mathbb{R}$ for n odd.
- $f(x)$ is invertible only for $x \geq 0$ and its inverse is the radical function $f^{-1}(x) = x^{1/n}$ **that will be defined later on**.
- We will prove that for n odd the power function is increasing, while for n even is increasing for $x \geq 0$ and decreasing for $x \leq 0$.

In order to proceed with the definition we must state a theorem (that we do not show) that is fundamental to guarantee that the equation $y^n = a$, for given strictly positive a, has always a solution.

Theorem 7.1. *Let $a \in \mathbb{R}$ be a strictly positive real number, that is $a > 0$. Therefore for any $n \in \mathbb{N}$, with $n > 0$, there exists a unique $y \in \mathbb{R}$ with $y > 0$ such that*

$$y^n = a.$$

We call that y the n-th root of a and we indicate it with

$$y = a^{\frac{1}{n}}.$$

The theorem above can be formulated equivalently as

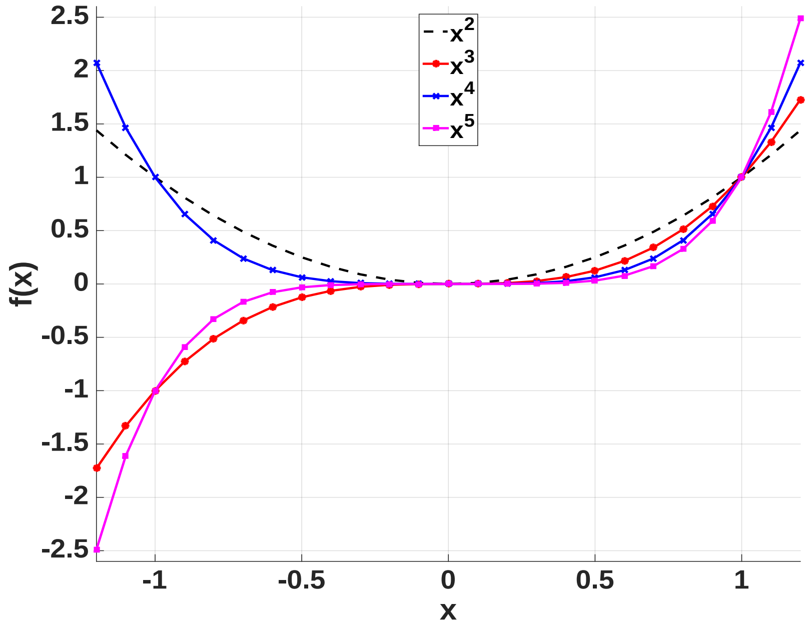


Fig. 1: The power functions.

Theorem 7.2. *For any $n \in \mathbb{N}$ with $n > 0$ the function*

$$f : \mathbb{R}^+ \to \mathbb{R}^+$$

defined as

$$f(x) = x^n$$

is surjective.

Since $f(x) = x^n$ as a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ is injective (note that we are restricting the domain to positive real numbers! For example $f(x) = x^2$ is NOT injective on all $\mathbb{R}$) it follows that $f(x) = x^n$ is a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$ and then is invertible. The inverse function si called the n-th root function or radical function.

$$f(x) = x^n : \mathbb{R}^+ \to \mathbb{R}^+ \Rightarrow f^{-1}(x) = x^{1/n} : \mathbb{R}^+ \to \mathbb{R}^+.$$

Remark. Note that it is fundamental in this context the introduction of the real numbers $\mathbb{R}$. In fact the equation

$$y^2 = 2$$

has no solution in $\mathbb{Q}$ in the sense that

$$\nexists y \in \mathbb{Q} : y^2 = 2.$$

Besides, the fact that the solution of $y^n = a$ exists does not mean that it is easy to compute it. The theorem does not give us any numerical routines.

Given the theorem above we can proceed with the definition of radicals.

Definizione 7.2. *Let $a > 0$ be a strictly positive real number and let n and m be two positive integers, i.e. $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Assume further that $m \neq 0$. We define*

$$a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n = (a^n)^{\frac{1}{m}}.$$

Definizione 7.3. For any $n > 1$, a function of the form

$$f(x) = x^{\frac{1}{n}}$$

is called a radical function.

Properties. Let $f(x) = x^{\frac{1}{n}}$ be a radical function. Then

- $f(x)$ is defined only for $x \geq 0$.
- $f(x) \geq 0$ in all its domain.
- The image of $f(x)$ is $\mathbb{R}^+$.
- $f(x)$ is invertible and its inverse is the power function $f^{-1}(x) = x^n$.
- Let x be strictly positive. We want to find for which x it is true that $0 < x^{\frac{1}{n}} \leq x$. Hence, assuming that $0 < x^{\frac{1}{n}} \leq x$, we arrive at

  $$0 < x \leq x^n$$

  which implies that $x \geq 1$, in fact it is not possible that $x < 1$ since this would imply $x^n < x$ (if you multiply any number smaller than one by itself, at every multiplication we obtain a smaller number).

  Summing up

  $$x^{\frac{1}{n}} \leq x \Leftrightarrow x \geq 1$$

and similarly

$$x^{\frac{1}{n}} \geq x \Leftrightarrow x \leq 1.$$

item Fix now $x \in (0,1)$. We want to find for which n and m it holds that

$$x^{1/n} > x^{1/m}.$$

By taking the n-power we get,

$$x > x^{n/m}$$

by now taking the m power we get

$$x^m > x^n$$

but since $x\,(0,1)$ this is possible if and only if $m < n$. Similarly fix $x \in (1,\infty)$. We want to find for which n and m it holds that

$$x^{1/n} > x^{1/m}$$

again we take first the n-th power and then the m-th power and we arrive at

$$x^m > x^n$$

but since $x > 1$ this is possible if and only if $m > n$.

Remark. Note that the definition above is well-posed since both $a^{\frac{1}{m}}$ and $(a^n)^{\frac{1}{m}}$ are well-defined real numbers.

Definition 14. A polynomial function is any function of the type

$$f(x) = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0$$

where n is an integer and a_i for $i = 0,...,n$ are real numbers. The number n is called degree of the polynomial while $a_n\,x^n$ is called the leading term and a_n the leading coefficient.

Exercize. Determine where

$$f(x) = x^3 - 2$$

is positive and negative, increasing or decreasing, and where $f(x) = 0$.

Solution. The function is positive if and only if $x^3 > 2$. Since the function x^3 is increasing this means that $x > 2^{1/3}$. Similarly the function is negative for $x < 2^{1/3}$ and zero if and only if $x = 2^{1/3}$. Note that since $g(x) = x^3$ is increasing then also $f(x) = x^3 - 2$ is increasing since adding or subtracting a constant is immaterial from this point of view.

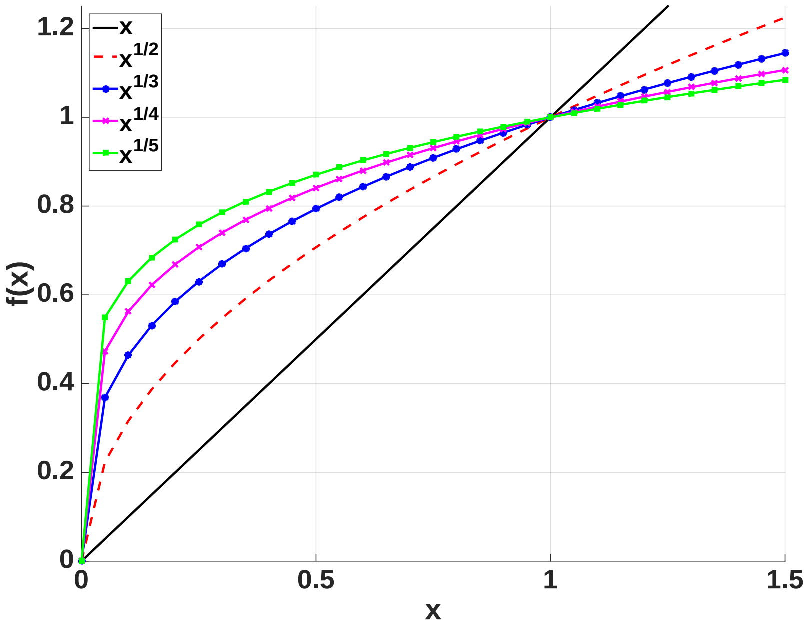


Fig. 2: The radical functions.

We have now a problem to solve. Given that we now the meaning of x^n which is the meaning of

$$3^\pi = 3^{3.14159265359.....} = ??? \tag{7.1}$$

that is exponentials whose exponent is a purely irrational number, that is a number of $\mathbb{R} \setminus \mathbb{Q}$. Remember that these kinds of number **cannot** be represented as ratios of the type $\frac{m}{n}$ with m and n integers and they have an infinite decimal representations (as the example reported for π in equation (7.1)). Let's focus on the case, for example, of $\sqrt{2}$, suppose we want to compute $a^{\sqrt{2}}$ with $a > 0$, $a \neq 1$. Recall that

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769480731766797379\ldots$$

The idea is that any of these "special" numbers, such as $\sqrt{2}$, can be approximated with a sequence of elements of $\mathbb{Q}$, that is there is with an infinite collections $\{q_1, q_2, q_3, ...\}$ of elements of $\mathbb{Q}$ such that

$$\lim_{n\to\infty} q_n = \sqrt{2},$$

where the last equation means that, the larger the n the better the approximation. For example consider

$$q_1 = \frac{14}{10} = \frac{7}{5} = 1.4, \; q_2 = \frac{141}{100} = 1.41, \; q_3 = \frac{1414}{1000} = 1.414, \cdots$$

for each of these rational numbers q_n we can compute, for example, a^{q_n}. The positive news is that, no matter which sequence of $q_n \to \sqrt{2}$ we take the limit of a^{q_n} is the same. So we define

$$a^{\sqrt{2}} \stackrel{\text{def}}{=} \lim_{n\to\infty} a^{q_n}.$$

We can now proceed with the "formal" definition:

Definition 15. *Let $a \in \mathbb{R}$ with $a > 0$ and $a \neq 1$. For any $x \in \mathbb{R} \setminus \mathbb{Q}$, $x > 0$, let $q_n \in \mathbb{Q}$, for all n, be any sequence of rational numbers such that $q_n \to x$ as $n \to \infty$ (we will define more formally this writing later on), then we define*

$$a^x \stackrel{def}{=} \lim_{n\to\infty} a^{q_n}.$$

If $x < 0$ we define

$$a^x \stackrel{def}{=} \frac{1}{a^{-x}}.$$

Definition 16. For any $a > 0$ with $a \neq 1$ the function

$$f(x) = a^x$$

is called the exponential function.

Properties. Let $a > 0$ and $a \neq 1$. Consider the exponential function $f(x) = a^x$.

- $f(x)$ is defined on $\mathbb{R}$ and its image is $\mathbb{R}^+ \setminus \{0\}$. In fact $a^x > 0$ for all x.
- It can be proved that

$$a^{x+y} = a^x\, a^y, \quad (a^x)^y = (a^y)^x = a^{x\,y}.$$

- If $a > 1$ then $a^x > 1$ for all x positive and $a^x < 1$ for all x negative. Viceversa if $0 < a < 1$ then $a^x > 1$ for all x negative and $a^x < 1$ for all x positive.
- If $a > 1$ the function is increasing:

$$f(y) - f(x) = a^y - a^x = a^x\, (a^{y-x} - 1)$$

so if $x < y$ then $a^{y-x} > 1$ and then $f(x) < f(y)$.

If $0 < a < 1$ then consider $g(x) = \frac{1}{f(x)} = \left(\frac{1}{a}\right)^x$, since $\frac{1}{a} > 1$ then $g(x)$ is increasing and so $f(x) = a^x$ is decreasing.

- The function is injective and, as can be seen graphically from Figure 5, is also surjective on $\mathbb{R}^+ \setminus \{0\}$, hence there exists the inverse function which is called $\log_a(x)$.

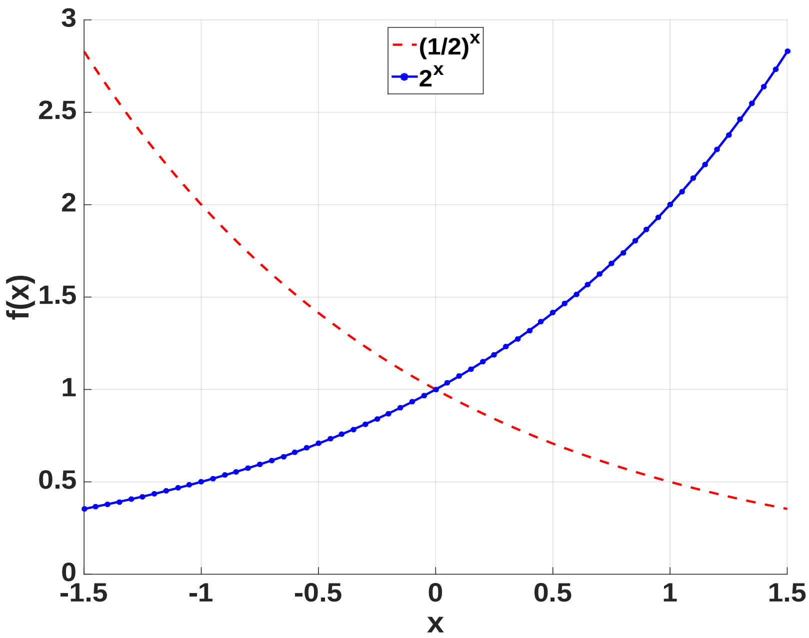


Fig. 3: The exponential functions.

Definition 17. Let a be a real number with $0 < a < 1$ and $a > 1$. For any $x > 0$ we call $\log_a(x)$ the real number y such that

$$a^y = x$$

The function $f(x) = \log_a(x)$ is called the logarithm with base a of x.

Properties. Let a be a real number with $0 < a < 1$ and $a > 1$ and let $f(x) = \log_a(x)$.

- The function $f(x)$ is defined from $\mathbb{R}^+ \setminus \{0\}$ to $\mathbb{R}$.
- The image is $\mathbb{R}$.
- Since $a^0 = 1$ then $\log_a(1) = 0$. Besides, if $a > 1$ then $\log_a(x) > 0$ for $x > 1$, in fact if we write

$$a^y = x$$

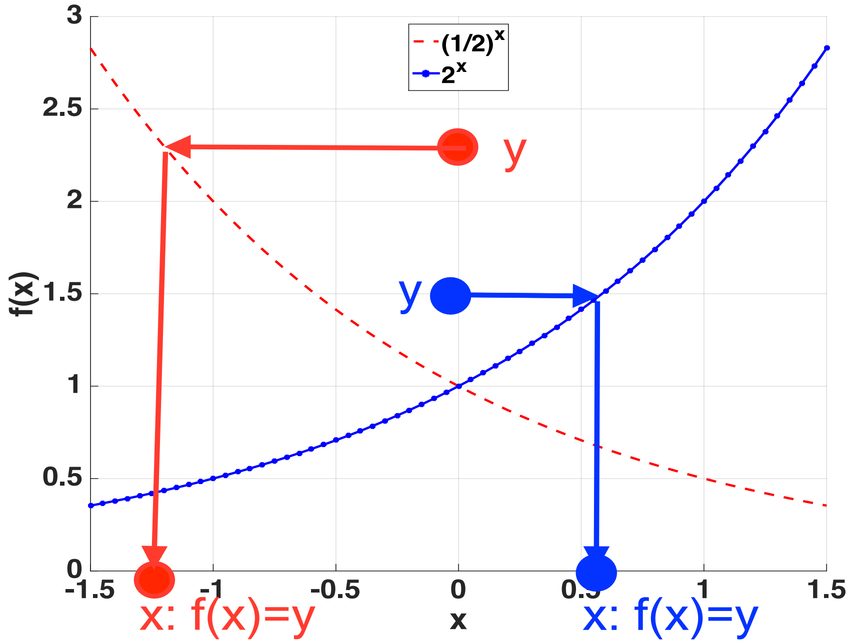


Fig. 4: The exponential functions: inversion.

then if $a > 1$ and $x > 1$ it can't be that $y < 0$ because otherwise we could write $y = -|y|$ and then

$$\left(\frac{1}{a}\right)^{|y|} = x$$

but $\left(\frac{1}{a}\right)^{|y|} < 1$ only because $a > 1$ (look at the red dotted line in Figure 3, they say that $b^y < 1$ if $y > 0$ and $0 < b < 1$), so it must be $y > 0$. Similarly if $0 < a < 1$ then $\log_a(x) < 0$ for $x > 1$.

- For any x and y we have that

$$\log_a(x) + \log_a(y) = \log_a(x\,y)\,.$$

In order to verify the last identity note that, by using the properties of the exponential function,

$$a^{\log_a(x)+\log_a(y)} = a^{\log_a(x)}\,a^{\log_a(y)} = x\,y$$

so, by definition, the quantity $\log_a(x) + \log_a(y)$ is the logarithm with base a of $x\,y$.

- For any b it holds that

$$b\,\log_a(x) = \log_a\left(x^b\right).$$

In order to verify the last identity note that, by using the properties of the exponential function,

$$a^{b\,\log_a(x)} = \left(a^{\log_a(x)}\right)^b = x^b,$$

where we have used $a^{\log_a(x)} = x$. So, by definition, the quantity $b \log_a(x)$ is the logarithm with base a of x^b.

- The function is increasing for $a > 1$ and decreasing for $a < 1$. Suppose for example that $a > 1$ then
$$\log_a y - \log_a x = \log_a y + \log_a x^{-1} = \log_a y\, x^{-1} = \log_a \frac{y}{x}.$$
Nevertheless if $x < y$ then $\frac{y}{x} > 1$ and then $\log_a\left(\frac{y}{x}\right) > 0$ if $a > 1$ and $\log_a\left(\frac{y}{x}\right) < 0$ if $0 < a < 1$.

- We can change base of the logarithm according to the rule
$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}.$$

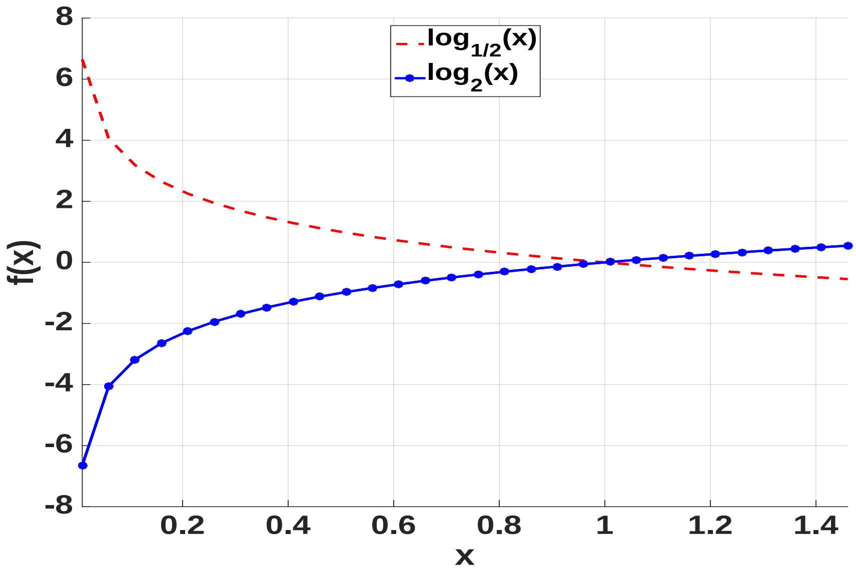


Fig. 5: The exponential functions: inversion.

Exercizes

- Consider the function
$$f(x) = \log_a(x) + \log_a(x-1)$$
with $a > 1$. Find the domain D of the function and characterize the sets $I_+ = \{x \mid f(x) \geq 0\}$ and $I_- = \{x \mid f(x) \leq 0\}$.

Solution.

The function is a sum of two elementary functions:

$$f_1(x) = \log_a(x)\,,\ f_2(x) = \log_a(x-1)\,.$$

The function f_1 is defined for all $x > 0$ while f_2 for all $x - 1 > 0$, hence $x > 1$. Since both f_1 and f_2 must be defined simultaneously in order to define f then the function f is defined for

$$D = \{x \mid x > 0\} \bigcap \{x > 1\} = \{x > 1\}\,.$$

Concerning the two sets, let's focus first on I_+. Note that we can write

$$f(x) = \log_a(x\,(x-1))$$

since $a > 1$ we have to look for those $x \in D$ such that $x\,(x-1) > 1$ hence $x < \frac{1-\sqrt{5}}{2}$ or $x > \frac{1+\sqrt{5}}{2}$, but we have to exclude $x < \frac{1-\sqrt{5}}{2}$ because $\frac{1-\sqrt{5}}{2} < 1$.

Finally since $\frac{1+\sqrt{5}}{2} > 1$ we get

$$I_+ = \{x \in D \mid x\,(x-1) > 1\} = \left(\frac{1+\sqrt{5}}{2}, \infty\right)$$

Now in order to find I_- we have to look for those $x \in D$ such that $x\,(x-1) < 1$, hence

$$I_- = \{x \in D \mid x\,(x-1) < 1\} = \left(1, \frac{1+\sqrt{5}}{2}\right)$$

- Consider the function $f(x) = \frac{x}{x+1}$. Find the domain D of f, where $f = 0$, $f > 0$ and $f < 0$. Prove that $f(x)$ is increasing in $(-\infty, -1)$ and in $(-1, \infty)$ but not in its entire domain.

  Solution. The function is defined in $D = \mathbb{R} \setminus \{-1\}$. The function is zero if and only if

$$\frac{x}{x+1} = 0 \to x = 0.$$

The function is positive where $x > 0$ and $x + 1 > 0$ that is $x > 0$ and where $x < 0$ and $x + 1 < 0$ that is $x < -1$. The function is negative where $x < 0$ and $x > -1$ hence in $(-1, 0)$.

Consider x and y with $x < y$. Let's distinguish the two cases.

1) If x and y are in $(1, \infty)$ then the condition

$$\frac{x}{x+1} < \frac{y}{y+1},$$

since $x + 1$ and $y + 1$ are both **positive** is equivalent to

$$x\,(y+1) < y\,(x+1) \Leftrightarrow x\,y + x < y\,x + y \Leftrightarrow x < y,$$

which is true by hypothesis, so the function is increasing.

2) If x and y are in $(-\infty,-1)$ then $x+1<0$ and $y+1<0$. Hence the condition

$$\frac{x}{x+1}<\frac{y}{y+1},$$

is equivalent to

$$x>(x+1)\,\frac{y}{y+1}$$

which is equivalent to

$$x\,(y+1)<(x+1)\,y$$

that is $x\,y+x<x\,y+y$, which gives again the hypothesis $x<y$. To prove that it is not increasing on D it is enough to consider $x=-2$ and $y=2$ then we get

$$f\,(-2)=\frac{-2}{-1}=2>f\,(2)=\frac{2}{2+1}=\frac{2}{3}.$$

- For any $a>1$ consider the function $f\,(x)=\log_a\left(\frac{x}{x+1}\right)$. Find the domain D of f and establish where f is increasing or decreasing in its domain.

  Solution.

  Note that $f\,(x)$ is the composition of $x\to\frac{x}{x+1}$ and $\frac{x}{x+1}\to\log_a\left(\frac{x}{x+1}\right)$. The first is defined for all $x\neq-1$ while the second only for $\frac{x}{x+1}>0$, so the domain is

$$D=\{x\mid x\neq-1\}\bigcap\left\{x\;\middle|\;\frac{x}{x+1}>0\right\}=\{x\mid x\neq-1\}\bigcap\left(\{x\mid x>0\}\bigcup\{x\mid x<-1\}\right)=\mathbb{R}\smallsetminus[-1,0]\,,$$

  since the first is increasing (see the exercize before) in $(-1,\infty)$ and increasing in $(-1,\infty)$ and the second is an increasing function, then by theorem (6.2) the composite function is increasing in $(-1,\infty)$ and $(0,\infty)$. Nevertheless if I take $x=-2$ and $y=2$ I get

$$f\,(-2)=\log_a(2)>f\,(2)=\log_a\left(\frac{2}{3}\right),$$

  so the function is not increasing in its entire domain.

- Let $f\,(x)=\left(\log_2\left(\left(\log_2\left(x^2-1\right)\right)^{1/2}\right)\right)^{1/2}$. Determine the domain of the function.

  Solution. By the properties of the logarithm and of the radical we have

$$f\,(x)=\left(\frac{1}{2}\,\log_2\left(\log_2\left(x^2-1\right)\right)\right)^{1/2}=\frac{1}{\sqrt{2}}\,\left(\log_2\left(\log_2\left(x^2-1\right)\right)\right)^{1/2}$$

  so the function is the composition of

$$x\to x^2-1\to\log_2\left(x^2-1\right)\to\log_2\left(\log_2\left(x^2-1\right)\right)\to 2^{-1/2}\,\sqrt{\log_2\left(\log_2\left(x^2-1\right)\right)}$$

The first is defined everywhere. The second is defined for all x such that $x^2-1>0$ which is true for all $x\in\mathbb{R}$ such that $x>1$ or $x<-1$. The third is defined for all x such that $\log_2\left(x^2-1\right)>0$ hence $x^2-1>1$ that is $x^2>2$ so $x>\sqrt{2}$ or $x<-\sqrt{2}$. The fourth is defined for all x such that $\log_2\left(\log_2\left(x^2-1\right)\right)>0$, hence $\log_2\left(x^2-1\right)>1$ hence $x^2-1>2$ or $x^2>3$ that is either $x<-\sqrt{3}$ or $x>\sqrt{3}$. In summary the domain of the function is

$$D=(-\infty,-\sqrt{3})\bigcup(\sqrt{3},\infty).$$

- Let $a>1$. Is the function $f(x)=a^{a^x}$ increasing or not? What can be said about

  $$f(x)=a^{a^{a^{\cdot^{\cdot^{\cdot^{a^x}}}}}}$$

  where the dots mean that the procedure is iterated n-times?

  Solution. Let $h(x)=a^x$. The function provided is simply

  $$f=h\circ h,$$

  so the function f is increasing. In the general case

  $$a^{a^{a^{\cdot^{\cdot^{\cdot^{a^x}}}}}}=\underbrace{(h\circ h\circ\cdots\circ h)(x)}_{n-\text{times}},$$

  hence the function $a^{a^{a^{\cdot^{\cdot^{\cdot^{a^x}}}}}}$ is increasing.

- For all $n\in\mathbb{N}$ consider the function $f(x)=\log_n\left(\frac{x-\frac{1}{n}}{x^{1/n}+2}\right)$. Determine the domain D_n of the function f_n. Determine then the set D where all the f_n's are defined.

  Solution. First we need to impose that $x>0$ because of the radical $x^{1/n}$. Then we have to impose

  $$x^{1/n}+2\neq 0$$

  which is always true for $x>0$ (i.e. no more conditions).

  We have to impose that

  $$\frac{x-\frac{1}{n}}{x^{1/n}+2}>0,$$

  hence $x>\frac{1}{n}$. Hence

  $$D_n=\left(\frac{1}{n},\infty\right)$$

  and so

  $$\bigcap_{n\in\mathbb{N}}D_n=D_1=(1,\infty)$$

  is the set where all the f_n's are defined.

- Consider the polynomial function

$$f(x) = 2x^{10} + x^4 + 3x^2 + 2$$

determine the maximum and minimum of the function in $[-1,1]$, that is the maximum and the minimum of the set

$$S = \left\{2x^{10} + x^4 + 3x^2 + 2 \;\middle|\; x \in [-1,1]\right\}.$$

Solution. First note that $f(x) \geq 2$ for all x and $f(0) = 2$ so the minimum is 2. Consider then that $2x^{10}$, x^4, $3x^2$ are all increasing in $[0,1]$ hence the sum is increasing in $[0,1]$ and then the maximum is achieved in $x = 1$ and is equal to $2 + 1 + 3 + 2 = 7$. Similarly all the monomials are decreasing in $[-1,0]$ and hence the maximum is achieved in $x = -1$ and it is equal to the value in $x = -1$. So the maximum in $[-1,1]$ is 7.

8 Euclidean Geometry

The set $\mathbb{R}^2$ can be represented by associating every point $(x,y) \in \mathbb{R}^2$ a point on the Cartesian plan with horizontal coordinate x and vertical coordinate y. A straight line in $\mathbb{R}^2$ is the locus of points $(x,y) \in \mathbb{R}^2$ such that $ax + by + c = 0$, where a, b, and c are given constants. Written explicitly

$$L = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; ax + by + c = 0\right\}.$$

For example with $b = 1$ and $a = -1$ and $c = 0$ we get the locus of points $y = x$ that is the secant of the first quadrant. Note that, a straight line can be described also as

$$L = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; y = mx + c\right\},$$

or, in a parametric fashion,

$$L = \{(t, mt + c) \mid t \in (-\infty, \infty)\}.$$

Exercise. Given the point $P = (1,1)$ (sometimes also indicated simply with $P(1,1)$) determine all the straight lines passing through P. Determine then all the straight lines that pass through $P = (1,1)$ and $Q = (-1,1)$.

Solution. We want to impose find all the straight lines L such that $P \in L$, so we impose

$$a+b+c=0 \Rightarrow c=-(a+b)$$

Hence all lines with equation $a\,x+b\,y-(a+b)=0$ pass through P, that is we have an infinite number of lines satisfying this condition. Let's now impose that also the point Q belong to the line. We have to **further** impose that

$$-a+b-(a+b)=0 \Rightarrow -a+b-a-b=0 \Rightarrow -2\,a=0 \Rightarrow a=0.$$

Then the equation of the straight lines becomes

$$a\,x+b\,y-(a+b)=0 \Rightarrow b\,y-b=0 \Rightarrow y=1$$

which is the horizontal line passing through the point $y=1$.

Definition 18. Two lines with equations

$$\begin{aligned} L_1: \quad & a_1\,x+b_1\,y+c_1=0, \\ L_2: \quad & a_2\,x+b_2\,y+c_2=0 \end{aligned} \tag{8.1}$$

are said **parallel** if

$$a_1 \cdot b_2 = a_2 \cdot b_1$$

and **perpendicular** if

$$a_1 \cdot a_2 + b_1 \cdot b_2 = 0$$

Examples. The lines $L_1: y=0$ and $L2: y=1$ are parallel since $a_1=0$ and $a_2=0$ so the condition of parallelism is verified. The lines $L1: x=0$ and $L2: y=1$ are perpendicular since

$$a_1=1, b_1=0, a_2=0, b_2=1$$

Exercise. Find all the straight lines L that pass through the point $P=(-1,-1)$ and that are parallel to the line $x+y+1=0$.

Solution. First we impose that the generic line $a\,x+b\,y+c=0$ passes through P, hence

$$-a-b+c=0 \Rightarrow c=a+b.$$

Hence the generic equation must be of the form

$$a\,x + b\,y + a + b = 0$$

Now we impose the condition of parallelism with $x + y + 1 = 0$ that is

$$a\,1 = 1\,b \Rightarrow a = b$$

that is

$$a\,x + a\,y + 2\,a = 0 \Rightarrow x + y + 2 = 0.$$

Exercise. Determine where the two lines

$$x + 2\,y = 0, \quad 2\,x + y = 0$$

intersect.

Solution. We have to find, if it exists, a point that belongs to both lines, so we have to solve the linear system

$$\begin{cases} x + 2\,y = 0, \\ 2\,x + y = 0 \end{cases}$$

from the first equation we get $x = -2\,y$ which plugged into the second one gives $-4\,y + y = 0$ so $y = 0$ and then $x = 0$. We can conclude that the two lines have a unique intersection point which is the origin.

Definition 19. The distance between two points $P = (x_p, y_p)$ and $Q = (x_q, y_q)$ is defined as

$$d\,(Q, P) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2} = d\,(P, Q)\,.$$

The distance of any point form the origin is the given by

$$d\,(Q, 0) = \sqrt{x_p^2 + y_p^2}$$

Exercise. Compute the distance between $(1, 1)$ and $(1, -1)$.

Solution.

$$d\left(\left(1,1\right),\left(1,-1\right)\right)=\sqrt{\left(1-1\right)^{2}+\left(1+1\right)^{2}}=2.$$

Definition 20. A circumference with center $P=\left(x_0,y_0\right)$ and radius ρ is the locus of points

$$C_P\left(\rho\right)=\left\{\left(x,y\right)\in\mathbb{R}^2 \,\middle|\, d\left(\left(x,y\right),P\right)=\rho\right\}.$$

Exercises. Find all the intersection points between the circumference $C_{(0,0)}\left(1\right)$ and the line $x+\frac{1}{2}y=0$.

Solution. We have to impose that

$$\begin{cases} x^2+y^2=1, \\ x+\frac{1}{2}y=0 \end{cases}$$

for the first we get $y=\pm\sqrt{1-x^2}$ which gives

$$x\pm\frac{1}{2}\sqrt{1-x^2}=0\Rightarrow\pm\frac{1}{2}\sqrt{1-x^2}=-x\Rightarrow\frac{1}{4}\left(1-x^2\right)=x^2\Rightarrow\frac{5}{4}x^2=\frac{1}{4}\Rightarrow x=\pm\frac{1}{\sqrt{5}}$$

and so, for example, if $x=+\frac{1}{\sqrt{5}}$ from the equation of the line we get

$$\frac{1}{\sqrt{5}}+\frac{1}{2}y=0\Rightarrow y=-\frac{2}{\sqrt{5}}$$

if $x=-\frac{1}{\sqrt{5}}$ from the equation of the line we get

$$-\frac{1}{\sqrt{5}}+\frac{1}{2}y=0\Rightarrow y=+\frac{2}{\sqrt{5}}$$

The intersection points are then

$$\left(\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}}\right),\quad\left(-\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)$$

7 Trigonometry

7.1 Angles and radiants

In planar geometry, an *angle* is the figure formed by two rays, called the *sides* of the angle, sharing a common endpoint, called *vertex*. Angles are measured in *grades* or *radiants*. An angle of one grade, denoted by $1°$, corresponds to the 360^{th} part of the round angle. The right angle measures 90°.

However, in mathematical analysis another method is used to measure angles. Let $A\hat{O}B$ an angle with vertex O and let AB the arc individuated by the circumference with center O and radius r. The ratio between the length a of the arc AB and the measure of the radius r is the *measure in radiants* of the angle $A\hat{O}B$. This measure is a *pure number* because it is a ratio between two quantities having the same unit of measure. If $r = 1$ then the length of the arc is equal to the length of the angle (see Figure 7.1).

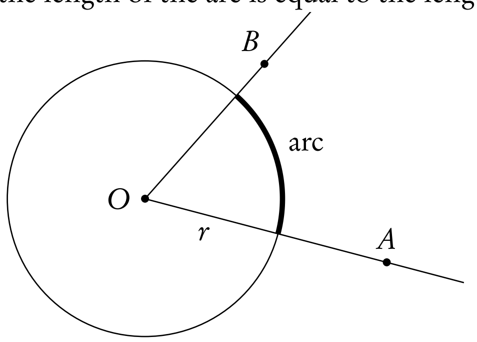


Figure 7.1 *Angles and their measure in radiants*

Table 7.1 reports some of the most used measures of angles (symbol $\alpha°$ is used to indicate the measure in grade whereas α to indicate the measure in radiants).

$\alpha°$	α
0°	0
30°	$\pi/6$
45°	$\pi/4$
60°	$\pi/3$
90°	$\pi/2$
180°	π
270°	$3\pi/2$
360°	2π

Table 7.1 *Angles and their measure in grades and radiants*

Although the definition of the measurement of an angle does not support the concept of negative angle, in applications, it is frequently useful to impose a convention that allows positive and negative angular values, to represent orientations and/or rotations in opposite directions relative to some reference. A positive sign is attributed to angles oriented anti-clockwise and negative to angles oriented clockwise.
There exists a direct correspondence between circumference arcs and angles. To measure an angle one moves (clockwise or anti-clock-wise) from the point of intersection between the circumference and the first side to the point of intersection between the circumference and the second side. In particular, it is possible to “travel” the circumference more than one time, obtaining angles “larger” than 2π. These angles are named *generalized angles*.
For example, with reference to Figure 7.2, you can imagine to start from the point P and “travel” the arc (of length 1) to join the point Q. At this point, you continue along the circumference to reach again the point Q. In this case the length of the “journey” will be $2\pi + 1$.

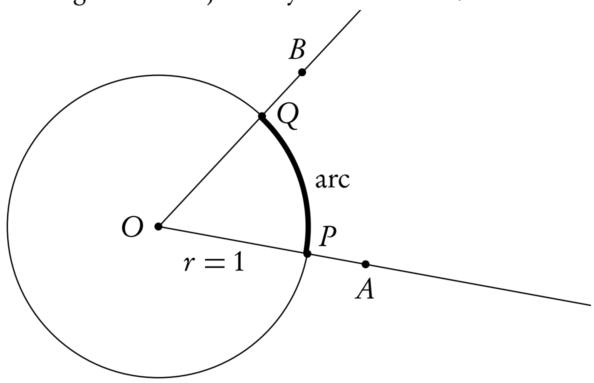


Figure 7.2 *Generalized angles*

When working with a Cartesian plane, it is always possible to assume that the vertex of the angle coincides with the origin and the first side with the positive semi-positive x-axis. In this situation, to measure angles, it is necessary to draw the circumference described by equation $x^2 + y^2 = 1$. This circumference, with unit radius, is named *geometric circumference*. At this point, angles are identified with the arcs of this circumference. Moreover, it is possible to associate with any real number a point on the circumference (this association is *not* unique because of the definition of generalized angles), by “travelling” the circumference (clockwise or anti-clockwise), starting from the point $(0, 1)$, for an arc of length the absolute value of the real number.

7.2 Sine and cosine functions

Le $P = (x_P, y_P)$ the point on the geometric circumference associated with the real number x. The abscissa, x_P, and the ordinate, y_P, of P have a great impact on applications. In particular, the following definition holds.

Definition 7.1. *The abscissa of the point P is named* cosine *of the real number x; the ordinate of the point P is named* sine *of the real number x. Precisely:*

$$x_P = \cos(x), \quad y_P = \sin(x), \text{ or, simply} x_P = \cos x, \quad y_P = \sin x. \tag{7.1}$$

Figures 7.3 and 7.4 represent the functions introduced above.

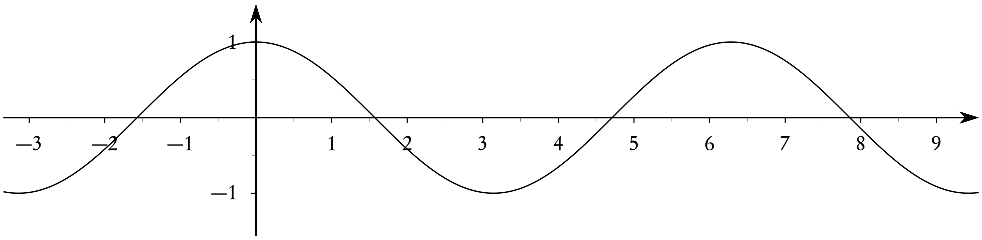


Figure 7.3 *The cosine function*

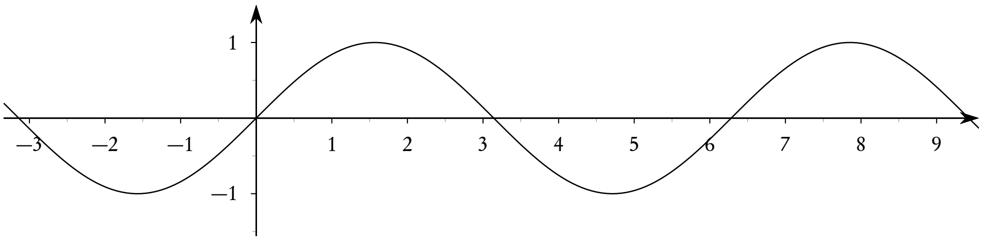


Figure 7.4 *The sine function*

Both sine and cosine functions (hereafter *trigonometric functions*) are *periodic*. In mathematics, a periodic function is a function that repeats its values in regular intervals or periods. Trigonometric functions repeat over intervals of 2π. Periodic functions are used throughout science to describe oscillations, waves, and other phenomena that exhibit periodicity. Any function which is not periodic is called *aperiodic*. Figure 7.5 represents a function obtained by opportunely mixing trigonometric functions.

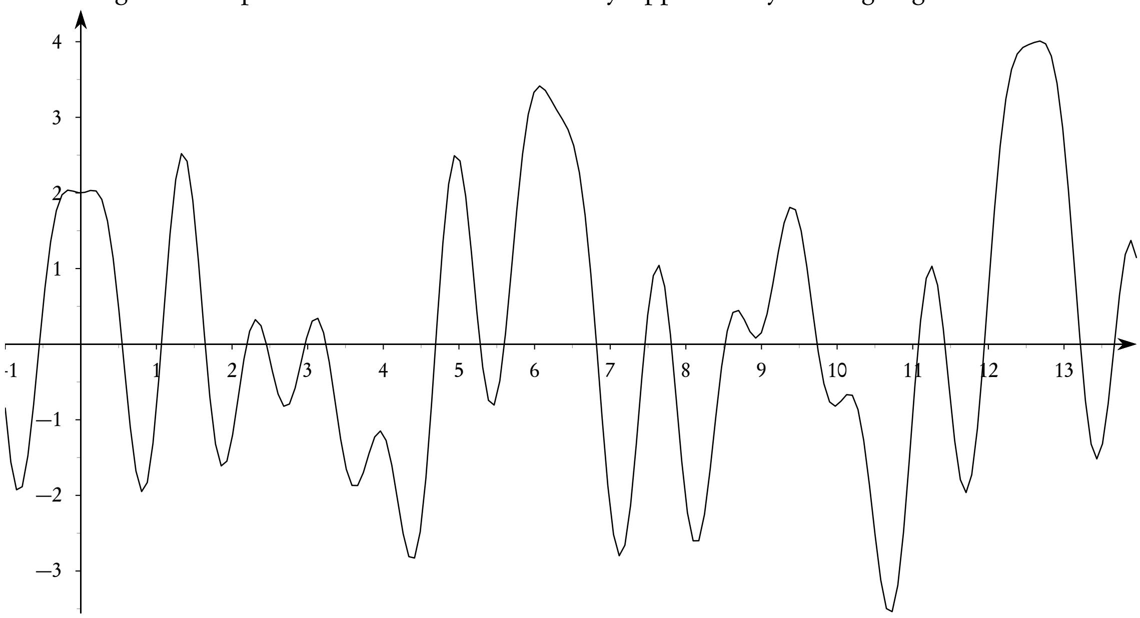


Figure 7.5 *Oscillatory function*

7.3 Addition formulae

In this section some important formulae linked to trigonometric functions are given. In particular, the *sum and difference formulae*.

(7.2) $$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y, \quad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y\,.$$

For instance, from

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}\,,\ \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\,,\ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\,,\ \sin\frac{\pi}{6} = \frac{1}{2}\,,$$

one obtains

$$\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}\,.$$

In particular, setting $x = y$:

(7.3) $$\cos(2x) = \cos^2 x - \sin^2 x\,, \quad \sin(2x) = 2\sin x \cos x\,.$$

9 Inverse trigonometric functions.

The sine and cosine function cannot be inverted on the entire real line, however we can find an inverse by restricting the dominion of the function. So consider the “restricted functions”

$$\cos : [0, \pi] \to [-1, 1], \quad \sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1].$$

With this restriction the cosine is a strictly decreasing function and the sine is a strictly increasing function, so it is possible to define the inverse functions

$$\arccos : [-1, 1] \to [0, \pi], \quad \arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

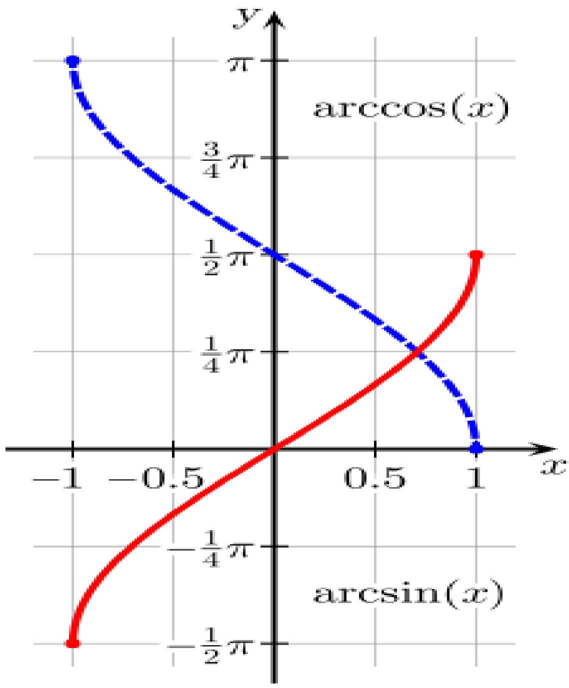


Fig. 6: Red: graph of the arcsin, blue: graph of the arccos.

Equivalently, the sin and cos function can be defined, by looking at the triangle in Figure 7, as

- $\sin(x) = \frac{a}{h}$.
- $\cos(x) = \frac{b}{h}$

Hence, by looking at Figure 8 we get that, the so-called secant function is such that

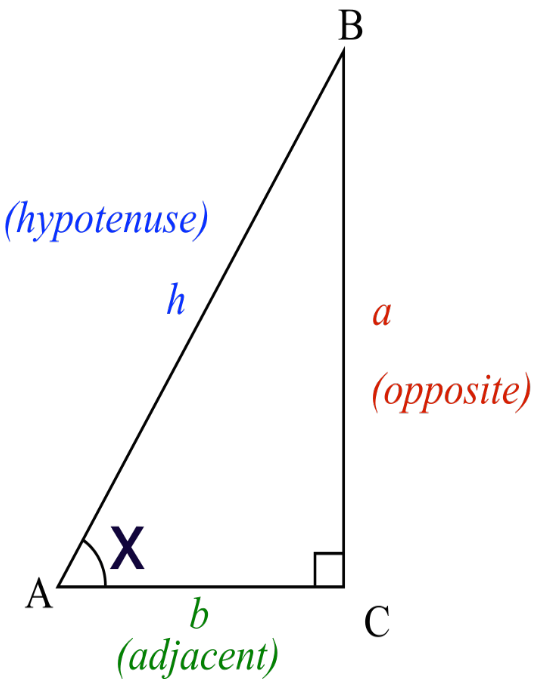


Fig. 7: Trigonometric triangle.

$$\text{secant}\,(\theta)\cdot\cos\,(\theta) = 1 \Rightarrow \text{secant}\,(\theta) = \frac{1}{\cos\,(\theta)}$$

and, finally, the tangent, as defined in Figure 8, must be such that

$$\text{secant}\,(\theta)\cdot\sin\,(\theta) = \tan\,(\theta) \Rightarrow \tan\,(\theta) = \frac{\sin\,(\theta)}{\cos\,(\theta)}.$$

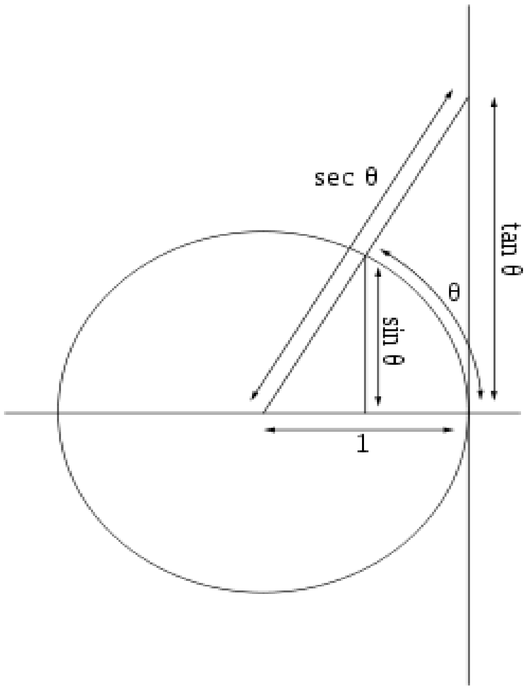


Fig. 8: Sine, cosine and the tangent.

10 Principle of Induction

Axiom 1. $\forall$ proposition $\mathcal{P}_n$ defined on the set of integer number $\mathbb{N}$:

$$[\mathcal{P}_{n^*} \wedge (\forall n > n^* : \mathcal{P}_n \Rightarrow \mathcal{P}_{n+1})] \Rightarrow (\forall n > n^* : \mathcal{P}_n)$$

Esercize 1. Compute the sum of the first n integer as a function of n.

$$S_n = 1 + 2 + ... + n.$$

Therefore:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + 2 = 3 = (2 \cdot 3)/2 \\ S_3 &= 1 + 2 + 3 = 6 = (3 \cdot 4)/2 \\ S_4 &= 1 + 2 + 3 + 4 = 10 = (4 \cdot 5)/2 \end{aligned}$$

Guess: $S_n = n \cdot (n+1)/2$. Let's show it by induction.

- $S_1 = 1 \Rightarrow$ true.
- Assume $S_n = n \cdot (n+1)/2$. Compute:

$$S_{n+1} = S_n + n + 1 = \frac{n \cdot (n+1)}{2} + n + 1 = (n+1) \left[\frac{n}{2} + 1\right] = \frac{(n+1) \cdot (n+2)}{2}. \Rightarrow \text{ok!}$$

Esercize 2. Show that:

$$(1+x)^n \geq 1 + n\,x, x \geq -1, n \in \mathbb{N}.$$

- $(1+x)^0 = 1 \geq 1 \Rightarrow$ ok!.
- Assume $(1+x)^n \geq 1 + n\,x$ and compute:

$$\begin{aligned} (1+x)^{n+1} &= (1+x)\ (1+x)^n \geq \\ &\geq (1+x)\ (1+n\,x) = 1 + n\,x + x + n\,x^2 = \\ &= 1 + (n+1)\ x + n\,x^2 \geq 1 + (n+1)\ x \Rightarrow \text{ok!} \end{aligned}$$

For all $n \in \mathbb{N}$ we define

$$n! = n\,(n-1)\,(n-2) \cdots 1.$$

Moreover we define for any couple of integers

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

With this definition the following properties hold

$$\begin{aligned}\binom{n}{k+1} + \binom{n}{k} &= \binom{n+1}{k+1} \\ \binom{n}{0} &= \binom{n+1}{0} = 1 \\ \binom{n}{n} &= \binom{n+1}{n+1} = 1\end{aligned}$$

Esercize 3. Show that:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k,$$

where:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

- For $n = 1$ we get:

$$(a+b)^1 = \sum_{k=0}^{1} \binom{1}{k} a^{n-k}\, b^k = \binom{1}{0} a + \binom{1}{1} b = a + b \Rightarrow \text{ok!}$$

- Assume $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k$ Compute:

$$(a+b)^{n+1} = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k = \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1}\, b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^{k+1}$$

The first term is:

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} a^{n-k+1}\, b^k &= \binom{n}{0} a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1}\, b^k \overset{q=k-1}{=} \binom{n}{0} a^{n+1} + \sum_{q=0}^{n-1} \binom{n}{q+1} a^{n-q}\, b^{q+1} \\ &\overset{k=q}{=} \binom{n}{0} a^{n+1} + \sum_{k=0}^{n-1} \binom{n}{k+1} a^{n-k}\, b^{k+1}.\end{aligned}$$

The second term is:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^{k+1} = \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k}\, b^{k+1} + \binom{n}{n} b^{n+1}$$

Then:

$$\begin{aligned}(a+b)^{n+1} &= \binom{n}{0} a^{n+1} + \sum_{k=0}^{n-1} \binom{n}{k+1} a^{n-k}\, b^{k+1} + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k}\, b^{k+1} + \binom{n}{n} b^{n+1} \\ &= \binom{n}{0} a^{n+1} + \sum_{k=0}^{n-1} \left[\binom{n}{k+1} + \binom{n}{k}\right] a^{n-k}\, b^{k+1} + \binom{n}{n} b^{n+1}\end{aligned}$$

Remember that:

$$\begin{aligned}\binom{n}{k+1}+\binom{n}{k} &= \binom{n+1}{k+1}\\ \binom{n}{0} &= \binom{n+1}{0}=1\\ \binom{n}{n} &= \binom{n+1}{n+1}=1\end{aligned}$$

Thus:

$$\begin{aligned}(a+b)^{n+1} &= \binom{n}{0}a^{n+1}+\sum_{k=0}^{n-1}\binom{n}{k+1}a^{n-k}b^{k+1}+\sum_{k=0}^{n-1}\binom{n}{k}a^{n-k}b^{k+1}+\binom{n}{n}b^{n+1}\\ &= \binom{n}{0}a^{n+1}+\sum_{k=0}^{n-1}\left[\binom{n}{k+1}+\binom{n}{k}\right]a^{n-k}b^{k+1}+\binom{n}{n}b^{n+1}\\ &= \binom{n}{0}a^{n+1}+\sum_{k=0}^{n-1}\binom{n+1}{k+1}a^{n-k}b^{k+1}+\binom{n}{n}b^{n+1}\\ &\overset{k+1\to q=k}{=} \binom{n}{0}a^{n+1}+\sum_{k=1}^{n}\binom{n+1}{k}a^{n-k+1}b^{k}+\binom{n}{n}b^{n+1}\\ &= \binom{n+1}{0}a^{n+1}+\sum_{k=1}^{n}\binom{n+1}{k}a^{n-k+1}b^{k}+\binom{n+1}{n+1}b^{n+1}\\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n-k+1}b^{k}\Rightarrow \text{ok!}\end{aligned}$$

11 Sequences

Definition 21. A sequence is any application:

$$s:E\subset\mathbb{N}\to\mathbb{R},$$

from a subset of $\mathbb{N}$ to $\mathbb{R}$. We indicate usually the image of a natural number sub s as:

$$s(n)=s_n.$$

Definition 22. A sequence p_n is said to converge if there exists a real number $p\in\mathbb{R}$ such that:

$$\forall\varepsilon>0\exists\bar{n}_\varepsilon\in\mathbb{N}:\forall n\geq\bar{n}_\varepsilon\Rightarrow|p_n-p|<\varepsilon.$$

In this case we write:

$$p_n\to p \text{ or } \lim_{n\to\infty}p_n=p.$$

We add also the two following definitions:

$$\forall M>0\exists\bar{n}_\varepsilon\in\mathbb{N}:\forall n\geq\bar{n}_\varepsilon\Rightarrow p_n>M,$$

and

$$\forall M>0\exists\bar{n}_\varepsilon\in\mathbb{N}:\forall n\geq\bar{n}_\varepsilon\Rightarrow p_n<-M,$$

and we write, respectively, that:

$$p_n \to \pm\infty \text{ or } \lim_{n\to\infty} p_n = \pm\infty.$$

and we say that the sequence is divergent. If the sequence does not diverge or converge we say that it has no limit.

Theorem 11.1. Suppose that (p_n) is a sequence. If it exists the limit $\ell = \lim_{n\to\infty} p_n$ then this limit is unique.

Proof. Suppose that there exist two distinct limits ℓ and ℓ'. Hence it must happen that

$$\begin{aligned} \forall\varepsilon > 0 \exists \bar{n}_\varepsilon \in \mathbb{N} &\;:\; \forall n \geq \bar{n}_\varepsilon \Rightarrow |p_n - \ell| < \varepsilon. \\ \forall\varepsilon > 0 \exists \bar{m}_\varepsilon \in \mathbb{N} &\;:\; \forall m \geq \bar{m}_\varepsilon \Rightarrow \left|p_m - \ell'\right| < \varepsilon. \end{aligned}$$

Hence, for an arbitrary small $\varepsilon > 0$ take any integer $k \geq \max(\bar{n}_\varepsilon, \bar{m}_\varepsilon)$ and consider

$$\left|\ell - \ell'\right| \leq |\ell - p_k| + \left|p_k - \ell'\right| < 2\,\varepsilon.$$

Since ε is arbitrary small we get $\ell = \ell'$. □

The following theorem clarify why a limit point is called a limit point:

Theorem 11.2. If p is a limit point (accumulation point) of $E \subset \mathbb{R}$ then there exists a sequence $p_n \in E$ such that $p_n \to p$.

For all $n \in \mathbb{N}$ we known that there exists a point p_n in E such that:

$$|p - p_n| < \frac{1}{n}.$$

Now for all $\varepsilon > 0$ take $\bar{n} > \frac{1}{\varepsilon}$, therefore for all $n \geq \bar{n} > \frac{1}{\varepsilon}$ we have that:

$$|p - p_n| < \frac{1}{n} \leq \frac{1}{\bar{n}} < \varepsilon,$$

which is our statement. □

Esercize 4. Show that $\forall a, b \in \mathbb{R}$:

$$|a - b| \geq ||a| - |b||\,, \quad \forall a, b \in \mathbb{R}..$$

(Reverse Triangular Inequality).

Let's write:

$$a = a - b + b = (a - b) + b.$$

Therefore:

$$|a| = |(a-b)+b| \leq |a-b| + |b|, \Rightarrow |a| - |b| \leq |a-b| \tag{11.1}$$

where we have used the triangular inequality. Now exchange a with b:

$$|b| = |(b-a)+a| \leq |b-a| + |a| \Rightarrow |b| - |a| \leq |b-a|. \tag{11.2}$$

Equations (11.1)-(11.2) imply that:

$$||a| - |b|| \leq |a-b|. \tag{11.3}$$

Definition 23. Suppose that $\mathcal{P}(n)$ is a proposition on $\mathbb{N}$. Instead of saying

$$\exists \bar{n} : \forall n \geq \bar{n} \Rightarrow \mathcal{P}(n),$$

we will say that $\mathcal{P}(n)$ holds for n "sufficiently large".

Esercize 5. Prove that if $s_n \to p$ then $|s_n| \to q = |p|$. Is the converse true?

We know that:

$$\forall \quad \varepsilon > 0 \quad \exists \quad \bar{n} > 0 \quad : \forall \quad n > \bar{n} \to |p_n - p| < \varepsilon \tag{11.4}$$

The quantity $||p_n| - |p||$ can be maximized using the reverse triangular inequality by:

$$||p_n| - |p|| \leq |p_n - p| \tag{11.5}$$

Then we have that:

$$\forall \quad \varepsilon > 0 \quad \exists \quad \bar{n} > 0 \quad \text{s.c.} \quad \forall \quad n > \bar{n} \to ||p_n| - |p|| \leq |p_n - p| < \varepsilon, \tag{11.6}$$

which is the thesys. The converse is not true. Take $s_n = (-1)^n \Rightarrow |s_n| = 1 \to 1$, nevertheless s_n doesn't converge.

Theorem 11.3. Let p_n be a sequence such that is it exists $p = \lim_{n\to\infty} p_n$. Therefore p_n is bounded, in the sense that $p_n \in N_\rho(p)$ for all n.

Proof. Consider the definition of limit of a sequence and take $\varepsilon = 1$. Hence there exists N such that for all $n > N$ we have $|p_n - p| < 1$ that is $p_n \in N_1(p)$. Now consider any number ρ such that

$$\rho \geq \max(|p_1 - p|, |p_2 - p|, |p_3 - p|, \ldots, |p_{N-1} - p|, 1)$$

hence $p_n \in N_\rho(p)$ for all n. □

Theorem 11.4. Suppose s_n and t_n are two sequences in $\mathbb{R}$ and assume further than $s_n \to s$ and $t_n \to t$, thus:

1. $(s_n + t_n) \to s + t$.
2. $(s_n\, t_n) \to s\, t$.
3. $(c + s_n) \to c + s$ and $(c\, s_n) \to c\, s$ for any number c.
4. if $s_n \to s$ and $s \neq 0$ then for n sufficiently large we have $s_n \neq 0$ and moreover we have $\frac{1}{s_n} \to \frac{1}{s}$.

Proof.

1. Take an arbitrary small $\varepsilon > 0$. Since both p_n and t_n converge then there will be two integers $n_{1,\varepsilon}$ and $n_{2,\varepsilon}$ such that for all $n \geq n_{1,\varepsilon}$ we have $|s_n - s| < \frac{\varepsilon}{2}$ and for all $n \geq n_{2,\varepsilon}$ we have $|t_n - t| < \frac{\varepsilon}{2}$. Hence for all $n \geq \max(n_{1,\varepsilon}, n_{2,\varepsilon})$ we get
$$|s_n + t_n - (s + t)| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
2. We want to evaluate the difference
$$|s_n\, t_n - s\, t| = |s_n\, t_n - s_n\, t + s_n\, t - s\, t| = |s_n\,(t_n - t) + t\,(s_n - s)| \leq |s_n|\;|t_n - t| + |t|\;|s_n - s|$$
Since $s_n \to s$ we know that s_n is limited that is there exists $A \geq 0$ such that $|s_n| \leq A$ for all n. Let's assume that $A > 0$ since the case $A = 0$ (which implies $s_n = 0$ for all n) is trivial. Besides we know that, for all ε there exist $n_{1,\varepsilon}$ and $n_{2,\varepsilon}$ such that for all $n \geq n_{1,\varepsilon}$ we have $|s_n - s| < \frac{\varepsilon}{\lambda}$ and for all $n \geq n_{2,\varepsilon}$ we have $|t_n - t| < \frac{\varepsilon}{\lambda}$, where $\lambda > 0$ is to be chosen, whence
$$|s_n\, t_n - s\, t| \leq |A|\;|t_n - t| + |t|\;|s_n - s| < A\,\frac{\varepsilon}{\lambda} + |t|\;\frac{\varepsilon}{\lambda},$$
If now we choose λ such that
$$A\,\frac{\varepsilon}{\lambda} + |t|\;\frac{\varepsilon}{\lambda} = \varepsilon \Leftrightarrow \lambda = A + |t|\,.$$
we get that for all $\varepsilon > 0$ there exists $\bar{n}_\varepsilon = \max(n_{1,\varepsilon}, n_{2,\varepsilon})$ such that for all $n \geq \bar{n}_\varepsilon$ we get
$$|s_n\, t_n - s\, t| < \varepsilon.$$
3. Immediate form 2) and 3) since the constant sequence $c_n = c$ converges to c.
4. We know that for all $\varepsilon > 0$ there exists a $\bar{n}_\varepsilon$ such that $\forall n \geq \bar{n}_\varepsilon$ we have $|s_n - s| < \varepsilon$. Since $s \neq 0$ it cannot happen that definitively $s_n = 0$ otherwise we will get $|s| < \varepsilon$ which is impossible since $s \neq 0$. Hence there must exists a constant $c > 0$ such that $|s_n| \geq c$ for n sufficiently large. Whence, for n sufficiently large, we have
$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{s\, s_n} \leq \frac{|s - s_n|}{s\, c} < \frac{\varepsilon}{s\, c}$$
hence $\frac{1}{s_n} \to \frac{1}{s}$. ✠

Esercize 6. Let p_n be a sequence in $\mathbb{R}$ such that $p_n \to p$ and $p_n \geq 0$ for all n. Prove that

1. $p \geq 0$.
2. $p_n^{\frac{1}{2}} \to p^{\frac{1}{2}}$

1) Suppose by contradiction that $p < 0$. Then there exists $\varepsilon > 0$ such that $p + \varepsilon < 0$ and for n sufficiently large we have $|p_n - p| < \varepsilon$ which is equivalent to $-\varepsilon < p_n - p < \varepsilon$ which is equivalent to

$$p - \varepsilon < p_n < p + \varepsilon < 0$$

which is impossible since $p_n \geq 0$ for all n.

2) Let's distinguish two cases. If $p = 0$ then for n sufficiently large

$$p_n < \varepsilon^2$$

hence $p_n^{\frac{1}{2}} < \varepsilon$, whence $p_n^{\frac{1}{2}} \to 0$.

If $p > 0$ then

$$|\sqrt{p_n} - \sqrt{p}| = |\sqrt{p_n} - \sqrt{p}|\, \frac{|\sqrt{p_n} + \sqrt{p}|}{|\sqrt{p_n} + \sqrt{p}|} = \frac{|p_n - p|}{\sqrt{p_n} + \sqrt{p}} < \frac{|p_n - p|}{\sqrt{p}} < \frac{\varepsilon\sqrt{p}}{\sqrt{p}} = \varepsilon$$

Esercize 7. Compute $\lim_{n\to\infty} \sqrt{n^2 + n} - n$.

We first note that: $\sqrt{n^2 + n} \to \infty$ and of course $n \to \infty$. The idea is that for large n the quantity $n^2 + n$ is dominated by n^2 and therefore the limit under study should be finite. Let's try to rationalize the sequence:

$$\begin{aligned} \sqrt{n^2 + n} - n &= \frac{\left(\sqrt{n^2 + n} - n\right)\left(\sqrt{n^2 + n} + n\right)}{\sqrt{n^2 + n} + n} \\ &= \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \frac{n}{\sqrt{n^2 + n} + n} = \frac{n}{n\left(\sqrt{1 + \frac{1}{n}} + 1\right)} \\ &= \frac{1}{\left(\sqrt{1 + \frac{1}{n}} + 1\right)} \to \frac{1}{2}. \end{aligned} \tag{11.7}$$

Theorem 11.5. The following two assertions hold:

1. Let x_n and y_n be two convergent sequences in $\mathbb{R}$. Assume that $x_n \leq y_n$ for n sufficiently large. Hence $\lim_n x_n \leq \lim_n y_n$

2. Let x_n, y_n and z_n be three sequences in $\mathbb{R}$. Suppose that $x_n \le y_n \le z_n$ for n sufficiently large and that $\lim_n x_n = \lim_n z_n = a$. Hence there exists $\lim_n y_n$ and it is equal to a.

Proof. The 1) is trivial. Let's prove assertion 2). We know that for all $\varepsilon > 0$ there exists n_1 and n_2 such that for all $n \ge n_1$ we have $|x_n - a| < \varepsilon$ and for all $n \ge n_2$ we have $|z_n - a| < \varepsilon$. Hence if $n \ge \max(n_1, n_2)$ we have

$$a - \varepsilon < x_n \le y_n \le z_n < a + \varepsilon,$$

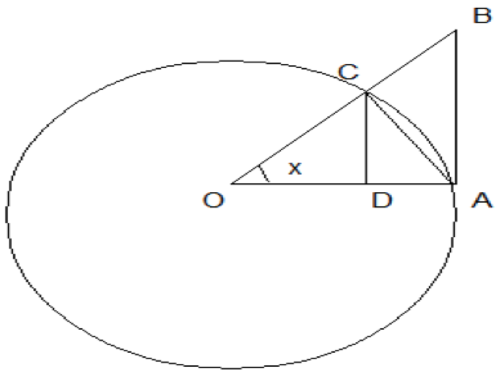
whence the thesis. ✠

A notable limit.

We want to compute:

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 0 \cdot \infty.$$

Consider a unit circle and an angle $x \in \left(0, \frac{\pi}{2}\right)$.



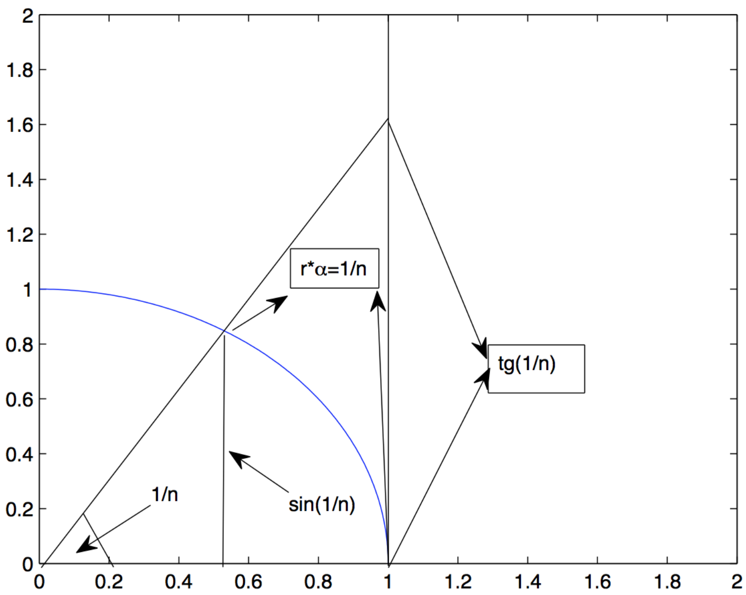
The length of the arc $\widetilde{AC}$ is $\widetilde{AC} = 1 \times x = x$, the length of the segment $\bar{CD}$ is $\sin(x)$ while the length $\bar{OD} = \cos(x)$. Now is clear that the area of the triangle $\widehat{OAC} = \frac{1 \times \sin(x)}{2}$ is less than the area of the circular sector $\widehat{OAC}$ which is $\frac{1}{2} 1^2 x = \frac{x}{2}$ which is less than the area of the triangle OAB which is $\frac{1 \times \tan(x)}{2} = \frac{1}{2} \tan(x)$ or in formula:

$$0 \le \frac{1}{2} \sin(x) \le \frac{1}{2} x \le \frac{1}{2} \tan(x) \Rightarrow 0 \le \sin(x) \le x \le \tan(x).$$

or graphically:

For $x = \frac{1}{n}$ we have:

$$0 \le \sin\left(\frac{1}{n}\right) \le \frac{1}{n} \le \tan\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\cos\left(\frac{1}{n}\right)},$$



i.e.,

$$\frac{1}{\sin\left(\frac{1}{n}\right)} \geq n \geq \frac{\cos\left(\frac{1}{n}\right)}{\sin\left(\frac{1}{n}\right)} \Rightarrow$$

$$1 \geq \sin\left(\frac{1}{n}\right) n \geq \cos\left(\frac{1}{n}\right) \geq 0 \Rightarrow$$

$$0 \leq \cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1.$$

Nevertheless:

$$\cos\left(\frac{1}{n}\right) \to 1.$$

Thus:

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1.$$

11.1 Monotonicity

Definition 24. A sequence s_n of real numbers is said to be:

- monotonically increasing if $s_n \leq s_{n+1}$.
- monotonically decreasing if $s_n \geq s_{n+1}$.

Theorem 11.6. Every increasing (respectively decreasing) sequence s_n converges to

$$\lim_{n\to\infty} s_n = \sup\{s_n | n \in \mathbb{N}\} \;\; (\text{respectively } \lim_{n\to\infty} s_n = \inf\{s_n | n \in \mathbb{N}\}\;).$$

Proof. Suppose s_n is increasing and let $\ell = \sup\{s_n | n \in \mathbb{N}\}$. If $\ell = \infty$ then s_n is not bounded and hence $\forall M > 0$ I can find $\bar{n}_M$ such that for $n \geq \bar{n}_M$ we have $s_n > M$ (otherwise it would be bounded from above), whence $s_n \to \infty$. Suppose now that $\ell < \infty$. By the definition of supremum we know that $s_n \leq \ell$ for all n and that for all $\epsilon > 0$ the exist $\bar{n}_\varepsilon$ such that $\ell - \varepsilon < s_{\bar{n}_\varepsilon}$. Since s_n is increasing then for all $n \geq \bar{n}_\varepsilon$ we have

$$\ell - \varepsilon < s_{\bar{n}_\varepsilon} \leq s_n \leq \ell < \ell + \varepsilon,$$

whence $|s_n - \ell| < \varepsilon$, i.e. $s_n \to \ell$ and a similar reasoning applies for the decreasing sequences. ✠

Corollario 11.7. Suppose that s_n is a monotonically increasing (resp. decreasing) sequence. Hence s_n converges to a finite limit if and only if it is bounded from above (resp. below).

Proof. Let's prove before implication $\Rightarrow$. If s_n is increasing and it converges to a finite limit therefore for the Theorem 11.6 we must have $\sup_n s_n < \infty$ otherwise we would have $s_n \to \infty$ which is impossible. Hence s_n is bounded from above.

Let's prove now implication $\Leftarrow$. Since s_n is bounded from above we know that exists and it is finite the supremum $\ell = \sup_n s_n$. By the theorem 11.6 we know that $s_n \to \ell$.✠

11.2 Subsequences

Definition 25. Given a sequence s_n and a second sequence of strictly increasing positive integers $n_1 < n_2 < ... < n_k < ...$ we call the sequence s_{n_k} a sub-sequence of s_n. If s_{n_k} converges its limit is called a subsequential limit of s_n. In particular note that it must happens that $n_k \geq k$.

Theorem 11.8. A sequence s_n converges to s in and only if every subsequence of s_n converges to s.

Proof.

The implication $\Leftarrow$ is trivial. If every subsequence converges to s hence since s_n is a subsequence of itself it converges to s.

Let's now consider the other implication $\Rightarrow$. Suppose that $s_n \to s$. Let U be a neighborhood of s. Then there exists N such that for all $n \geq N$ it happens that $s_n \in U$. Since $n_k \geq k$ therefore if $k \geq N$ we get $n_k \geq k \geq N$ and hence $s_{n_k} \in U$, whence $s_{n_k} \to s$. ✠

Exercizes.

- Show that $s_n = (-1)^n$ has no limit.

  Solution. Since $s_{2n} = 1 \to 1$ and $s_{2n+1} = -1 \to -1$ the sequence has no limit.

- Prove that $s_n = \log_a(n)$ with $a > 1$ is such that $s_n \to \infty$.

  Solution. Consider an $M > 0$. We want to find a n_M such that for all $n \geq n_M$ it holds

  $$\log_a(n) \geq M.$$

  Since $\log_a$ with $a > 1$ is an increasing function it is enough to consider take any n_M such that

  $$n_M \geq a^M.$$

  So now consider a generic n with $n \geq n_M \geq a^M$, then by the monotonicity of the logarithm we get

  $$\log_a(n) \geq \log_a(n_M) \geq \log_a\left(a^M\right) = M.$$

- Find the limit of the sequence $s_n = 3 - \frac{1}{\log_2(n)}$.

  Solution. The guess is that the limit is 3. Let $\varepsilon > 0$ be arbitrarily small. The condition

  $$|s_n - 3| < \varepsilon$$

  is equivalent to

  $$\left|\frac{1}{\log_2(n)}\right| < \varepsilon.$$

  Since $n \geq 1$ we get

  $$\frac{1}{\log_2(n)} < \varepsilon.$$

  So it is enough to take any n_ε such that $n_\varepsilon > 2^{1/\varepsilon}$.

11.3 Some Notable Limits

- Let $a \in \mathbb{R}$ with $a \neq 1$. Then

  $$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if} \quad |a| < 1 \\ \infty & \text{if} \quad a > 1 \\ \nexists & \text{if} \quad a \leq -1 \end{cases}$$

  Case $|a| < 1$. Hence $|a| = \frac{1}{1+h}$ for some $h > 0$. By the binomial inequality

  $$(1+h)^n \geq 1 + n\,h \geq n\,h$$

  whence

  $$0 \leq \frac{1}{(1+h)^n} \leq \frac{1}{n\,h}$$

  the results follows from the fact that $\frac{1}{n} \to 0$.

  Case $a > 1$. Hence $a = 1 + h$ for some $h > 0$. Therefore

$$a^n = (1+h)^n \geq 1 + n\,h.$$

Since $n \to \infty$ and $h > 0$ we have the result.

Suppose that $a = -1$. Let $s_n = (-1)^n$. Therefore I can define two subsequences $s_{2n} = 1 \to 1$ and $s_{2n+1} = -1 \to -1$, hence s_n does not converge.

Finally, suppose $a < -1$, whence $a = -|a|$ with $|a| > 1$. Hence $a^n = (-1)^n\,|a|^n$ with $|a|^n \to \infty$. Let $s_n = a^n$. As a consequence $s_{2k} \to \infty$ and $s_{2k+1} \to -\infty$ hence s_n does not converge.

- Let $a \in \mathbb{R}$ be such that $a > 1$ and let k be an integer number $k \geq 1$. We want to compute

$$\lim_{n\to\infty} \frac{n^k}{a^n}.$$

Write $a = 1 + h$. For n sufficiently large we will have $n > k+1$ hence

$$a^n = (1+h)^n = \sum_{m=0}^{n} \binom{n}{m} h^m \geq \binom{n}{k+1} h^{k+1} = \frac{n\,(n-1)\cdots(n-k)}{(k+1)!}\,h^{k+1},$$

whence

$$0 \leq \frac{n^k}{a^n} \leq \frac{(k+1)\,n^k}{h^{k+1}\,\underbrace{n\,(n-1)\cdots(n-k)}_{k+1\text{ terms}}} = \frac{(k+1)}{h^{k+1}\,n\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k}{n}\right)} \to 0.$$

- Find a formula for the sum the first n-powers of any real number x

$$S_n = 1 + x + x^2 + \ldots + x^n$$

We multiply both side of the equation by $1 - x$ obtaining

$$\begin{aligned}
(1-x)\,S_n &= (1+x)\left(1 + x + x^2 + \ldots + x^n\right)\\
&= 1 + x + x^2 + \ldots + x^n - x\left(1 + x + x^2 + \ldots + x^n\right)\\
&= 1 + x + x^2 + \ldots + x^n - x - x^2 - \ldots - x^{n+1}\\
&= 1 - x^{n+1},
\end{aligned}$$

whence

$$S_n = \frac{1 - x^{n+1}}{1 - x}.$$

As a consequence, by looking backward in the section dedicated to the notable limits, we can claim that

$$\lim_{n\to\infty} S_n = \begin{cases} \frac{1}{1-x} & \text{if} \quad |x| < 1,\\ +\infty & \text{if} \quad x \geq 1,\\ \nexists & \text{if} \quad x \leq -1. \end{cases}$$

- We want to compute
$$\lim_n n^{1/n}.$$
It is immediate to see that for all $n \geq 1$ then $n^{1/n} \geq 1$ since the inequality $n^{1/n} < 1$ would lead to
$$\underbrace{n^{1/n} \cdots n^{1/n}}_{n \text{ times}} < 1 \cdots 1 = 1,$$
nevertheless $\underbrace{n^{1/n} \cdots n^{1/n}}_{n \text{ times}} = n$ which is impossible. Hence $n^{1/n} = 1 + a_n$ with $a_n \geq 0$ that is $(1 + a_n)^n = n$. Now we have
$$n = (1 + a_n)^n = 1 + \binom{n}{1} a_n + \binom{n}{2} a_n^2 + \ldots \geq 1 + \binom{n}{2} a_n^2 = 1 + \frac{n\,(n-1)}{2}\, a_n^2.$$
The inequality $1 + \frac{n\,(n-1)}{2}\, a_n^2 \leq n$ re-arranged gives
$$0 \leq a_n^2 \leq \frac{2}{n} \to 0.$$
Hence $a_n \to 0$ and thus $n^{1/n} \to 1$.

- We want to compute
$$\lim_{n\to\infty} \frac{n^n}{n!}$$
Consider that
$$\frac{n^n}{n!} = \frac{n \cdot n \cdots n}{n \cdot (n-1) \cdots 1} = \underbrace{\frac{n}{n}}_{=1} \cdot \underbrace{\frac{n}{n-1}}_{>1} \cdots \underbrace{\frac{n}{2}}_{>1} \cdot n > n$$
hence
$$\lim_{n\to\infty} \frac{n^n}{n!} = \infty$$
or
$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

- For $a > 1$ compute the limit of the sequence $s_n = \frac{\log_a(n)}{n}$.

  Consider that $s_n = \log_a\left(n^{\frac{1}{n}}\right)$ but $n^{\frac{1}{n}} \to 1$ hence $s_n \to \log_a(1) = 0$.

  More generally for $a > 1$ and $b > 0$ we have
$$\lim_{n\to\infty} \frac{\log_a n}{n^b} = 0.$$

- Let $a > 1$. We want to compute the limit
$$\lim_{n\to\infty} \frac{a^n}{n!}.$$
Let $x_n = \frac{a^n}{n!}$. Consider that

$$\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{(n+1)!}\frac{n!}{a^n} = \frac{a\,a^n}{a^n}\frac{n!}{(n+1)n!} = \frac{a}{n+1} \to 0$$

which means that

$$\forall \epsilon > 0, \exists n_\varepsilon : \forall n \geq n_\varepsilon \Rightarrow \left|\frac{x_{n+1}}{x_n}\right| < \varepsilon.$$

In particular, there exists $\varepsilon \in (0,1)$ and an N such that for all $n > N$ we have

$$0 < \frac{x_{n+1}}{x_n} < \varepsilon \Rightarrow 0 < x_{n+1} < \varepsilon\, x_n, \tag{11.8}$$

where we also have used the fact that $x_n \geq 0$ for all n. By iteration of the (11.8) we get

$$0 < x_{n+1} < \varepsilon\, x_n < \varepsilon^2\, x_{n-1} < \varepsilon^3\, x_{n-2} < \ldots < \varepsilon^{n-N}\, x_{N+1}.$$

Consider now that N is a fixed given number, hence we can let $n \to \infty$ obtaining $\varepsilon^{n-N} \to 0$ then

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

so the factorial grows faster than the exponential.

Orders of Infinity.

Suppose that $a_n \to \infty$ and $b_n \to \infty$. We say that $a_n \ll b_n$ if $\frac{b_n}{a_n} \to \infty$, that is both a_n and b_n diverge, however b_n diverges faster than a_n. Summing up the results of the limits above we can say that, for all $a > 1$ and $b > 0$, that the following orders of infinity hold:

$$\log_a(n) \ll n^b \ll a^n \ll n! \ll n^n.$$

Definition 26. We say that $a_n \sim b_n$ for $n \to \infty$ if:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

In this case we also say that a_n and b_n are asymptotically equivalent. For example if $a_n \to L$ then $a_n \sim L$.

Observation 1. The following properties hold:

- If $a_n \sim b_n$ and $b_n \to l \in \mathbb{R} \cup \{\pm\infty\}$ then $a_n \to l$.

- If $a_n \sim b_n$ and $b_n \sim c_n$ then $a_n \sim c_n$.
- If $a_n \sim b_n$ then for every sequence c_n such that $c_n \neq 0 \forall n$ we have:

$$a_n\, c_n \sim b_n\, c_n, \quad \frac{a_n}{c_n} \sim \frac{b_n}{c_n}, \quad \frac{c_n}{a_n} \sim \frac{c_n}{b_n}. \tag{11.9}$$

Proof. We prove just the case $b_n \to \ell$ with ℓ a finite number and $\ell > 0$. Since $a_n \sim b_n$ we have for sufficiently large n

$$1 - \varepsilon < \frac{a_n}{b_n} < 1 + \varepsilon,$$

or

$$b_n\,(1-\varepsilon) < a_n < b_n\,(1+\varepsilon)$$

Now if $\varepsilon \to 0$ then $n \to \infty$ and so $b_n \to \ell$ which implies, by the comparison theorem, that $a_n \to \ell$. □

Consider for example $n\sqrt{1 + \frac{1}{n} + \frac{1}{n^2}} \sim n$.

Esercize 8. Compute:

$$\lim_{n\to\infty} \frac{n\sqrt{n} - n^2}{n+1}.$$

Note that:

$$n\sqrt{n} - n^2 = n^{\frac{3}{2}} - n^2 \sim -n^2 \text{ because } \frac{3}{2} < 2,$$

moreover:

$$n + 1 \sim n.$$

Threrfore we have:

$$\frac{n\sqrt{n} - n^2}{n+1} \sim \frac{-n^2}{n} = -n \to -\infty. \tag{11.10}$$

Esercize 9. Show that:

$$\lim_{n\to\infty} p^{\frac{1}{n}} = 1, \forall \quad p > 0 \tag{11.11}$$

Assume $p \geq 1$. Consider $x_n = p^{\frac{1}{n}} - 1$, thus $x_n \geq 0$ and:

$$(1 + x_n)^n = p.$$

Using binomial inequality:

$$\begin{aligned} (1+x_n)^n &\geq 1+n\,x_n \Rightarrow \\ p &\geq 1+n\,x_n \Rightarrow \\ 0 \leq x_n &\leq \frac{p-1}{n} \Rightarrow \\ x_n &\to 0 \Rightarrow \\ p^{\frac{1}{n}} &\to 1. \end{aligned}$$

If $0<p<1$ then consider $q=\frac{1}{p}>1$. Therefore $q^{\frac{1}{n}}\to 1$ and:

$$p^{\frac{1}{n}} = \frac{1}{q^{\frac{1}{n}}} \to \frac{1}{1} = 1.$$

Esercize 10. Compute:

$$\lim_{n\to\infty} \frac{4\,n+\frac{2}{n}}{\frac{1}{n^2}+5\,n}.$$

$$4\,n+\frac{2}{n} \sim 4\,n.$$

$$\frac{1}{n^2}+5\,n \sim 5\,n.$$

As a consequence:

$$\lim_{n\to\infty} \frac{4\,n+\frac{2}{n}}{\frac{1}{n^2}+5\,n} = \lim_{n\to\infty}\frac{4\,n}{5\,n} = \frac{4}{5}.$$

Esercize 11. Compute:

$$\lim_{n\to\infty} n-\sqrt{n+n^2}.$$

When dealing with square roots it's wise to rationalize:

$$n-\sqrt{n+n^2} = \frac{\left(n-\sqrt{n+n^2}\right)\left(n+\sqrt{n+n^2}\right)}{n+\sqrt{n+n^2}} = \frac{n^2-\left(n+n^2\right)}{n+\sqrt{n+n^2}} = -\frac{n}{n\left(1+\sqrt{\frac{1}{n}+1}\right)} \to -\frac{1}{2} \tag{11.12}$$

Esercize 12. Compute:

$$\lim_{n\to\infty} \left(3^n+4^n\right)^{\frac{1}{n}}.$$

$$\left(3^n+4^n\right)^{\frac{1}{n}} = \left\{4^n\left[1+\left(\frac{3}{4}\right)^n\right]\right\}^{\frac{1}{n}} = 4\left[1+\left(\frac{3}{4}\right)^n\right]^{\frac{1}{n}}.$$

Neverthelss for $0<p<1$, $p^n\to 0$, thus:

$$\left(3^n+4^n\right)^{\frac{1}{n}} = 4\left[1+\left(\frac{3}{4}\right)^n\right]^{\frac{1}{n}} \to 4. \tag{11.13}$$

The Euler Sequence

Let
$$a_n = \left(1 + \frac{1}{n}\right)^n$$
be the a sequence of real numbers (also known as the *Euler* sequence). First notice that, by the binomial inequality, we have
$$a_n \geq 1 + n\,\frac{1}{n} = 2.$$
Using the binomial theorem we get
$$\begin{aligned}
a_n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \\
&= 1 + \sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \\
&= 1 + \sum_{k=1}^{n} \frac{n\,(n-1)\,\cdots\,(n-k+1)}{n^k}\,\frac{1}{k!} \\
&= 1 + 1 + \sum_{k=2}^{n} \frac{n\,(n-1)\,\cdots\,(n-k+1)}{n^k}\,\frac{1}{k!} \\
&= 2 + \sum_{k=2}^{n} \left[1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)\right] \frac{1}{k!} \\
&< 2 + \sum_{k=2}^{n} \left[1 \cdot \left(1 - \frac{1}{n+1}\right) \cdot \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{k-1}{n+1}\right)\right] \frac{1}{k!} \\
&< 2 + \sum_{k=2}^{n+1} \left[1 \cdot \left(1 - \frac{1}{n+1}\right) \cdot \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{k-1}{n+1}\right)\right] \frac{1}{k!} = a_{n+1}.
\end{aligned} \tag{11.14}$$
Now we show that the sequence is bounded from above. Note that
$$k! = 1\,2\,3 \cdots k \geq 1\,2\,2 \cdots 2 = 2^{k-1},$$
whence
$$\begin{aligned}
2 < a_n &= 1 + \sum_{k=1}^{n} \left[1 \cdot \underbrace{\left(1 - \frac{1}{n}\right)}_{<1} \cdot \underbrace{\left(1 - \frac{2}{n}\right)}_{<1} \cdots \underbrace{\left(1 - \frac{k-1}{n}\right)}_{<1}\right] \frac{1}{k!} \\
&< 1 + \sum_{k=1}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + 2\sum_{k=1}^{n} \frac{1}{2^k} < 1 + 2\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = 1 + 2\left(\frac{1}{1-\frac{1}{2}} - 1\right) = 3.
\end{aligned}$$
Hence $\exists e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ and $2 < e \leq 3$.

Exercise. Compute the limit of the sequence $s_n = \left(1 + \frac{5}{n^2}\right)^{n^2}$.

Solution. Call $m = \frac{n}{\sqrt{5}}$, hence

$$\lim_{n\to\infty}\left(1+\frac{5}{n^2}\right)^{n^2} = \lim_{m\to\infty}\left(1+\frac{1}{m^2}\right)^{5\,m^2} = \lim_{m\to\infty}\left(\left(1+\frac{1}{m^2}\right)^{m^2}\right)^5$$

Call $k = m^2$, hence

$$\lim_{m\to\infty}\left(\left(1+\frac{1}{m^2}\right)^{m^2}\right)^5 = \lim_{k\to\infty}\left(\left(1+\frac{1}{k}\right)^{k}\right)^5 = e^5.$$

So that

$$s_n \to e^5.$$

Economic Application

Suppose that, at time n, the price of a good is p_n. Suppose further that the demand, for a given price, is given by $D(p_n) = -b\,p_n + a$ while the supply is given by $S(p_n) = s\,p_n - m$. Assume that the price evolves according to

$$p_{n+1} = p_n + (D(p_n) - S(p_n)),$$

We want to study the limit $\lim_{n\to\infty} p_n$.

$$\begin{aligned}
p_{n+1} &= p_n + (D(p_n) - S(p_n)) = p_n + a - b\,p_n + m - s\,p_n \Rightarrow\\
p_{n+1} &= a + m + (1 - b - s)\,p_n \Rightarrow\\
p_{n+1} &= \alpha + \beta\,p_n \Rightarrow\\
p_1 &= \alpha + \beta\,p_0 \Rightarrow\\
p_2 &= \alpha + \beta\,\alpha + \beta^2\,p_0 \Rightarrow\\
p_3 &= \alpha + \beta\,\alpha + \beta^2\,\alpha + \beta^3\,p_0 \Rightarrow\\
&\vdots \qquad \vdots\\
p_n &= \alpha\left(1 + \beta + \beta^2 + \cdots + \beta^{n-1}\right) + \beta^n\,p_0 \Rightarrow\\
p_n &= \alpha\left(\frac{1-\beta^n}{1-\beta}\right) + \beta^n\,p_0.
\end{aligned}$$

Simplifying:

$$
\begin{aligned}
p_n &= \frac{\alpha\,(1-\beta^n)+(1-\beta)\,\beta^n\,p_0}{1-\beta} \Rightarrow \\
p_n &= \frac{\alpha\,(1-\beta^n)+\beta^n\,p_0-\beta^{n+1}\,p_0}{1-\beta} \Rightarrow \\
p_n &= \frac{\beta^n\,(p_0\,(1-\beta)-\alpha)+\alpha}{1-\beta} \Rightarrow \\
p_n &= \frac{(1-b-s)^n\,(p_0\,(b+s)-(a+m))+a+m}{b+s} \Rightarrow \\
p_n &= \frac{(1-(b+s))^n\,(b+s)\,(p_0-p_e)+a+m}{b+s} \Rightarrow \\
p_n &= (1-(b+s))^n\,(p_0-p_e)+p_e.
\end{aligned}
$$

Now define $b+s=r$, the quantity r represents the sum of the rate of decrasing demand and the rate of increasing supply:

$$
\lim_{n\to+\infty} p_n = \begin{cases} p_e & \text{if } 0<r\le 1 \text{ (convergence with monotonic behaviour)} \\ p_e & \text{if } 1<r<2 \text{ (convergence with non-monotonic behaviour)} \end{cases}
$$

For $r\ge 2$ there is no limit.

Esercize 13. The return of an investment in a time horizon $[0,1]$ is R, i.e. if we invest a quantity of money M_0 at time $t_0=0$ we arrive at $M_0\,(1+R)$ at time $t_f=1$. Suppose that the quantity of money is continuously invested, i.e. we invest money at time t_0 wait for maturity at time $t_0+\Delta t$ and re-invest the original amount plus maturity at time $t_0+\Delta t$, then wait for maturity at time $t_0+2\,\Delta t$ and so on. Compute which of the two strategies (in absence of transaction costs) is the one with the largest return in the limit $\Delta t\to 0$.

Solution. If we divide the interval $[0,1]$ in two part the return is:

$$
\left(1+\frac{R}{2}\right)\left(1+\frac{R}{2}\right)=\left(1+\frac{R}{2}\right)^2.
$$

More generally if we continuously invest:

$$
\lim_{n\to\infty}\left(1+\frac{R}{n}\right)^n=e^R.
$$

Remember that the sequence $a_n=\left(1+\frac{R}{n}\right)^n$ is increasing hence

$$
e^R=\lim_{n\to\infty} a_n \ge a_1 = 1+R.
$$

Therefore (as expected) continuously investing is the winning strategy. In practice transaction costs make this choice unfeasible.

Esercize 14. Let $k > 1$ be a given integer. Establish if the sequence

$$x_n = \left(k \sin\left(\frac{1}{n^2}\right) + \frac{1}{k} \cos n\right)^n.$$

has a limit or note and in the affermative case compute it.

Solution. We know that for $0 < x < \frac{\pi}{2}$ then $0 < \sin(x) < x$ and $0 < \cos x < 1$, hence

$$|x_n| \leq \left(\frac{k}{n^2} + \frac{1}{k}\right)^n.$$

Since $k > 1$ there exists a δ with $0 < \delta < 1$ such that, for n sufficiently large

$$\frac{k}{n^2} + \frac{1}{k} \leq 1 - \delta$$

whence

$$|x_n| \leq (1 - \delta)^n \to 0.$$

11.4 "Advanced" topics on sequences.

This section discusses some advanced topics on sequences. **It is an optional section that can be skipped**.

Theorem 11.9. Every sequence contains a monotone subsequence.

Proof. Consider a sequence x_n. We call the m-th term x_m a *peak* if $x_m \geq x_n$ for all $n \geq m$. There are two cases.

- There are infinitely many peaks. In this case I list them according to the increasing subscript $\{x_{n_1}, x_{n_2}, ..., x_{n_k}, ...\}$ and since they are peaks by definition it must be

  $$x_{n_1} \geq x_{n_2} \geq ... \geq x_{n_k} \geq ...,$$

  hence a monotonically decreasing sub-sequence.

- There are finitely many peaks. Again I list them according to the increasing subscript $\{x_{m_1}, x_{m_2}, ..., x_{m_k}\}$. Consider $n_1 = m_k + 1$. Hence x_{n_1} it is not a peak and therefore it must exist a index $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Nevertheless since $n_2 > n_1 > m_k$ then x_{n_2} it is not a peak, hence it must exists an index $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$. Iterating this we get a monotonically increasing sub-sequence x_{n_k}.

✠

Definition 27. Let s_n be a sequence and $\mathcal{E}$ the set defined as:

$$\mathcal{E} = \{x \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} \,|\exists n_k : \; s_{n_k} \to x\},$$

i.e. the collection of all limits of the subsequences of s_n. We define the upper and lower limit of s_n as:

$$\limsup_{n\to\infty} s_n = \sup \mathcal{E}$$
$$\liminf_{n\to\infty} s_n = \inf \mathcal{E}.$$

Theorem 11.10. It x_n and y_n are real sequences then

1. $\limsup(-x_n) = -\liminf x_n$ and $\liminf(-x_n) = -\limsup x_n$.
2. For any $a > 0$ $\limsup(a\,x_n) = a\,\limsup(x_n)$ and $\liminf(a\,x_n) = a\,\liminf(x_n)$.
3. Suppose that $\limsup x_n$ and $\limsup y_n$ are finite then: $\limsup(x_n + y_n) \le \limsup(x_n) + \limsup(y_n)$. Suppose that $\liminf x_n$ and $\liminf y_n$ are finite then: $\liminf(x_n+y_n) \ge \liminf x_n + \liminf y_n$.
4. $\liminf x_n \le \limsup x_n$ where the equality holds if and only if the sequence converges and in this case $\lim x_n = \liminf x_n = \limsup x_n$.

Proof. 1) and 2) are obvious while 4) follows from Theorem 11.8. Let's prove assertion 3) for the $\limsup$ case, the other one is similar. Hence, let $X = \limsup x_n$ and $Y = \limsup y_n$. Given $\varepsilon > 0$ then there exist two integers n_1 and n_2 such that $\forall n \ge n_1$ we have $x_n < X + \frac{\varepsilon}{2}$ and $\forall n \ge n_2$ we have $y_n < Y + \varepsilon 2$. Hence if $n \ge \max(n_1, n_2)$ we have

$$x_n + y_n < X + Y + \varepsilon.$$

We can take the $\lim_{n\to\infty}$ on both side of the inequality, we do not know in general if $x_n + y_n$ converges, but we know that the $\limsup$ always exists and hence it must be

$$\limsup(x_n + y_n) \le X + Y. ✠$$

Theorem 11.11. Bolzano-Weiestrass Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $(x_n)_{n\in\mathbb{N}}$ is bounded then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Proof. According to Theorem 11.9 any sequence has a monotone-subsequence. Since $(x_n)_{n\in\mathbb{N}}$ is bounded then also the monotonic sub-sequence is bounded, but then by Theorem 11.6 it is convergent.

12 Some exercise on sequences

Try to solve these exercises by yourself, then look at the solutions.

Esercize 15. Let s_n be the sequence defined recursively as:

$$\begin{aligned} s_0 &= a \\ s_{n+1} &= \tfrac{1}{2}\left(s_n + \tfrac{\alpha}{s_n}\right), \end{aligned}$$

with $a \geq \alpha > 0$. Establish if s_n converge and in this case compute its limit.

First of all notice that $s_0 > 0$ and if $s_n > 0$ immediately follows $s_{n+1} > 0$, hence by induction $s_n > 0$ for all n. From the definition of the sequence we get

$$s_n^2 - 2s_{n+1}s_n + \alpha = 0$$

hence the second-degree polynomial has a real root whence the delta must be positive

$$4\,s_{n+1}^2 - 4\,\alpha \geq 0$$

that is $s_{n+1}^2 \geq \alpha$ for all n. Now consider

$$s_{n+1} - s_n = \frac{1}{2}\,\frac{s_n^2 + \alpha}{s_n} - s_n = \frac{s_n^2 + \alpha - 2\,s_n^2}{2\,s_n} = \frac{\alpha - s_n^2}{2\,s_n} \leq 0$$

whence $s_{n+1} \leq s_n$, therefore the sequence is monotonically decreasing and since it is bounded from below therefore it must exist $L = \lim_{n\to\infty} s_n$. Besides every sub-sequence converge to the same limit, hence also $s_{n+1} \to L$. By the recursive rule the only possibility for L is to satisfy the equation

$$L = \frac{1}{2}\left(L + \frac{\alpha}{L}\right) \Leftrightarrow L^2 = \frac{1}{2}\left(L^2 + \alpha\right) \Leftrightarrow L^2 = \alpha$$

whence $L = \alpha^{\frac{1}{2}}$.

The exercise just solved gives a rule to approximate the square root of any positive real number.

Esercize 16. Let s_n be the sequence defined recursively as:

$$\begin{aligned} s_1 &= \sqrt{2} \\ s_{n+1} &= \sqrt{2 + \sqrt{s_n}} \end{aligned}$$

Establish if s_n converge and in this case compute its limit.

Let's write out some terms of the sequence:

$$\begin{aligned} s_1 &= \sqrt{2} \\ s_2 &= \sqrt{2+\sqrt{\sqrt{2}}} > \sqrt{2} = s_1 \end{aligned}$$

Apparently the sequence seems increasing. Assume that $s_{n+1} > s_n$ and try to show that $s_{n+2} > s_{n+1}$:

$$s_{n+2} = \sqrt{2+\sqrt{s_{n+1}}} \overset{\text{(by hypothesis)}}{>} \sqrt{2+\sqrt{s_n}} = s_{n+1} \Rightarrow \text{ok!}$$

Does there exist an upper limit?

$$\begin{aligned} s_1 &= \sqrt{2} < 2 \\ s_2 &= \sqrt{2+\sqrt{\sqrt{2}}} < 2 \end{aligned}$$

The last inequality is equivalent to $\sqrt{2} < 4$ which is more evident. Now suppose that $s_n < 2$ and show that $s_{n+1} < 2$:

$$s_{n+1} = \sqrt{2+\sqrt{s_n}} < \sqrt{2+\sqrt{2}} < 2 \tag{12.1}$$

The last inequality is equivalent to $\sqrt{2} < 2$ which is more evident. Increasing + Bounded from above $\Rightarrow$:

$$\exists \quad L = \lim_{n\to\infty} s_n \tag{12.2}$$

and it must satisfies:

$$L = \sqrt{2+\sqrt{L}} \Rightarrow L^4 - 4L^2 - L + 4 = 0 \tag{12.3}$$

Numerically:

$$L \approx 1.8312....$$

Esercize 17. Consider:

$$\begin{cases} a_1 = 2 \\ a_{n+1} = \frac{1}{2}\,(a_n + 6) \end{cases}.$$

Establish if a_n admits limit and, in this case, compute it.

Let's write out some term of the sequence:

$$\begin{aligned} a_1 &= 2 \\ a_2 &= \tfrac{1}{2}\,(2+6) = 4 \\ a_3 &= 5 \\ a_4 &= \tfrac{11}{2} < \tfrac{11+1}{2} = 6 \\ a_5 &= \tfrac{1}{2}\left(\tfrac{11}{2}+6\right) = \tfrac{23}{4} < \tfrac{23+1}{4} = 6 \\ a_6 &= \ldots = \tfrac{47}{8} < \tfrac{47+1}{8} = 6 \end{aligned}.$$

The induction hypothesis is that $a_n < 6$:

$$a_{n+1} = \frac{1}{2}\left(a_n + 6\right) < \frac{1}{2}\left(6 + 6\right) = 6.$$

Therefore $a_n < 6, \forall n$. Is a_n increasing? $a_1 < a_2$. Now assume $a_n < a_{n+1}$:

$$a_{n+2} = \frac{1}{2}\left(a_{n+1} + 6\right) > \frac{1}{2}\left(a_n + 6\right) = a_{n+1}.$$

Therefore a_n is limited from above and increasing $\Rightarrow$

$$\exists l : \lim_{n\to\infty} a_n = l.$$

Note that:

$$\lim_{n\to\infty} a_{n+1} = l.$$

Therefore l must verify:

$$l = \frac{1}{2}\left(l + 6\right) \Rightarrow l = 6 \Rightarrow \lim_{n\to\infty} a_n = 6.$$

Esercize 18. Consider:

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 3 - \frac{1}{a_n} \end{cases}.$$

Establish if a_n admits limit and, in this case, compute it.

Let's write out some term of the sequence:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3 - 1 = 2 \\ a_3 &= 3 - \tfrac{1}{2} = \tfrac{5}{2} < \tfrac{5+1}{2} = 3 \\ a_4 &= 3 - \tfrac{2}{5} = \tfrac{13}{5} < \tfrac{13+2}{5} = 3 \end{aligned}.$$

The induction hypothesis is that $a_n < 3 \Rightarrow \frac{1}{3} < \frac{1}{a_n}$:

$$a_{n+1} = 3 - \frac{1}{a_n} < 3 - \frac{1}{3} = \frac{8}{3} < \frac{8+1}{3} = 3.$$

Is a_n increasing? $a_1 < a_2$. Now assume $a_n < a_{n+1}$:

$$a_{n+2} = 3 - \frac{1}{a_{n+1}} > 3 - \frac{1}{a_n} = a_{n+1}.$$

Therefore a_n is limited from above and increasing $\Rightarrow$

$$\exists l : \lim_{n\to\infty} a_n = l.$$

Note that:

$$\lim_{n\to\infty} a_{n+1} = l.$$

Therefore l must verify:

$$l = 3 - \frac{1}{l} \Rightarrow l^2 - 3\,l + 1 = 0 \Rightarrow l_\pm = \frac{3 \pm \sqrt{9-4}}{2}.$$

Neverthelss $l_- = \frac{3-\sqrt{5}}{2} < 1$ and $a_1 > 1$ and the sequence is increasing $\Rightarrow l_-$ cannot be the limit of l. Therefore:

$$\lim_{n\to\infty} a_n = \frac{3+\sqrt{5}}{2}.$$

Esercize 19. Find upper and lower limit of:

$$s_1 = 0, \quad s_{2\,m} = \frac{s_{2\,m-1}}{2}, \quad s_{2\,m+1} = \frac{1}{2} + s_{2\,m}.$$

Let's write out some term:

$$\begin{array}{ccc}
s_1 = 0 & \rightarrow & s_2 = 0 \\
 & \swarrow & \\
s_3 = \frac{1}{2} & \rightarrow & s_4 = \frac{1}{4} = \frac{2^1-1}{2^2} \\
 & \swarrow & \\
s_5 = \frac{3}{4} & \rightarrow & s_6 = \frac{3}{8} = \frac{2^2-1}{2^3} \\
 & \swarrow & \\
s_7 = \frac{7}{8} & \rightarrow & s_8 = \frac{7}{16} = \frac{2^3-1}{2^4}
\end{array}.$$

Similarly for the odd sequence note that:

$$s_1 = 0, \quad s_3 = \frac{1}{2^1}, \quad s_5 = \frac{2^2-1}{2^2}, \quad s_7 = \frac{2^3-1}{2^3}.$$

Now induction implies that:

$$s_{2\,m} = \frac{2^{m-1}-1}{2^m} \left(\Rightarrow s_{2\,m+1} = \frac{1}{2} + \frac{2^{m-1}-1}{2^m} = \frac{2^m-1}{2^m} \right).$$

Therefore:

$$\lim_{m\to\infty} s_{2\,m} = \lim_{m\to\infty} \frac{2^{m-1}-1}{2^m} = \frac{1}{2}.$$

$$\lim_{m\to\infty} s_{2\,m+1} = \lim_{m\to\infty} \frac{2^m-1}{2^m} = 1.$$

Note that every subsequence can be splitted in its odd and even sub-subsequence:

$$\begin{array}{ccc}
 & & s_{2\,n_k+1} \\
 & \nearrow & \\
s_{n_k} & & \\
 & \searrow & \\
 & & s_{2\,n_k}
\end{array}$$

Therefore it is enough to check if the odd and even sequences converge somewhere. In our case we can conclude that:

$$\limsup s_n = 1, \quad \liminf s_n = \frac{1}{2}. \tag{12.4}$$

As a direct consequence s_n doesn't converge.

13 Series

Definition 28. Given a sequence a_n, the sequence of the partial sums of a_n is defined as

$$S_n = \sum_{k=1}^{n} a_k.$$

We call the sum of the series of the a_n the limit, if it exists, of the partial sums and we write:

$$S = \lim_{n\to\infty} S_n = \sum_{k=1}^{+\infty} a_k.$$

Observation 2. The following properties hold:

1. *(Necessary condition). If*

$$S_n = \sum_{k=0}^{\infty} a_k < \infty,$$

   then

$$\lim_{k\to\infty} a_k = 0.$$

   The proof is immediate since $a_n = S_n - S_{n-1}$. So if $S_n \to \ell$ for some finite ℓ then also $S_{n-1} \to \ell$ and thus $a_n \to \ell - \ell = 0$.

2. *If $0 \leq a_k \leq c_k$ for all $k \geq k_0$, with k_0 given, then the convergence of $\sum_{k=1}^{\infty} c_k$ implies the convergence of $\sum_{k=1}^{\infty} a_k$.*

3. *If $a_k \geq c_k \geq 0$ for all $k \geq k_0$, with k_0 given, then the divergence of $\sum_{k=1}^{\infty} c_k$ implies the divergence of $\sum_{k=1}^{\infty} a_k$.*

   The proofs of the points 2 and 3 are an application of the comparison theorem to the sequence of partial sums S_n.

Esercize 20. For which values of x the series:

$$\sum_{k=0}^{\infty} x^k,$$

converges?

Note that:

$$1 + x = \frac{(1+x)\,(1-x)}{1-x} = \frac{1-x^2}{1-x}.$$

Therefore:

$$1 + x + x^2 = \frac{1-x^2}{1-x} + x^2 = \frac{1-x^2+x^2-x^3}{1-x} = \frac{1-x^3}{1-x}.$$

Iterating again:

$$1 + x + x^2 + x^3 = \frac{1 - x^3}{1 - x} + x^3 = \frac{1 - x^3 + x^3 - x^4}{1 - x} = \frac{1 - x^3}{1 - x}.$$

Now it's easy to show by induction that:

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Hence:

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \begin{cases} \frac{1}{1-x} & |x| < 1 \\ \infty & x \geq 1 \\ \text{no limit} & x \leq -1. \end{cases}$$

13.1 Series of nonnegative terms

Theorem 13.1. Suppose that:

$$a_1 \geq a_2 \geq a_3 \geq \dots \geq 0.$$

Then the series:

$$\sum_{k=1}^{\infty} a_k,$$

converges if and only if the series:

$$\sum_{k=0}^{\infty} 2^k a_{2^k},$$

converges.

Proof.

It is immediate to see that

$$0 \leq \sum_{k=1}^{\infty} a_k \leq \sum_{k=0}^{\infty} 2^k\, a_{2^k} \leq 2 \sum_{k=1}^{\infty} a_k.$$

Concerning the first inequality note that

$$\begin{array}{llllllll} \sum_{k=1}^{\infty} a_k & = & a_1 & + & a_2 + a_3 & + & a_4 + a_5 + a_6 + a_7 & + \cdots \\ & \leq & a_1 & + & a_2 + a_2 & + & a_4 + a_4 + a_4 + a_4 & + \cdots \\ & = & a_1 & + & 2\, a_2 & + & 4\, a_4 & + \cdots \\ & = & \sum_{k=0}^{\infty} 2^k\, a_{2^k} & & & & & \end{array}$$

Concerning the second inequality note that

$$\begin{array}{llllllll} \sum_{k=0}^{\infty} 2^k a_{2^k} & = & a_1 & + & a_2 + a_2 & + & a_4 + a_4 + a_4 + a_4 & + \cdots \\ & = & a_1 + a_2 & + & a_2 + a_4 + a_4 + a_4 & + & a_4 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8 & + \cdots \\ & \leq & a_1 + a_1 & + & a_2 + a_2 + a_3 + a_3 & + & a_4 + a_4 + a_5 + a_5 + a_6 + a_6 + a_7 + a_7 & + \cdots \\ & = & 2\, \sum_{k=1}^{\infty} a_k & & & & & , \end{array}$$

which completes the proof.

Observation 3. The sum
$$\sum_{k=0}^{+\infty}\left(\frac{1}{k}\right)^p,$$
converges $\Leftrightarrow$ $p > 1$.

We show that it doesn't converge for $p = 1$:
$$\begin{aligned}\sum_{k=1}^{2^n}\frac{1}{k} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots\\ &> 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\cdots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\\ &= 1+\frac{n}{2}.\end{aligned}$$
Whence
$$\lim_{n\to\infty}\sum_{k=1}^{2^n}\frac{1}{k} > \lim_{n\to\infty}\left(1+\frac{n}{2}\right)=\infty.$$
Note that the necessary condition is verified!

More generally if $p \le 0$ the generic term is not infinitesimal and as a consequence the series diverges. If $p > 0$ we can apply the 2^k-criterion:
$$\sum_{k=1}^{\infty}2^k\frac{1}{2^{kp}}=\sum_{k=1}^{\infty}\left(2^{(1-p)}\right)^k,$$
The argument of the geometric series is < 1 if and only if:
$$2^{(1-p)} < 1 \Leftrightarrow p > 1,$$
whence the thesis.

Definition 29. A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if
$$\sum_{n=1}^{\infty}|a_n| < \infty.$$

Theorem 13.2. If a series is absolutely convergent then it is convergent.

Proof. Since
$$0 \le a_n + |a_n| \le 2\,|a_n|$$
we have
$$0 \le \sum_{n=1}^{m}(a_n+|a_n|) \le 2\sum_{n=1}^{m}|a_n| < 2\sum_{n=1}^{\infty}|a_n|$$

so the sequence $s_m = \sum_{n=1}^{m}(a_n + |a_n|)$ is a monotonic increasing bounded sequence and hence it must converge. Now note that the original series

$$S_m = \sum_{n=1}^{m} a_n = s_m - \sum_{n=1}^{m} |a_n|$$

is the difference of two converging sequences, hence it must converge.

13.2 Convergence Criteria for Series

1. $\sum a_k$ converges if and only if for every $\epsilon > 0$ there is an integer N such that for all $m \geq n \geq N$ we have that:
$$\left|\sum_{k=n}^{m} a_k\right| \leq \epsilon.$$

2. Root test: given $\sum_k a_k$ consider $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Therefore if $\alpha < 1$ the series converges, if $\alpha > 1$ the series diverges, if $\alpha = 1$ the test gives no information.

   Proof. We prove only the case $\alpha < 1$. Assume, hence, that $\alpha = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$. Therefore, there exists an r such that, for n sufficiently large
$$|a_n|^{1/n} \leq r < 1,$$
   whence
$$0 \leq |a_n| \leq r^n.$$
   Since $\sum_n r^n$ converges so does $\sum_n |a_n|$, hence the series converges absolutely and thus converges.

3. Ratio test: $\sum_k a_k$ converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ and diverges if $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for some $n > n_0$.

   Proof. If $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then there exists r so that $0 < r < 1$ such that for n sufficiently large (say $n > N$) we get $|a_{n+1}| < r\,|a_n|$ hence by induction $|a_{n+i}| < r^i\,|a_n|$. Hence
$$\sum_{i=N}^{\infty} |a_i| = \sum_{i=0}^{\infty} |a_{N+i}| < \sum_{i=1}^{\infty} r^i\,|a_N| = |a_N|\,\frac{r}{1-r} < \infty,$$
   whence the series converges absolutely and hence converges.

   On the other side if $L > 1$ then definitively $a_{n+1} > a_n$ and therefore it is not possible that $a_n \to 0$ hence the series diverges.

Esercize 21. Show that the series:
$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$
converges.

Use ratio test:

$$\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0.$$

For the ratio test the series converges. We call the sum of the series e.

Economic Application

Suppose that, at time $t = 0$, I enter in a contract that pays me 1\$ in one year. At the moment of the investment the value of the contract is less than receiving now 1\$ because, in this case I would invest 1\$ immediately in the bank account receiving, at the end of the year, $1 \cdot (1 + R) > 1$ dollars, where R is the yearly linear interest rate. So the present value of the investment is

$$V_0^{(1)} = \frac{1}{1+R},$$

where, for simplicity, we have omitted the \$ symbol. Similarly, if the contract pays me 1\$ in two year the present value is

$$V_0^{(2)} = \frac{1}{(1+R)^2}.$$

Now suppose that, at time $t = 0$, I enter in a contract that pays me 1\$ every year perpetually. The present value is

$$V = \sum_{n=1}^{\infty} V_0^{(n)} = \sum_{n=1}^{\infty} \frac{1}{(1+R)^n} = \frac{1}{1 - \frac{1}{1+R}} - 1 = \frac{1}{R}.$$

More generally the present value of x\$ invested in a annuity in perpetuity with interest rate R is $\frac{x}{R}$.

Esercize 22. Show that:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Define $s_n = \left(1+\frac{1}{n}\right)^n$. Use newton binomial:

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \frac{1}{n^k} \\
&= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!\,n^k} \\
&= \sum_{k=0}^{n} \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)\,(n-k)!}{k!\,(n-k)!\,n^k} \\
&= \sum_{k=0}^{n} \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k!\,n^k} \\
&= \sum_{k=0}^{n} \frac{1}{k!} \frac{\overbrace{n\,(n-1)\,(n-2)\cdots(n-k+1)}^{k \text{ terms}}}{\underbrace{n\,n\cdots n}_{k \text{ terms}}} \\
&= \sum_{k=0}^{n} \frac{1}{k!} 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) < \sum_{k=0}^{n}\frac{1}{k!}.
\end{aligned}$$

Thus:

$$\limsup_{n\to\infty} s_n \leq e.$$

Now take an $m < n$, m fixed. Every term entering in the binomial formula for s_n is a positive term, therefore if we stop the sum at m we obtain a smaller quantity:

$$s_n > \sum_{k=0}^{m} \frac{1}{k!} 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right).$$

As a consequence:

$$\liminf_{n\to\infty} s_n \geq \sum_{k=0}^{m}\frac{1}{k!}.$$

We finally have that:

$$\sum_{k=0}^{m}\frac{1}{k!} \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq e.$$

Now let $m\to\infty$:

$$e \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq e.$$

That is:

$$\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_n = e.$$

Esercize 23. Find an upper bound for the error:

$$e - \sum_{k=0}^{n}\frac{1}{k!}.$$

Note that:

$$\begin{aligned}
e - \sum_{k=0}^{n} \frac{1}{k!} &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\
&= \frac{1}{(n+1)!} + \frac{1}{(n+2)\,(n+1)!} + \frac{1}{(n+3)\,(n+2)\,(n+1)!} + \cdots \\
&< \frac{1}{(n+1)!} + \frac{1}{(n+1)\,(n+1)!} + \frac{1}{(n+1)\,(n+1)\,(n+1)!} + \cdots \\
&= \frac{1}{(n+1)!}\left[1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \cdots\right] = \frac{1}{(n+1)!}\,\frac{1}{1 - \frac{1}{(n+1)}} = \frac{1}{n!\,n}.
\end{aligned}$$

Therefore:

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n!\,n}.$$

Theorem 13.3. The number e is irrational.

Suppose, by contradiction, that there exist two integers q and p such that $e = p/q$. We know that, for all n:

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n!\,n}.$$

In particular for $n = q$ we get:

$$0 < e - \sum_{k=0}^{q} \frac{1}{k!} < \frac{1}{q!\,q},$$

or:

$$0 < q! \left(e - \sum_{k=0}^{q} \frac{1}{k!}\right) < \frac{1}{q} \le 1. \tag{13.1}$$

By assumption:

$$q!\,e = q!\,\frac{p}{q} = (q-1)!\,p,$$

i.e. $q!\,e$ is an integer. Nevertheless:

$$q! \sum_{k=0}^{q} \frac{1}{k!} = q! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!}\right) \in \mathbb{N}.$$

As a consequence $q!\left(e - \sum_{k=0}^{q} \frac{1}{k!}\right)$. Moreover $q!\left(e - \sum_{k=0}^{q} \frac{1}{k!}\right)$ is in $(0,1)$ as stated by (12.1) . Nevertheless the open interval $(0,1)$ does not contain integer numbers. □

Observation 4. The sequence:

$$Q_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

is a sequence of rational numbers, i.e. $Q_n \in \mathbb{Q}$ for all n. Nevertheless:

$$e = \lim_{n\to\infty} Q_n \notin \mathbb{Q}.$$

This is another example showing that $\mathbb{Q}$ does not contain the limits of all its Cauchy sequences.

Esercize 24. In a population of N individuals every element has two alleles A_1 and A_2. Each new generation is obtained from the parent generation by repeating $2\,N$ times the following steps :

- Choose an allele at random from among the $2\,N$ alleles in the parent generation.
- Make an exact copy of the allele.
- Place the copy in the offspring generation.

Compute, in the limit of an infinite population, the probability that a particular allele gets a copy into the next generation.

Solution. The probability that a particular allele is not chosen on a single draw is $1 - \frac{1}{2\,N}$. As each draw is with replacement, the probability that the allele is not drawn at all is $\left[1 - \frac{1}{2\,N}\right]^{2\,N}$. In the limit of infinite population we find:

$$\left[1 - \frac{1}{2\,N}\right]^{2\,N} \to e^{-1} \approx 37\%.$$

As a consequence the probability that an allele gets a copy into the next generation is 63%.

Theorem 13.4. Limit Comparison Test.

Suppose that a_n and b_n are two positive sequences, $a_n \geq 0$ and $b_n \geq 0$. Then if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

and in this case we write $a_n \sim b_n$, then either both series ($\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$) converge or they both diverge.

Proof.

We know that for all $\varepsilon > 0$ there exists a $n_0 > 0$ such that for all $n \geq n_0$ we have that $\left|\frac{a_n}{b_n} - 1\right| < \varepsilon$ which can be re-written as

$$-\varepsilon < \frac{a_n}{b_n} - 1 < \varepsilon \iff 1 - \varepsilon < \frac{a_n}{b_n} < 1 + \varepsilon.$$

If ε is sufficiently small we have $1 - \varepsilon > 0$ then we can say that there exist two positive constants $c_1 > 0$ and $c_2 > 0$ such that for n sufficiently large

$$c_1 < \frac{a_n}{b_n} < c_2 \iff c_1\,b_n < a_n < c_2\,b_n.$$

Hence if $\sum_{n=0}^{\infty} b_n$ diverges then $\sum_{n=0}^{\infty} a_n$ diverges (first inequality), if $\sum_{n=0}^{\infty} a_n$ diverges then $\sum_{n=0}^{\infty} b_n$ diverges (second inequality). Similarly if $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges (second inequality), if $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} b_n$ converges (first inequality). □

Esercize 25. Determine whether the following series are convergent or divergent.

- $\sum_{n=0}^{\infty} \frac{n^4+7\,n^3+\sin(n)}{\pi\,n^5+8\,n^2+1}$.
- $\sum_{n=0}^{\infty} \frac{n^2}{2\,n^3+1}$.

Esercize 26. Establish if the series

$$\sum_{n=0}^{\infty} \frac{n^2+2^n}{n^3+3^n}$$

converges or not.

Solution. Use ratio test.

Esercize 27. Study the convergence/divergence of the following series:

$$\sum_{k=0}^{+\infty} \frac{1}{k+\sqrt{k}}$$

$$\sum_{k=0}^{+\infty} \frac{1}{k+k^2} \tag{13.2}$$

Asymptotically:

$$\frac{1}{k+\sqrt{k}} \sim \frac{1}{k} \Rightarrow \text{ div.}$$

$$\frac{1}{k+k^2} \sim \frac{1}{k^2} \Rightarrow \text{ conv.} \tag{13.3}$$

Theorem 13.5. Absolute convergence implies convergence, i.e.:

$$\sum_{k=0}^{+\infty} |a_k| < \infty \Rightarrow \sum_{k=0}^{+\infty} a_k < \infty$$

Esercize 28. Investigate the convergence of:

$$\sum_{k=0}^{+\infty} \frac{k^2}{k!}$$

Necessary condition:

$$a_k = \frac{k^2}{k!} \to 0 \Rightarrow \text{ok!}$$

Use ratio test:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(k+1)\ k!}\,\frac{k!}{k^2} = \frac{(k+1)^2}{k^2\ (k+1)} \sim \frac{k^2}{k^3} \to 0.$$

Ratio test implies convergence.

Esercize 29. Investigate the convergence of:

$$\sum_{k=0}^{+\infty} (-1)^{k-1} \frac{k^2}{k!}$$

Ratio test implies absolute convergence that implies convergence.

Esercize 30. Investigate the convergence of:

$$\sum_{k=0}^{+\infty} \frac{k}{(k+1)!}$$

Clearly:

$$0 < \frac{k}{(k+1)!} < \frac{k^2}{k!}.$$

Nevertheless we know that $\sum_k \frac{k^2}{k!} < \infty$ therefore $\sum_{k=0}^{+\infty} \frac{k}{(k+1)!} < \infty$.

Esercize 31. Investigate the convergence of:

$$\sum_{k=0}^{+\infty} \frac{k^k}{k!}$$

The necessary condition is not satisfied:

$$\frac{k^k}{k!} \to \infty.$$

Esercize 32. Investigate the convergence of:

$$\sum_{k=0}^{\infty} (-1)^{k-1} \frac{k}{3^k}$$

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{k+1}{3\,3^k}\,\frac{3^k}{k} = \frac{k+1}{3\,k} \to \frac{1}{3} < 1.$$

Ratio test implies convergence.

Esercize 33. Investigate the convergence of:

$$\sum_{k=0}^{+\infty} \frac{(k+1)!}{2^k\,k!}$$

Use ratio test to prove convergence.

Esercize 34. Investigate the convergence of:

$$\sum_{k=0}^{+\infty} \frac{4\,k}{1+k^2}$$

Ratio test is unuseful! Use asymptotic behaviour.

Esercize 35. Investigate the convergence of:

$$\sum_{k=0}^{+\infty} \frac{2^k}{k\,3^k}$$

$$\frac{2^k}{k\,3^k} < \frac{2^k}{3^k} \Rightarrow \text{ conv.}$$

Esercize 36. Investigate the convergence of:

$$\sum_{k=0}^{+\infty} \left(\frac{3\,k}{k+3}\right)^k$$

Use root test:

$$a_k^{\frac{1}{k}} = \frac{3\,k}{k+3} \to 3 > 1 \Rightarrow \text{ div.}$$

Esercize 37. Compute the sum:

$$\sum_{k=1}^{+\infty} \frac{1}{k\,(k+1)}$$

We try to split the generic term as:

$$\frac{1}{k\,(k+1)} = \frac{A}{k} + \frac{B}{k+1}.$$

Straightforward computations show that $A = 1$ and $B = -1$:

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

We write explictly some term of the series:

$$\begin{array}{lcl} S_1 & = & 1 - \frac{1}{2} \\ S_2 & = & 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} \\ S_3 & = & 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4} \\ \vdots & & \vdots \\ S_n & = & 1 - \frac{1}{n+1} \\ S_{n+1} & = & S_n + \frac{1}{n+1} - \frac{1}{n+2} \Rightarrow S_{n+1} = 1 - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} = 1 - \frac{1}{(n+1)+1} \Rightarrow \text{ok!} \end{array} \tag{13.4}$$

Therefore:

$$\forall n, S_n = 1 - \frac{1}{n+1} \to 1.$$

Esercize 38. Compute the sum:

$$\sum_{k=0}^{+\infty} \left(\frac{1}{2\,k+1} - \frac{1}{2\,k+3}\right)$$

We write explictly some term of the series:

$$\begin{array}{rcl}
S_1 & = & 1 - \frac{1}{3} \\
S_2 & = & 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} = 1 - \frac{1}{5} \\
S_3 & = & 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} = 1 - \frac{1}{7} \\
\vdots & & \vdots \\
S_n & = & 1 - \frac{1}{2n+3} \\
S_{n+1} & = & S_n + \frac{1}{2(n+1)+1} - \frac{1}{2(n+1)+3} \Rightarrow \\
S_{n+1} & = & 1 - \frac{1}{2n+3} + \frac{1}{2(n+1)+1} - \frac{1}{2(n+1)+3} = 1 - \frac{1}{2n+3} + \frac{1}{2n+3} - \frac{1}{2(n+1)+3} = 1 - \frac{1}{2(n+1)+3} \Rightarrow \text{ok!}
\end{array} \tag{13.5}$$

Therefore:

$$\forall n, S_n = 1 - \frac{1}{2\,(n+1)+3} \to 1.$$

Esercize 39. Compute the sum of the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}}$$

Solution. Since

$$\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

we get

$$\sum_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + ... + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{n+1}} \to 1.$$

Theorem 13.6. Suppose that

- $\sum_{n=0}^{\infty} a_n$ converges absolutely.
- $\sum_{n=0}^{\infty} a_n = A$.
- $\sum_{n=0}^{\infty} b_n = B$.

Let

$$c_n = \sum_{k=0}^{n} a_k\, b_{n-k},$$

then

$$\sum_{n=0}^{\infty} c_n = A\,B.$$

Theorem 13.7. Alternating series

A series of the form

$$\sum_{n=1}^{\infty} (-1)^n\, a_n$$

where the $a_n \geq 0$ for all n or $a_n \leq 0$ for all n is called an alternating series. If a_n decreases monotonically and $\lim_{n\to\infty} a_n = 0$ then the alternating series converges.

Proof. Suppose, without loss of generality, that we are in the case $a_n \geq 0$. The odd partial sums $O_m = S_{2m+1}$ increases monotonically, i.e.

$$O_{m+1} = S_{2(m+1)+1} = S_{2m+3} = S_{2m+1} + a_{2m+2} - a_{2m+3} \geq S_{2m+1} + a_{2m+3} - a_{2m+3} = O_m.$$

Similarly the series of even partial sum $E_m = S_{2m}$ decreases

$$E_{m+1} = S_{2(m+1)} = S_{2m+2} = S_{2m} - a_{2m+1} + a_{2m+2} \leq S_{2m} - a_{2m+1} + a_{2m+1} = E_m.$$

Note that $S_{2m+1} - S_{2m} = -a_{2m+1} \leq 0$ that is $S_{2m+1} \leq S_{2m}$ Moreover

$$-a_1 = S_1 \leq S_{2m+1} \leq S_{2m} \leq S_1$$

hence S_{2m+1} is increasing and bounded from above, thus $S_{2m+1} \to L$ and S_{2m} is decreasing and bounded from below, thus $S_{2m} \to L'$. Nevertheless $S_{2m+1} - S_{2m} = -a_{2m+1} \to 0$ hence $L = L'$. Since all sub-subsequences of a sequence can be split into the even and odd subsequence we conclude that $S_n \to L$. □

Definizione 13.1. We define, for all $x \in \mathbb{R}$ the exponential function e^x as the sum of the series

$$e^x \triangleq \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The definition is well-posed since by the ratio test the series converges absolutely and hence it converges.

Theorem 13.8. The exponential function has the following properties

- For all x and y in $\mathbb{R}$ it holds that
$$e^{x+y} = e^x\, e^y.$$
- For all $y \in \mathbb{R}$ with $x > 0$ there exits a unique $x \in \mathbb{R}$ such that
$$e^x = y$$
we call that x the natural logarithm of x and we indicate it as
$$x = \ln y.$$

The natural logarithm of a number is thus the inverse of the exponential function and it has the following properties

- For all x and y in $\mathbb{R}$ it holds that
$$\ln(x\,y) = \ln(x) + \ln(y).$$

Proof. We prove only the multiplicative property of the exponential function.

$$
\begin{aligned}
e^{x+y} &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} x^k\, y^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} x^k\, y^{n-k} \\
\text{(by Theorem (12.6))} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} = e^x\, e^y.
\end{aligned} \tag{13.6}
$$

Definizione 13.2. For all $a > 0$ and $a \neq 1$ we define the exponential function with base a the function

$$a^x \triangleq e^{x \ln(a)}.$$

Theorem 13.9. Let a be a real number such that $a > 0$ and $a \neq 1$. For all $y > 0$ there exists a unique $x \in \mathbb{R}$ such that

$$y = a^x$$

and we put $y = \log_a x$.

The following proposition completes the list of properties of the exponential and logarithmic function:

Theorem 13.10. Let $a > 0$ and $b > 0$ be two strictly positive real numbers, $a \neq 1$ and $b \neq 1$.

- For all $x \in \mathbb{R}$ it holds $\log_a (b^x) = x \log_a b$.
- If $x = 1$ then $\log_a x = 0$ and $a^x = a$.
- If $0 < a < 1$ then the functions $f(x) = a^x$ and $f(x) = \log_a x$ are decreasing. If $a > 1$ they are both increasing.
- $\lim_{n\to\infty} \log_a n = \infty$ if $a > 1$ while $\lim_{n\to\infty} \log_a n = -\infty$ if $0 < a < 1$.
- If $s_n \to s$ is a converging sequence then $a^{s_n} \to a^s$ and $\log_a s_n \to \log_a s$.

Esercize 40. For what values of x does the series

$$\sum_{n=0}^{\infty} e^{n\,x}$$

converge?

Esercize 41. Establish if the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{2 + \ln n^2},$$

converges or not and, when it converges, compute the sum.

Esercize 42. Establish for which $x \in \mathbb{R}$ the series

$$\sum_{n=0}^{\infty} (n+1)\, x^n$$

converges.

Solution.

$$\begin{aligned} 1 + 2x + 3x^2 + 4x^3 + \dots &= 1 + x + x^2 + x^3 + \dots \\ &\quad +x + x^2 + x^3 + \dots \\ &\quad +x^2 + x^3 + \dots \\ &\quad +x^3 + \dots \\ &\quad + \dots \\ &= 1 + x + x^2 + x^3 + \dots \\ &\quad +x(1 + x + x^2 + \dots) \\ &\quad +x^2(1 + x + \dots) \\ &\quad +x^3(1 + \dots) \\ &\quad + \dots \\ = (1 + x + x^2 + x^3 + \dots)(1 + x + x^2 + x^3 + \dots) &= \frac{1}{(1-x)^2} \end{aligned}$$

Esercize 43. Compute the sum of the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}}$$

Hint.

$$\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}.$$

Esercize 44. Establish if the series

$$\sum_{n=1}^{\infty} \ln(2(n+1)) - \ln(2n)$$

coverges or not.

Solution. First note that

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n = \ln e = 1.$$

Hence

$$\ln(2(n+1)) - \ln(2n) = \ln\left(1 + \frac{1}{n}\right) \sim \frac{1}{n}$$

so that

$$\sum_{n=1}^{\infty} \ln(2(n+1)) - \ln(2n) = \infty.$$

Esercize 45. Does the series

$$\sum_{n=2}^{\infty} \frac{\sqrt{n+1}}{(2n^2-3n+1)(\ln n + (\ln n)^2)}$$

converge or not?

Solution. For n sufficiently large $\ln > 1$ and hence

$$0 < \frac{\sqrt{n+1}}{(2n^2-3n+1)(\ln n + \ln^2 n)} < \frac{\sqrt{n+1}}{n^2-3n+1}.$$

Now note that

$$\sqrt{n+1} \leq 2\sqrt{n}$$

and for $n \geq 6$ we have $n^2 - 3n + 1 \geq \frac{1}{2}n^2$ hence

$$\frac{\sqrt{n+1}}{n^2-3n+1} < \frac{4}{n^{3/2}},$$

since

$$\sum_{n=2}^{\infty} \frac{4}{n^{3/2}} < \infty$$

thus the series converges.

Esercize 46. Let a_n be a positive sequence such that

$$\sum_{n\geq 1} a_n = +\infty.$$

Prove that

$$\sum_{n\geq 1} \frac{a_n}{1+a_n} = +\infty.$$

Proof. Suppose, by contradiction, that

$$\sum_{n\geq 1} \frac{a_n}{1+a_n} < \infty.$$

Hence in particular

$$\frac{a_n}{1+a_n} \to 0$$

so that for sufficiently large n

$$\frac{a_n}{1+a_n} < \frac{1}{2} \Rightarrow a_n < \frac{1}{2} + \frac{a_n}{2} \Rightarrow a_n < 1.$$

This implies that

$$\frac{a_n}{1+1} = \frac{a_n}{2} < \frac{a_n}{a_n+1},$$

but then for the comparison theorem we will have $\sum_{n\geq 1} a_n < \infty$, which is an absurd.

Esercize 47. Prove that
$$\sum_{n=1}^{\infty} \frac{1}{n(1+1/2+\cdots+1/n)}$$
diverges.

Solution. Use condensation criterion:
$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \frac{1}{1+1/2+\cdots+1/2^k}.$$
Nevertheless
$$1+1/2+\cdots+1/2^k \leq 1+2\cdot 1/2+4\cdot 1/4+\cdots+2^{k-1}\cdot 1/2^{k-1}+1/2^k \leq k+1,$$
hence
$$\sum_{n=1}^{\infty} \frac{1}{n(1+1/2+\cdots+1/n)} = \infty.$$

Esercize 48. Suppose that $a_n \geq 0$ and that $\sum_{n=1}^{\infty} a_n < \infty$. Prove that
$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} < \infty.$$

Solution. From Cauchy-Schwarz inequality $|\langle x,y\rangle|^2 \leq \langle x,x\rangle\ \langle y,y\rangle$ we get
$$\left(\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n}\right)^2 \leq \sum_{n=1}^{N} (\sqrt{a_n})^2 \cdot \sum_{n=1}^{N} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$
therefore
$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \leq \sqrt{\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}} < \infty.$$

Esercize 49. Establish if the series
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{3^k+\sqrt{k}}$$
converges or not. Does the sequence converge absolutely?

Solution.
$$\sum_{k=1}^{n} \left|\frac{(-1)^k}{3^k+\sqrt{k}}\right| = \sum_{k=1}^{n} \frac{1}{3^k+\sqrt{k}} \leq \sum_{k=1}^{n} \frac{1}{3^k-\frac{1}{2}3^k} < 2\sum_{k=1}^{n} \left(\frac{1}{3}\right)^k.$$

Esercize 50. Establish for which value of $\alpha > 0$ the series
$$\sum_{k=1}^{\infty} \left(1-\cos\left(\frac{1}{k^\alpha}\right)\right)$$
converges.

Solution. Since

$$1 - \cos\left(\frac{1}{n^\alpha}\right) = 2\sin^2\left(\frac{1}{2n^\alpha}\right)$$

we have that

$$\lim_{n\to\infty} \frac{1-\cos\left(\frac{1}{n^\alpha}\right)}{\frac{1}{2n^\alpha}} = 2$$

Hence

$$1 - \cos\left(\frac{1}{n^\alpha}\right) \sim \frac{1}{2n^\alpha},$$

whence the series converges if and only if $\alpha > 1$.

Esercize 51. Suppose that $\sum_{n=1}^{\infty} a_n < \infty$ and $a_n \geq 0$ for all n. Prove that

$$\sum_{n=1}^{\infty} \sqrt{a_n \cdot a_{n+1}} < \infty.$$

Solution. It is enough to note that for all positive x and y real numbers it holds that

$$2\sqrt{x\,y} \leq x + y$$

hence

$$\sqrt{a_n\,a_{n+1}} \leq \frac{a_n + a_{n+1}}{2}$$

then by the comparison criterion $\sum_{n=1}^{\infty} \sqrt{a_n \cdot a_{n+1}} < \infty$.

Esercize 52. Establish if the series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}, \quad \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

converge or not.

Solution. It is enough to apply the condensation criterion:

$$a_n := \frac{1}{n\log n} \implies 2^n a_{2^n} = \frac{2^n}{2^n \ln 2^n} = \frac{1}{\ln 2}\frac{1}{n}$$

hence the first series diverges. Since

$$\frac{1}{\ln n} \geq \frac{1}{n\,\ln n},$$

also the second one diverges.

14 Limits and Continuity

Definition 30. Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be a function defined on the set E, subset of $\mathbb{R}$, with value in $\mathbb{R}$. Let p be a limit point of E.

- We write $f(x) \to q$ as $x \to p$, or
$$\lim_{x\to p} f(x) = q,$$
if
$$\forall \epsilon > 0 \exists \delta > 0 : \forall x \text{ such that } x \in E \text{ and } 0 < |x - x_0| < \delta \Rightarrow |f(x) - q| < \epsilon.$$

- If E is not limited from above we write $f(x) \to q$ as $x \to +\infty$, or
$$\lim_{x\to +\infty} f(x) = q,$$
if
$$\forall \epsilon > 0 \exists M > 0 : \forall x \in E \wedge x \geq M \Rightarrow |f(x) - q| < \epsilon.$$

- If E is not limited from above we write $f(x) \to +\infty$ as $x \to +\infty$, or
$$\lim_{x\to +\infty} f(x) = +\infty,$$
if
$$\forall K > 0 \exists M > 0 : \forall x \in E \wedge x \geq M \Rightarrow f(x) > K.$$

Similar definitions holds for all the other possibilities, for example $\lim_{x\to -\infty} f(x) = p$ etc..etc..

Theorem 14.1. Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be a function defined on the set E, subset of $\mathbb{R}$, with value in $\mathbb{R}$. Let x_0 be a limit point of E. Then:
$$\exists \lim_{x\to x_0} f(x) = L \Leftrightarrow \forall x_n \in E : (x_n \to x_0 \wedge x_n \neq x_0) \Rightarrow f(x_n) \to L.$$

Proof. First we assume that the limit exists. This means that
$$\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : |x - x_0| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon.$$
Let x_n be a sequence in E such that $x_n \neq x_0$ and $x_n \to x_0$, hence for large n we have $|x_n - x_0| < \delta_\varepsilon$ whence $|f(x_n) - L| < \varepsilon$, that is $f(x_n) \to L$.

Suppose now that for all x_n in E such that $x_n \neq x_0$ and $x_n \to x_0$ we have $f(x_n) \to L$. We proceed by contradiction. So let's assume that the limit $\lim_{x\to x_0} f(x)$ does not exist. Hence there exists a $\varepsilon_0 > 0$ such that for all $\delta > 0$ there is a point x such that $0 < |x - x_0| < \delta$ but $|f(x) - L| \geq \varepsilon$. So for example, since δ can be chosen arbitrarily, we can choose to put $\delta = \frac{1}{n}$ for any $n \in \mathbb{N}$. And hence for all $n \in \mathbb{N}$ there is a x_n such that
$$0 < |x_n - x_0| < \delta \text{ and } |f(x_n) - L| \geq \varepsilon$$
This means that there exists a sequence such that $x_n \to x_0$, $x_n \neq x_0$ for all n, but $f(x_n) \not\to L$. $\square$

Remark. The above theorem holds also if $x_0 = \pm\infty$ or $L = \pm\infty$. For example $f(x) = \sin(x)$ has no limit for $x \to \infty$ because $f\left(n\frac{\pi}{2}\right) = 1$ for $n = 1, 5, 9, \ldots$ and $f\left(n\frac{\pi}{2}\right) = -1$ for $n = 3, 7, 11, \ldots$.

Theorem 14.2. Let $f : E \subseteq \mathbb{R} \to \mathbb{R}$ and x_0 be a limit point of E. If $f(x) \to L$ when $x \to x_0$ then the limit is unique.

Proof. Suppose by contradiction that there exist L_1 and L_2 with $L_1 \neq L_2$ such that

$$\lim_{x\to x_0} f(x) = L_1, \quad \lim_{x\to x_0} f(x) = L_2.$$

Hence for all $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - x_0| < \delta_1 \text{ and } x \in E \text{ implies } |f(x) - L_1| < \varepsilon/2,$$
$$0 < |x - x_0| < \delta_2 \text{ and } x \in E \text{ implies } |f(x) - L_2| < \varepsilon/2.$$

Now consider $\delta = \min(\delta_1, \delta_2)$. Since x_0 is a **limit point** of E then there exists $x \in E$ such that $0 < |x - x_0| < \delta$. Hence it follows that

$$|L_1 - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon,$$

whence $L_1 = L_2$. □

Rules for sum, multiplication and division of limits are the same valid for sequences.

Definition 31. Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be a function defined on the set E, subset of $\mathbb{R}$, with value in $\mathbb{R}$. The function f is said to be continuous in $p \in E$ if:

$$\forall \epsilon > 0 \exists \delta > 0 : x \in N_\delta(p) \cap E \Rightarrow |f(x) - f(p)| < \epsilon,$$

or equivalently

$$\forall \epsilon > 0 \exists \delta > 0 : |x - p| < \delta \wedge x \in E \Rightarrow |f(x) - f(p)| < \epsilon,$$

that is, small changes in x correspond to small changes in $f(x)$. If f is continuous for all $p \in E$ then f is said to be continuous in E.

Theorem 14.3. Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be a function defined on the set E, subset of $\mathbb{R}$, with value in $\mathbb{R}$. Suppose that $p \in E$ is a limit point of E. Then f is continuous at p if and only if:

$$\lim_{x\to p} f(x) = f(p).$$

- $f(x) = \sqrt{x}$ is continuous in its domain.

  Solution. Consider a $c > 0$. Then

$$|f(x) - f(c)| = \left|\sqrt{x} - \sqrt{c}\right| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right|$$

  Since $c \neq 0$ if we take x close enough to c we have $x \neq 0$ and so

$$|f(x) - f(c)| = \left|\sqrt{x} - \sqrt{c}\right| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| < \left|\frac{x - c}{\sqrt{c}}\right|$$

  so for all $\varepsilon > 0$ it is enough to consider any δ such that $\delta < \varepsilon\sqrt{c}$, if this is the case we have immediately that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. For the continuity in $c = 0$ it is enough, for all $\varepsilon > 0$ to take any $\delta < \varepsilon^2$ so that we have

$$|x| < \delta \Rightarrow \sqrt{|x|} < \sqrt{\delta} < \varepsilon.$$

- $f(x) = \sin x$ is continuous in its domain.

  Proof. Remember that $|\sin x| \leq |x|$. Hence

$$|\sin x - \sin c| = \left|2\cos\left(\frac{x + c}{2}\right)\sin\left(\frac{x - c}{2}\right)\right| \leq 2\left|\sin\left(\frac{x - c}{2}\right)\right| \leq |x - c|$$

  so any $\delta < \varepsilon$ does the job.

- The exponential function $f(x) = a^x$ is continuos in its domain.

  Proof. Consider first the case $x > c$ and $c > 0$, thus

$$0 < x - c < \delta \Rightarrow x < c + \delta \Rightarrow e^x < e^c e^\delta \Rightarrow e^x - e^c < e^c e^\delta - e^c = e^c\left(e^\delta - 1\right)$$

  so for all $\varepsilon > 0$ I take δ such that $e^c\left(e^\delta - 1\right) = \varepsilon$, that is

$$c + \ln\left(e^\delta - 1\right) = \ln(\varepsilon) \Rightarrow \ln\left(e^\delta - 1\right) = \ln(\varepsilon) - c \Rightarrow e^\delta - 1 = \varepsilon\, e^{-c} \Rightarrow e^\delta = \varepsilon\, e^{-c} + 1 \Rightarrow \delta = \ln\left(\varepsilon\, e^{-c} + 1\right)$$

  so that if $x - c < \delta$ then $e^x - e^c < \varepsilon$

Remark. Sums of continuous function are continuous. Products of continuous function are continuous. Inverse of an invertible continuous function is continuous. Composition of continuous function is continuous.

Esercize 53. Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that:

$$\lim_{h \to 0} \left(f(x+h) - f(x-h) \right) = 0, \forall x \in \mathbb{R}.$$

Does this imply that $f(x)$ is continuous?

Take:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Z} = \{-\infty, \cdots, -2, -1, 0, 1, 2, \cdots, +\infty\}, \\ 0 & x \in \mathbb{R}/\mathbb{Z}. \end{cases}$$

$f(x)$ is discontinuous, nevertheless:

$$\lim_{h \to 0} \left(f(x+h) - f(x-h) \right) = 0, \forall x \in \mathbb{R}.$$

because for each neighborhood $\mathcal{U}$ of $x \in \mathbb{Z}$, $\mathcal{U}$ contains infinite elements of $\mathbb{R}/\mathbb{Z}$.

Remarks. All elementary functions encountered up to now

$$\sin x, \cos x, a^x, \log_a x, x^a,$$

are **continuous in their domains of definition**.

Esercize 54. Compute:

$$\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x.$$

Consider the sequence:

$$a_n = \left(1 + \frac{1}{n}\right)^n,$$

we know that

$$\exists e = \lim_{n \to \infty} a_n, \quad e \in (2, 3].$$

Now let $n = \lfloor x \rfloor$ be the integer part of x. For example $\lfloor 2.34 \rfloor = 2$, $\lfloor 5.89 \rfloor = 5$, more generally

$$\lfloor x \rfloor = \max \{n \in \mathbb{N} | n \leq x\}.$$

Hence $n \leq x < n + 1$ and so

$$\begin{aligned} \frac{1}{n+1} &< \frac{1}{x} \leq \frac{1}{n} \\ 1 + \frac{1}{n+1} &< 1 + \frac{1}{x} \leq 1 + \frac{1}{n}. \end{aligned}$$

This implies that:

$$\begin{aligned}
\left(1+\frac{1}{n+1}\right)^n &\underset{(n \le x)}{\le} \left(1+\frac{1}{n+1}\right)^x \\
&\underset{\left(1+\frac{1}{n+1} < 1+\frac{1}{x}\right)}{<} \left(1+\frac{1}{x}\right)^x \\
&\underset{\left(1+\frac{1}{x} \le 1+\frac{1}{n}\right)}{\le} \left(1+\frac{1}{n}\right)^x \\
&\underset{(x < n+1)}{<} \left(1+\frac{1}{n}\right)^{n+1}
\end{aligned}$$

Summarizing:

$$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right).$$

Now note that:

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

The term on the left side of our inequality can be managed in the same way:

$$\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1} \left(1+\frac{1}{n+1}\right)^{-1} = e.$$

If $n \to \infty$ then $x \to \infty$. This shows that:

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e.$$

If $x \to -\infty$ then $x = -|x|$. Thus:

$$\begin{aligned}
\left(1+\frac{1}{x}\right)^x &= \left(1-\frac{1}{|x|}\right)^{-|x|} \\
&= \left(\frac{|x|-1}{|x|}\right)^{-|x|} \\
&= \left(\frac{|x|}{|x|-1}\right)^{|x|} \\
&= \left(1+\frac{1}{|x|-1}\right)^{|x|}.
\end{aligned}$$

Call $y = |x| - 1$:

$$\begin{aligned}
\left(1+\frac{1}{x}\right)^x &= \left(1+\frac{1}{y}\right)^{y+1} \\
&= \left(1+\frac{1}{y}\right)^y \left(1+\frac{1}{y}\right) \to e
\end{aligned} \tag{14.1}$$

14.1 Notable Limits

- Fraction of polynomials.

$$\lim_{x\to+\infty} \frac{a_r\, x^r + a_{r-1}x^r + \cdots + a_0}{b_q\, x^q + b_{q-1}x^{q-1} + \cdots + b_0} = \begin{cases} \operatorname{sign}\left[\frac{a_r}{b_q}\right]\infty & r > q \\ \frac{a_r}{b_q} & r = q \\ 0 & r < q \end{cases}.$$

$$\lim_{x\to-\infty} \frac{a_r\, x^r + a_{r-1}x^r + \cdots + a_0}{b_q\, x^q + b_{q-1}x^{q-1} + \cdots + b_0} = \begin{cases} \operatorname{sign}\left[\frac{a_r}{b_q}\right](-1)^{r-q}\infty & r > q \\ \frac{a_r}{b_q} & r = q \\ 0 & r < q \end{cases}.$$

ex:

$$\begin{aligned} \lim_{x\to-\infty} \frac{\pm 2\, x^2 + x}{3\, x} &= \mp\infty \\ \lim_{x\to-\infty} \frac{\pm 2\, x^2 + x}{3\, x^2} &= \pm\frac{2}{3} \\ \lim_{x\to-\infty} \frac{\pm 2\, x^2 + x}{3\, x^3} &= 0. \end{aligned}$$

- $$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

  Same demonstration for sequences.

- $$\lim_{x\to 0} \frac{1-\cos x}{x} = \frac{0}{0}.$$

  Try to go back to a known notable limit. Re-write the limit as:

$$\frac{1-\cos x}{x} = \frac{1-\cos x}{x}\frac{1+\cos x}{x}\frac{x}{1+\cos x} = \frac{1-\cos^2 x}{x^2}\frac{x}{1+\cos x} = \frac{\sin^2 x}{x^2}\frac{x}{1+\cos x} \to 1\cdot 0 = 0 \tag{14.2}$$

- $$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{0}{0}.$$

  Try to go back to a known notable limit. Similarly to the previous one:

$$\frac{1-\cos x}{x^2} = \frac{1-\cos x}{x^2}\frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{x^2}\frac{1}{1+\cos x} = \frac{\sin^2 x}{x^2}\frac{1}{1+\cos x} \to 1\cdot\frac{1}{2} = \frac{1}{2}.$$

- $$\lim_{x\to 0} \frac{\tan x}{x} = \frac{0}{0}.$$

  Try to go back to a known notable limit. Similarly to the previous ones:

$$\frac{\tan x}{x} = \frac{\sin x}{x}\frac{1}{\cos x} \to 1.$$

- $$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\frac{0}{0}.$$

  Exploit the properties of the logarithmic function:

  $$\frac{\ln(1+x)}{x}=\ln(1+x)^{\frac{1}{x}} \tag{14.3}$$

  Call $t=\frac{1}{x}\to\infty$:

  $$\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\lim_{t\to\infty}\ln\left(1+\frac{1}{t}\right)^t=\ln\left(\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t\right)=\ln e=1.$$

  Esercize 55. Compute the limit:

  $$\lim_{x\to\infty}\left(\frac{3x+2}{3x-4}\right)^{2x-4}=1^{\infty}. \tag{14.4}$$

  Use Ruffini's rule:

  $$\begin{array}{l|l} 3x+2 & 3x-4 \\ \hline (+1)\cdot(3x-4)=3x-4 & \frac{3x}{3x}=1\equiv Q(x) \\ \hline 3x+2-3x+4=6\equiv R(x) & \end{array}.$$

  Obtaining:

  $$\frac{3x+2}{3x-4}=1+\frac{6}{3x-4}.$$

  Therefore:

  $$\begin{aligned} \left(\frac{3x+2}{3x-4}\right)^{2x-4} &= \left(1+\frac{6}{3x-4}\right)^{2x-4} \\ &= \left(1+\frac{1}{\frac{3x-4}{6}}\right)^{2x-4} \\ (y\equiv\frac{3x-4}{6}) &= \left(1+\frac{1}{y}\right)^{\frac{2}{3}(6y+4)-4} \\ &= \left(1+\frac{1}{y}\right)^{4y+\frac{8}{3}-4} \\ &= \left(1+\frac{1}{y}\right)^{4y-\frac{4}{3}} \\ &= \underbrace{\left(1+\frac{1}{y}\right)^{4y}}_{\substack{\downarrow\\ e^4}}\underbrace{\left(1+\frac{1}{y}\right)^{-\frac{4}{3}}}_{\substack{\downarrow\\ 1}}\to e^4. \end{aligned}$$

Esercize 56. Compute:

$$\lim_{x\to 0}\left(1+x^2\right)^{\frac{1}{\tan x}}.$$

Try to go back to a known notable limit:

$$\left(1+x^2\right)^{\frac{1}{\tan x}}=\left[\left(1+x^2\right)^{\frac{1}{x^2}}\right]^{\frac{x^2}{\tan x}}.$$

Note that:

$$\begin{aligned}\lim_{x\to0}\left(1+x^2\right)^{\frac{1}{x^2}} &= \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y=e\\ \lim_{x\to}\frac{\tan x}{x} &= 1 \quad .\end{aligned}$$

Therefore:

$$\lim_{x\to0}\left(1+x^2\right)^{\frac{1}{\tan x}}=\lim_{x\to0}\left[\left(1+x^2\right)^{\frac{1}{x^2}}\right]^{\frac{x^2}{\tan x}}=\lim_{x\to0}[\underbrace{\left(1+x^2\right)^{\frac{1}{x^2}}}_{\substack{\downarrow\\ e}}]^{(\overbrace{\frac{\tan x}{x}}^{\substack{1\\ \uparrow}})\,x}=\lim_{x\to0}e^x=1.$$

Esercize 57. Compute:

$$\lim_{x\to0}\frac{e^x-1}{x}=\frac{0}{0}.$$

Define $y=e^x-1\Rightarrow x=\ln(1+y)$:

$$\lim_{x\to0}\frac{e^x-1}{x}=\lim_{y\to0}\frac{y}{\ln(1+y)}=\lim_{y\to0}\frac{1}{\frac{\ln(1+y)}{y}}=1. \tag{14.5}$$

This is a very important result, especially when computing derivatives.

Esercize 58. Compute:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x.$$

Change variable $y=\frac{x}{a}$:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{a\,y}=\lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^a=e^a.$$

Esercize 59. Compute:

$$\lim_{x\to0}\frac{\log_b(1+x)}{x}.$$

Recall:

$$\log_b(\cdot)=\frac{\log_a(\cdot)}{\log_a b}.$$

Therefore:

$$\lim_{x\to0}\frac{\log_b(1+x)}{x}=\lim_{x\to0}\frac{\ln(1+x)}{x\ln b}=\frac{1}{\ln b}.$$

Esercize 60. Compute:

$$\lim_{x\to0}\frac{a^x-1}{x}.$$

Define $y = a^x - 1 \Rightarrow x = \log_a (y+1)$:

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{y\to 0} \frac{y}{\log_a (y+1)} = \ln a \tag{14.6}$$

Esercize 61. Compute:

$$\lim_{x\to 0} \frac{e^{3x^2} - \cos(4x)}{\ln(1+9x^2)}.$$

Try to use notable limits:

$$\begin{aligned}
\frac{e^{3x^2} - \cos(4x)}{\ln(1+9x^2)} &= \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \frac{e^{3x^2} - 1 + 1 - \cos(4x)}{\ln(1+9x^2)} \\
&= \frac{\frac{e^{3x^2}-1}{x^2} + \frac{1-\cos(4x)}{x^2}}{\frac{\ln(1+9x^2)}{x^2}} \\
&= \frac{3\,\frac{e^{3x^2}-1}{3x^2} + 16\,\frac{1-\cos(4x)}{(4x)^2}}{9\,\frac{\ln(1+9x^2)}{9x^2}} \\
&= \frac{3\,\overbrace{\frac{e^{3x^2}-1}{3x^2}}^{\substack{1\\ \uparrow}} + 16\,\overbrace{\frac{1-\cos(4x)}{(4x)^2}}^{\substack{\frac{1}{2}\\ \uparrow}}}{9\,\underbrace{\frac{\ln(1+9x^2)}{9x^2}}_{\substack{\downarrow\\ 1}}} \to \frac{3+16\,\frac{1}{2}}{9} = \frac{11}{9}.
\end{aligned}$$

14.2 Types of Discontinuities

Let $f : \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}$ and let x_0 be a limit point of $\mathcal{D}$. Note that x_0 **may or may not** belong to $\mathcal{D}$, which is the domain of the function. We can have three types of discontinuities in x_0, depending on whether the left and right limits in x_0 exist or not. The following is the standard classification:

1. $\exists L_1 = \lim_{x\to x_0^-} f(x)$ and $\exists L_2 = \lim_{x\to x_0^+} f(x)$ but

   $$L_1 \neq L_2,$$

   in this case we say that the function has a **jump** discontinuity in x_0.

2. If it happens that

   $$\exists L = \lim_{x\to x_0} f(x),$$

   but either $x^* \notin \mathcal{D}$ or $x^* \in \mathcal{D}$ but $L \neq f(x^*)$. In this case we say that the function has a **removable** discontinuity in x_0.

3. Either $\nexists \lim_{x\to x_0^-} f(x)$ (or the limit exists but is not finite) or $\nexists \lim_{x\to x_0^+} f(x)$ (or the limit exists but is not finite) or both simultaneously. In this case we say that the function has an **essential** discontinuity in x_0.

   For example
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Esercize 62. Find the value of α s.t. the function:
$$f(x) = \begin{cases} \alpha \frac{\sin(x)}{x} & x > 0 \\ 2x^2 + 3 & x \leq 0 \end{cases}.$$
is continuous in $x = 0$.
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \alpha \frac{\sin(x)}{x} = \alpha.$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x^2 + 3 = 3.$$
Therefore $\alpha = 3$.

Esercize 63. Classify the type of discontinuity of:
$$f(x) = \begin{cases} \frac{\sin^2(x)\cos\left(\frac{1}{x}\right)}{e^x - 1} & x < 0 \\ \ln(1+x) & x \geq 0 \end{cases}.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin^2(x)\cos\left(\frac{1}{x}\right)}{e^x - 1} = \lim_{x\to 0^-} \frac{\frac{\sin^2(x)}{x^2} x^2 \cos\left(\frac{1}{x}\right)}{e^x - 1} = \lim_{x\to 0^-} \frac{\overbrace{\frac{\sin^2(x)}{x^2}}^{\to 1} \overbrace{x\cos\left(\frac{1}{x}\right)}^{\to 0}}{\underbrace{\frac{e^x-1}{x}}_{\to 1}} = 0.$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \ln(1+x) = 0.$$
Therefore f is continuous.

Esercize 64. Find α and β such that:
$$f(x) = \begin{cases} x - \alpha & x \leq 0 \\ \left|\beta - x^2\right| & x > 0 \end{cases}.$$
is a continuous function.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x - \alpha = -\alpha.$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left|\beta - x^2\right| = |\beta|.$$
Hence $|\beta| = -\alpha \Rightarrow \alpha < 0$ and moreover the function is continuous for each couple of parameters such that $\alpha = -|\beta|$.

Esercize 65. Find α such that:

$$f(x) = \begin{cases} x^2 - 2x + 3\alpha - 4 & x \le 0 \\ \frac{\sin(\alpha x)}{x} & x > 0 \end{cases}.$$

is a continuous function.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin(\alpha x)}{x} = \lim_{x\to 0^+} \alpha \frac{\sin(\alpha x)}{\alpha x} = \alpha.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 - 2x + 3\alpha - 4 = 3\alpha - 4.$$

Hence $f(x)$ is continuous for $\alpha = 3\alpha - 4 \Rightarrow \alpha = 2$.

Esercize 66. Classify the discontinuities of:

$$f(x) = \begin{cases} |1 - x| & |x| \ge 2 \\ \ln(2 - |x|) & |x| < 2 \end{cases}.$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} |1 - x| = |1 - 2| = 1.$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \ln(2 - |x|) = -\infty.$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \ln(2 - |x|) = \lim_{x\to -2^+} \ln(2 + x) = -\infty.$$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} |1 - x| = 3.$$

$x_\pm = \pm 2$ are discontinuity point of type I.

15 Main Theorems on Continuity

Theorem 15.1. Intermediate value theorem Let $I = [a, b]$ be an interval and $f : I \to \mathbb{R}$ be a continuous function. Suppose, without loss of generality, that $f(a) < f(b)$. Then

$$\forall \xi \in (f(a), f(b))\ \exists x_\xi \in (a, b) \text{ such that } f(x_\xi) = \xi.$$

Proof. Consider the set

$$S = \{x \in [a, b] \mid f(x) < \xi\}.$$

Since $f(a) < \xi$ we can say that $a \in S$. Besides $\forall x \in S$ then $x \le b$. Hence S is a non-empty bounded subset of $\mathbb{R}$, so it exists finite $c = \sup S$. Besides since c is the minimum upper bounds we get $c \le b$. Finally since $a \in S$ and c is an upper bound for S we get also that $a \le c$, whence in conclusion

$c \in [a, b]$. Now what I want to prove is that $f(c) = \xi$. Fix some $\varepsilon > 0$, hence by continuity of f there exists a $\delta > 0$ such that

$$\forall x \in (c - \delta, c + \delta) \Rightarrow f(x) - \varepsilon < f(c) < f(x) + \varepsilon. \tag{15.1}$$

Now recall that c is a supremum for S so there exists a $c^* \in (c - \delta, c]$ which is still in S. So this c^* has two important properties. First it belongs to S and then

$$f(c^*) < \xi.$$

Second c^* is in the interval $(c - \delta, c + \delta)$ and so

$$f(c^*) - \varepsilon < f(c) < f(c^*) + \varepsilon. \tag{15.2}$$

So we can use the following property

$$f(c^*) < \xi \Rightarrow f(c^*) + \varepsilon < \xi + \varepsilon$$

with the right inequality in (14.2) to get

$$f(c) < f(c^*) + \varepsilon < \xi + \varepsilon$$

Now I take a point $c^{**} \in [c, c + \delta)$ so for sure $f(c^{**}) \geq \xi$ because c is the supremum of S. But again using the continuity condition in (14.1) we can say that $f(c) > f(c^{**}) - \varepsilon$. So we can conclude by saying

$$\xi - \varepsilon \leq f(c^{**}) - \varepsilon < f(c) < f(c^*) + \varepsilon < \xi + \varepsilon$$

or more simply

$$\xi - \varepsilon < f(c) < \xi + \varepsilon.$$

Since ε is arbitrarily small we get $f(c) = \xi$. □

Remark. The Intermediate Value Theorem has an interesting application in economics, it is used to prove the existence of Nash equilibria.

Definition 32. Let $f : I \to \mathbb{R}$ be a function. Suppose that it exists $m \in I$ such that

$$\forall x \in I \Rightarrow f(x) \geq f(m).$$

In this case we call $f(m)$ the minimum value of f in I. An identical definition holds for the maximum.

Theorem 15.2. Let $f : [a, b] \to \mathbb{R}$ be a function defined on the closed limited interval $[a, b]$. If f is continuous on $[a, b]$ then f attains a minimum and a maximum on $[a, b]$.

Proof. First of all we prove that f is bounded both from below and form above. Suppose, by contradiction, that f is not bounded, for example is not bounded from above (an identical reasoning applies for the unboundedness from below). Hence for all $n \in \mathbb{N}$ there exits a $x_n \in [a, b]$ such that $f(x_n) > n$. For the Bolzano-Weierstrass theorem, that is Theorem 11.11, we can say that there exists x_{n_k} such that $x_{n_k} \to x^* \in [a, b]$. From continuity of f we derive that $f(x_{n_k}) \to f(x^*)$. But also $f(n_k) > n_k \geq k$ which implies that $f(x_{n_k}) \to +\infty$, which is a contradiction.

Hence we conclude that f is bounded from above, hence

$$M = \sup_{x \in [a,b]} f(x) < \infty.$$

Now let n be again a natural number. By the properties of the supremum we can say that $\exists x_n \in [a, b]$ such that

$$M - \frac{1}{n} \leq f(x_n) \leq M.$$

Hence $f(x_n) \to M$. Again by the Bolzano-Weierstrass theorem we can say that there exists x_{n_k} such that $x_{n_k} \to x^* \in [a, b]$. By continuity of f we get that $f(x_{n_k}) \to f(x^*)$. Nevertheless $f(x_{n_k})$ is a sub-sequence of $f(x_n)$ and hence $f(x^*) = M$, that is f attains a maximum. Similar reasonings hold for the minimum. □

16 Derivatives, L'Hopital, Taylor and Local Extrema

Definition 33. A function $f : \mathcal{D} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be differentiable in $x_0 \in \operatorname{int}(\mathcal{D})$ ($\operatorname{int}(\mathcal{D})$ denotes the set of the interior points of $\mathcal{D}$) if:

$$\exists L = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We call $L = f'(x_0)$. We define the left and right derivative of f in x_0 the two limits:

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h}, \quad \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h},$$

when they exist and we call them $f'(x_0^-)$ and $f'(x_0^+)$ respectively. Note that the left and right derivatives can be defined even if x_0 belongs to the closure of $\mathcal{D}$.

Theorem 16.1. If f is differentiable in x_0 then it is continuous in x_0.

Proof. Immediate since the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

exists and it is finite hence

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \cdot 0 = 0. \quad \square
\end{aligned} \tag{16.1}$$

Observation 5. Consider a function f differentiable in x_0 and consider the straight line that passes through the points $A = (x_0, f(x_0))$ and $B = (x_0 + h, f(x_0 + h))$. The generic equation of the line is $y = m\,x + b$, so imposing that the line passes through A and B is equivalent to impose the two conditions

$$\begin{cases} f(x_0) = m\,x_0 + b \\ f(x_0 + h) = m(x_0 + h) + b, \end{cases}$$

subtracting the first from the second we get

$$f(x_0 + h) - f(x_0) = m\,x_0 + m\,h + b - m\,x_0 - b = m\,h$$

that is

$$\frac{f(x_0 + h) - f(x_0)}{h} = m$$

when $h \to 0$ the line approaches the tangent to the graph of f in x_0, so the derivative $f'(x_0)$ represents the angular coefficient of the tangent of f in x_0.

Observation 6. Consider the set of functions defined in a open set $\mathcal{I}$ and differentiable infinite times on $\mathcal{I}$. We call this set $C^\infty(\mathcal{I})$. The operator:

$$D : C^\infty(\mathcal{I}) \to C^\infty(\mathcal{I}),$$

defined as:

$$D(f) = f'.$$

is a linear operator. That is

- $D(f + g) = D(f) + D(g)$,

- $D(c \cdot f) = c \cdot D(f)$, $c \in \mathbb{R}$ a given constant.

Proof. Take two $f, g \in C^{\infty}(\mathcal{I})$, $x \in \mathcal{I}$ and $\alpha, \beta \in \mathbb{R}$. Therefore:

$$\begin{aligned}
\lim_{h\to 0} \frac{(\alpha f + \beta g)(x+h) - (\alpha f + \beta g)(x)}{h} &= \lim_{h\to 0} \frac{\alpha f(x+h) + \beta g(x+h) - \alpha f(x) - \beta g(x)}{h} \\
&= \lim_{h\to 0} \frac{\alpha f(x+h) - \alpha f(x) + \beta g(x+h) - \beta g(x)}{h} \\
&= \lim_{h\to 0} \left[\alpha \frac{f(x+h) - f(x)}{h} + \beta \frac{g(x+h) - g(x)}{h}\right].
\end{aligned}$$

We know that:

$$\exists L = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

and we know that:

$$\exists P = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x).$$

Therefore:

$$\begin{aligned}
\lim_{h\to 0} \frac{(\alpha f + \beta g)(x+h) - (\alpha f + \beta g)(x)}{h} &= \alpha \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \beta \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \alpha f'(x) + \beta g'(x).
\end{aligned}$$

□

Observation 7. Consider two functions f and g both differentiable in the open set $\mathcal{I}$. Therefore:

$$\begin{aligned}
\lim_{h\to 0} \frac{(f g)(x+h) - (f g)(x)}{h} &= \lim_{h\to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x+h) + f(x)\, g(x+h) - f(x)\, g(x)}{h} \\
&= \lim_{h\to 0} \frac{g(x+h)\, (f(x+h) - f(x)) + f(x)\, (g(x+h) - g(x))}{h}.
\end{aligned}$$

Because we know that both functions are differentiable:

$$\begin{aligned}
\lim_{h\to 0} \frac{(f g)(x+h) - (f g)(x)}{h} &= \lim_{h\to 0} \frac{g(x+h)\, (f(x+h) - f(x))}{h} + \lim_{h\to 0} \frac{f(x)\, (g(x+h) - g(x))}{h} \\
&= g(x)\, f'(x) + f(x)\, g'(x).
\end{aligned}$$

That is:

$$(f g)' = f' g + f g'.$$

Moreover if it is possible to compose f and g, i.e. it is possible to define $f \circ g$ it can be shown that:

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

Esercize 67. Compute the derivative of $f(x) = K$ where K is a constant.

$$f'(x) = \lim_{h\to 0} \frac{K - K}{h} = 0.$$

Esercize 68. Compute the derivative of $f(x) = x$.

$$f'(x) = \lim_{h\to 0} \frac{x + h - x}{h} = 1.$$

Esercize 69. Compute the derivative of $f(x) = x^n$ for $n \in \mathbb{N}, n > 1$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\begin{pmatrix} n \\ 0 \end{pmatrix} x^n + \begin{pmatrix} n \\ 1 \end{pmatrix} x^{n-1} h + \begin{pmatrix} n \\ 2 \end{pmatrix} x^{n-2} h^2 + \cdots + \begin{pmatrix} n \\ n \end{pmatrix} h^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{x^n + \begin{pmatrix} n \\ 1 \end{pmatrix} x^{n-1} h + \begin{pmatrix} n \\ 2 \end{pmatrix} x^{n-2} h^2 + \cdots + h^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\begin{pmatrix} n \\ 1 \end{pmatrix} x^{n-1} h + \begin{pmatrix} n \\ 2 \end{pmatrix} x^{n-2} h^2 + \cdots + h^n}{h} \\
&= \lim_{h\to 0} \begin{pmatrix} n \\ 1 \end{pmatrix} x^{n-1} + \begin{pmatrix} n \\ 2 \end{pmatrix} x^{n-2} h + \cdots + h^{n-1}.
\end{aligned}$$

Now we have that: $\lim_{h\to 0} h = \lim_{h\to 0} h^2 = \lim_{h\to 0} h^3 = \cdots = \lim_{h\to 0} h^{n-1} = 0$. Therefore:

$$f'(x) = \begin{pmatrix} n \\ 1 \end{pmatrix} x^{n-1} = \frac{n!}{1!\,(n-1)!} x^{n-1} = n\, x^{n-1}.$$

Esercize 70. Compute the derivative of $f(x) = a^x$ for $a > 0, a \neq 1$.

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x \ln(a).$$

In particular $f(x) = e^x \Rightarrow f'(x) = e^x$.

Esercize 71. Compute the derivative of $f(x) = \sin x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin x \, \cos h + \cos x \, \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \left[\sin x \underbrace{\frac{\cos h - 1}{h}}_{\downarrow \atop 0} + \cos x \underbrace{\frac{\sin h}{h}}_{\downarrow \atop 1} \right] = \cos x. \end{aligned}$$

Esercize 72. Compute the derivative of $f(x) = \cos(x)$.

We know that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$:

$$\begin{aligned} f'(x) &= \frac{d\cos x}{dx} = \frac{d\sin\left(x+\frac{\pi}{2}\right)}{dx} \\ \left(g(x) = x + \frac{\pi}{2}\right) &= \frac{d\sin g}{dg} \frac{d\left(x+\frac{\pi}{2}\right)}{dx} = \cos u = \cos\left(x + \frac{\pi}{2}\right) = -\sin x. \end{aligned}$$

Esercize 73. Compute the derivative of $f(x) = |x|$.

Using the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h}.$$

If $x > 0$ then for h small enough I get:

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = 1.$$

Viceversa If $x < 0$ then for h small enough I get:

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \lim_{h\to 0} \frac{-x-h+x}{h} = -1.$$

For $x = 0$:

$$\begin{aligned} f'(0^+) &= \lim_{h\to 0^+} \frac{|h|}{h} = 1. \\ f'(0^-) &= \lim_{h\to 0^-} \frac{|h|}{h} = -1. \end{aligned} \tag{16.2}$$

Therefore the function is not differentiable in $x = 0$ and we can say that:

$$\frac{d}{dx}|x| = \frac{|x|}{x} = \frac{x}{|x|}.$$

The point $x = 0$ it's called an **angle point**. We say that the function $f(x)$ has an angle point in x_0 whenever the derivative has a jump discontinuity in x_0.

Esercize 74. Compute the derivative of $f(x) = \log_a(x)$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\log_a(x+h) - \log_a x}{h} \\
&= \lim_{h\to 0} \frac{\log_a\left(\frac{x+h}{x}\right)}{h} \\
&= \lim_{h\to 0} \log_a \left(\frac{x+h}{x}\right)^{\frac{1}{h}} \\
&= \lim_{h\to 0} \log_a \left[\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]^{\frac{1}{x}} \\
&= \lim_{h\to 0} \frac{1}{x} \log_a \left(1+\frac{h}{x}\right)^{\frac{x}{h}}.
\end{aligned}
$$

Now compute:

$$
\begin{aligned}
\lim_{h\to 0} \log_a \left(1+\frac{h}{x}\right)^{\frac{x}{h}} &= \lim_{t\to 0} \log_a (1+t)^{\frac{1}{t}} \\
&= \lim_{q\to\infty} \log_a \left(1+\frac{1}{q}\right)^q \\
&= \log_a\left(\lim_{q\to\infty}\left(1+\frac{1}{q}\right)^q\right) = \log_a e.
\end{aligned}
$$

As a consequence:

$$\frac{d \log_a x}{dx} = \frac{\log_a e}{x} = \frac{1}{x \ln a}.$$

In particular:

$$\frac{d \ln x}{dx} = \frac{1}{x}.$$

Esercize 75. Compute the derivative of $f(x) = x^\alpha$, with $\alpha \in \mathbb{R}$, $\alpha \neq 1$.

Let's write x^α as:

$$x^\alpha = e^{\alpha \ln x} = e^{g(x)} \Rightarrow g'(x) = \frac{\alpha}{x}.$$

Thus:

$$\frac{d}{dx} x^\alpha = \frac{d}{dx} e^{g(x)} = e^{g(x)} \frac{d}{dx} g(x) = x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

Esercize 76. Consider the function:

$$f(x) = \begin{cases} \sqrt{x} & x > 0 \\ -\sqrt{-x} & x \leq 0. \end{cases} \tag{16.3}$$

What can be said about the derivative of f in $x = 0$?

$$x > 0 \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}.$$

$$x < 0 \Rightarrow f'(x) = \frac{1}{2\sqrt{-x}}.$$

Therefore:

$$\lim_{x \to 0^{\pm}} f'(x) = +\infty.$$

This corresponds to a **flex point** in the graph of the function at $x = 0$.

Esercize 77. Consider the function:

$$f(x) = \sqrt{|x|}.$$

What can be said about the derivative of f in $x = 0$?

$$f'(x) = \frac{1}{2\sqrt{|x|}} \frac{|x|}{x}.$$

Therefore:

$$\lim_{h \to 0^{\pm}} f'(x) = \pm\infty.$$

In this case the graph of the function has a **cusp** in $x = 0$. We say that the function has a cusp in $x = x_0$ every time that both left and right derivatives diverge in $x = x_0$ one (left or right) to $+\infty$ and the other (right or left) to $-\infty$.

17 Main Theorems on Derivatives

Theorem 17.1. Rolle's Theorem Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$ then $\exists \xi \in (a,b)$ such that $f'(\xi) = 0$.

Proof. Since f is continuous on $[a,b]$ therefore f has a minimum and a maximum on $[a,b]$. If both are attained not in an interior point then since $f(a) = f(b)$ we get that f is constant and therefore $f'(\xi) = 0$ for all $\xi \in (a,b)$. Suppose then that at least the maximum is attained in an interior point $\xi \in (a,b)$. Hence for sufficiently small $h > 0$ we have that $\xi + h \in (a,b)$ and hence $f(\xi + h) \le f(\xi)$ or

$$\frac{f(\xi + h) - f(\xi)}{h} \le 0 \Rightarrow f'(\xi^{+}) \le 0.$$

Now if $h < 0$ is sufficiently small we have that $\xi + h \in (a,b)$ and still $f(\xi + h) \le f(\xi)$. But now since $h < 0$ the denominator in the previous inequality is negative and thus

$$\frac{f(\xi + h) - f(\xi)}{h} \ge 0 \Rightarrow f'(\xi^{-}) \ge 0.$$

Nevertheless f is differentiable in ξ hence $f'(\xi) = f'(\xi^-) = f'(\xi^+) = 0$. □

Theorem 17.2. Lagrange's Mean Value Theorem. If f is a real continuous function on $[a,b]$ which is differentiable in (a,b) then there is a point $\xi \in (a,b)$ at which

$$f(b) - f(a) = (b-a)\, f'(\xi).$$

Geometrically, the whole slope that the function has on the interval $[a,b]$ is reached by the derivative of the function in some point of the interval.

Proof. Consider the function

$$g(x) = f(x) - r\,x.$$

We want to find r such that $g(a) = g(b)$, hence

$$\begin{aligned} g(a) = g(b) \quad &\Leftrightarrow \quad f(a) - r\,a = f(b) - r\,b & (17.1)\\ &\Leftrightarrow \quad r\,(b-a) = f(b) - f(a) & (17.2)\\ &\Leftrightarrow \quad r = \frac{f(b)-f(a)}{b-a}. & (17.3) \end{aligned}$$

Since f is continuous in $[a,b]$ and differentiable in (a,b) it follows that g has the same properties. Besides since $g(a) = g(b)$ then g satisfies the Rolle's Theorem and hence $\exists \xi \in (a,b)$ such that $g'(\xi) = 0$ that is

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}. \quad \square$$

Theorem 17.3. Cauchy's Mean Value Theorem Let f and g be two functions continuous in $[a,b]$ and differentiable in (a,b). Then there exists a $\xi \in (a,b)$ such that

$$(f(b) - f(a))\, g'(\xi) = (g(b) - g(a))\, f'(\xi).$$

Proof. Suppose first that $g(b) \neq g(a)$. Define $h(x) = f(x) - r\,g(x)$ in such a way that $h(a) = h(b)$ that is

$$\begin{aligned} h(a) = h(b) \quad &\Leftrightarrow \quad f(a) - r\,g(a) = f(b) - r\,g(b) & (17.4)\\ &\Leftrightarrow \quad r\,(g(b) - g(a)) = f(b) - f(a) & (17.5)\\ &\Leftrightarrow \quad r = \frac{f(b)-f(a)}{g(b)-g(a)}. & (17.6) \end{aligned}$$

Since f and g are continuous in $[a,b]$ and differentiable in (a,b) then the same holds for h and so we can apply Rolle's Theorem to h obtaining that there exists a $\xi \in (a,b)$ such that $h'(\xi) = 0$. But $h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g'(x)$ and hence

$$0 = h'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(\xi)$$

whence the thesis. The case $g(a) = g(b)$ is trivial since it is enough to apply Rolle's Theorem to g to get the claimed identity (which is a void $0 = 0$). □

18 Characterization of discontinuities of derivatives.

The following theorems asserts that a derivative of a function cannot be "two much irregular":

Theorem 18.1. Let f be differentiable in (a, b), then f' cannot have any jump discontinuity on (a, b).

Proof. Let $c \in (a, b)$. We know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite. Suppose that the limits

$$\lim_{x \to c^+} f'(x) = A, \lim_{x \to c^-} f'(x) = B$$

exist and are finite (if one of the two is infinite or does not exist then it is an essential discontinuity). Now let's handle the case for $x \to c^+$ first. Clearly then $x > c$ and we have

$$\frac{f(x) - f(c)}{x - c} = f'(d)$$

for some $d \in (c, x)$. As $x \to c^+$ also $d \to c^+$, hence

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{d \to c^+} f'(d) = A$$

Similarly by considering $x \to c^-$ we can show that $B = f'(c)$ and then $A = B$ and $f'(x)$ is continuous at c and therefore does not have jump discontinuity. It may happen however that one or both of the limits A, B don't exist or are $\pm\infty$. □

Esercize 78. Find a function f differentiable on (a, b) but such that its derivative has an essential discontinuity.

Solution. Consider:

$$f(h) = \begin{cases} h^2 \sin\left(\frac{1}{h}\right) & h \neq 0, \\ 0 & h = 0. \end{cases}$$

It is obvious that f is continuous and differentiable for all $h \neq 0$ and observing that:

$$\lim_{h \to 0} f(h) = 0,$$

we find that $f(h)$ is continuous everywhere. Moreover it is obvious that $f(h)$ is differentiable for all $h \neq 0$ with derivative:

$$f'(h) = 2h \sin\left(\frac{1}{h}\right) - \cos\left(\frac{1}{h}\right).$$

Moreover noticing that:

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0.$$

we find that f is differentiable even in $h = 0$ with derivative $f'(0) = 0$. Nevertheless:

$$\lim_{h\to 0} f'(h) = \lim_{h\to 0} \left[2h \sin\left(\frac{1}{h}\right) - \cos\left(\frac{1}{h}\right)\right],$$

which does not exist, i.e. $f'(h)$ has an essential discontinuity in $x_0 = 0$.

19 Derivatives: Monotonicity and Local Extrema

Theorem 19.1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be defined in an open interval (that can be either limited or unlimited). Suppose f is differentiable in I. If $f'(x) \geq 0$ (resp. $f'(x) \leq 0$) for all x in I then f is increasing (resp. decreasing) in I. Vice versa if f is increasing (resp. decreasing) in I then $f'(x) \geq 0$ (resp. $f'(x) \leq 0$) for all x in I.

Proof. Consider $x_1, x_2 \in I$ with $x_1 < x_2$. For the mean value theorem there exists a $\xi \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(\xi)\,(x_2 - x_1).$$

If $f'(\xi) \geq 0$ it follows that $f(x_1) \leq f(x_2)$, vice versa if $f'(\xi) \leq 0$ it follows that $f(x_2) \leq f(x_1)$.

Now assume that f is increasing and let x_0 be an interior point. For positive h we have $x_0 \leq x_0 + h$

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0 \Rightarrow f'(x_0^+) \geq 0.$$

Similarly for negative h we have $h = -|h|$ and hence

$$-\frac{f(x_0 - |h|) - f(x_0)}{|h|} = \frac{f(x_0) - f(x_0 - |h|)}{|h|} \geq 0 \Rightarrow f'(x_0^-) \geq 0$$

whence $f'(x_0) \geq 0$ (the case f decreasing is identical). □.

Definition 34. We say that a function f reaches a local minimum in x_0 if $f(x) \geq f(x_0)$ for x sufficiently close to x_0 (similarly for a local maximum). More formally, a function f reaches a local minimum (resp. maximum) in x_0 if there exists a $\delta > 0$ such that for all x in the open interval $(x_0 - \delta, x_o + \delta)$ it happens that $f(x) \geq f(x_0)$ (resp.$f(x) \leq f(x_0)$).

Intuitively, local minima/maxima occur where the graph of a function f bottoms out/tops out (locally). Collectively, local minima and maxima are known as local extrema.

Theorem 19.2. (*Fermat's Theorem*) Suppose that $f(x)$ is a function defined on an interval I that surrounds x_0, that is x_0 is an interior point of I. If $f(x)$ reaches a local minimum or a local maximum in x_0, then or $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Proof. Suppose that f is differentiable in x_0 then $f'(x_0)$ cannot be strictly positive, because this would mean that f were increasing and the function values for arguments immediately on the left of x_0 would be smaller than $f(x_0)$ and values immediately on the right of x_0 would be larger than $f(x_0)$, thus f would neither peak nor bottom out at x_0. Trivially, with the same argument we can show that $f'(x_0)$ cannot be strictly negative. Therefore the remaining possibilities are $f'(x_0) = 0$ or $f'(x_0)$ does not exist. □

Definition 35. If a function f is differentiable in x_0 and $f'(x_0) = 0$ then the point x_0 is called a **critical point**.

Esercize 79. Find the critical points of:

$$f(x) = 2\cos x - x.$$

First compute:

$$f'(x) = -2\sin x - 1.$$

Then look for solutions of $f'(x) = 0$:

$$f'(x) = 0 \Leftrightarrow \sin x = -\frac{1}{2}.$$

There are infinite solutions:

$$x_m = \frac{7}{6}\pi \pm 2\,m\,\pi, \quad y_m = \frac{11}{6}\pi \pm 2\,m\,\pi.$$

More over $f'(x)$ is negative in a left neighborhood of x_m and positive in a right neighborhood of x_m. Vice versa for y_m. Therefore x_m are local minima and y_m are local maxima.

Esercize 80. Find the critical points of:

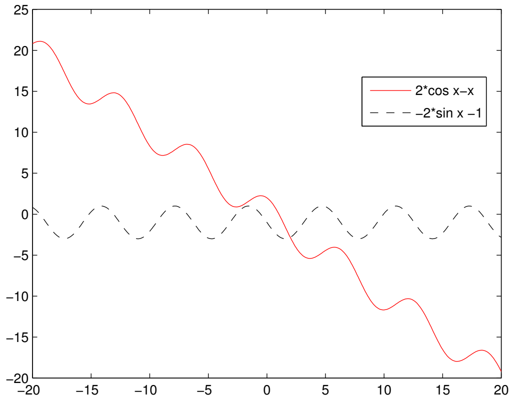
$$f(x) = x^{\frac{1}{x}}$$

In order to compute the derivative note that:

$$y(x) = \ln f(x) = \frac{1}{x}\ln x.$$

Taking the derivative on both sides:

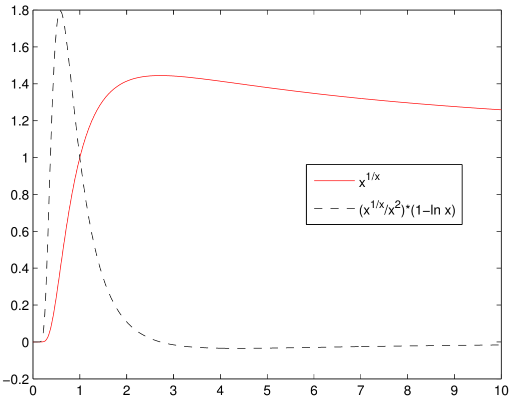
$$y'(x) = \frac{f'(x)}{f(x)} = \frac{1}{x^2} - \frac{1}{x^2}\ln x.$$



Obtaining:

$$f'(x) = f(x)\left(\frac{1}{x^2} - \frac{1}{x^2}\ln x\right) = \frac{x^{\frac{1}{x}}}{x^2}(1 - \ln x).$$

The equation $f'(x) = 0$ has the only solution $x = e$. Moreover $f'(x) > 0$ for $x < e$ and $f'(x) < 0$ for $x > e$. Therefore e is a local maximum.

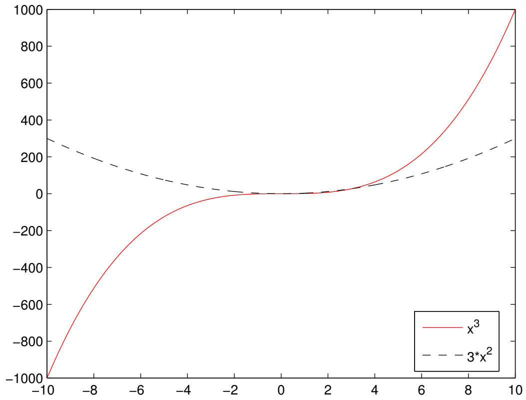


Esercize 81. Is the converse of the Fermat's theorem on local extrema true?

No. Take $f(x) = x^3$. Then $f'(x) = 3x^2$. Nevertheless $f'(x) = 3x^2 > 0$ bot for $x < 0$ and $x > 0$. Thus $x = 0$ is neither a local minimum nor a local maximum.

Remark.

In order to verify that a point x_0 is a local minimum/maximum we have to check not only that $f'(x_0) = 0$ but also that the derivative changes sign in x_0.



Esercize 82. Find a function f such that $f(x_0)$ is a local minimum or a local maximum, while $f'(x_0)$ does not exist.

The absolute value $f(x) = |x| \geq 0 \quad \forall x$ has a global (and hence also local) minimum in $x = 0$, nevertheless:

$$\begin{aligned} f'(0^+) &= \lim_{h \to 0^+} \frac{|0+h| - 0}{h} = 1 \\ f'(0^-) &= \lim_{h \to 0^+} \frac{|0+h| - 0}{h} = -1, \end{aligned}$$

i.e. $f'(0)$ does not exist.

Esercize 83. Find the critical points of:

$$f(x) = x^x$$

In order to compute the derivative note that:

$$y(x) = \ln f(x) = x \ln x.$$

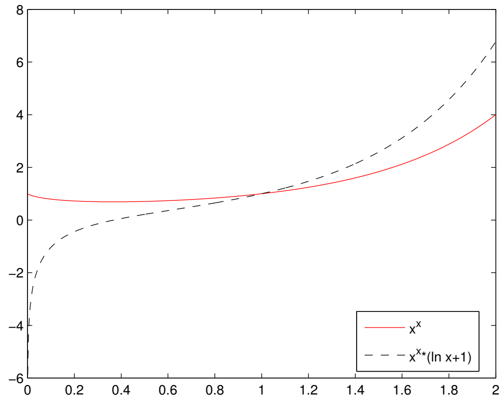
Taking the derivative on both sides:

$$y'(x) = \frac{f'(x)}{f(x)} = \ln x + 1.$$

Obtaining:

$$f'(x) = f(x)(\ln x + 1) = x^x (\ln x + 1).$$

Therefore $f'(x) = 0$ implies $x = \frac{1}{e}$. Moreover $f'(x) < 0$ if $x > \frac{1}{e}$ and $f'(x) > 0$ if $x < \frac{1}{e}$. As a consequence $x_0 = \frac{1}{e}$ is a local minimum.



20 Concavity and Convexity

Definition 36. A function $f : E \subseteq \mathbb{R} \to \mathbb{R}$ is said to be concave in the interval $I \subseteq E$ if for all x and y in I it holds that

$$f\left((1-\alpha)\,x + \alpha\,y\right) \geq (1-\alpha)\,f(x) + \alpha\,f(y)\,, \forall \alpha \in [0,1]$$

i.e. if the graph of the function is above the segment that joins $(x, f(x))$ with $(y, f(y))$.

Definition 37. A function $f : E \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex in the interval $I \subseteq E$ if for all x and y in I it holds that

$$f\left((1-\alpha)\,x + \alpha\,y\right) \leq (1-\alpha)\,f(x) + \alpha\,f(y)\,, \forall \alpha \in [0,1]$$

i.e. if the graph of the function is below the segment that joins $(x, f(x))$ with $(y, f(y))$.

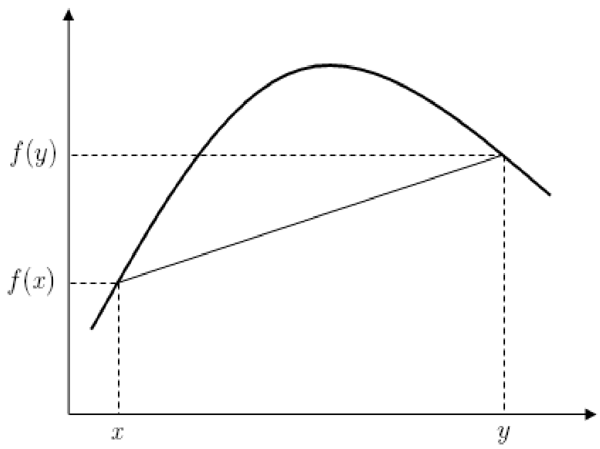


Fig. 9: A concave function

Let's focus on properties of convex functions, the analogous properties for concave functions are derived immediately by an appropriate change of the sign of inequalities.

Theorem 20.1. A function f is convex on the interval I if and only if

$$\forall x_1, x_2, x_3 \in I : x_1 < x_2 < x_3 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}. \tag{20.1}$$

or, equivalently, if and only if

$$\forall x_1, x_2, x_3 \in I : x_1 < x_2 < x_3 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}. \tag{20.2}$$

Proof. We want to use the definition, so we now that

$$\forall x, y \in I, \forall \alpha \in [0,1] \Rightarrow f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y). \tag{20.3}$$

Since $x_3 - x_2 < x_3 - x_1$ I define $\alpha = \frac{x_3 - x_2}{x_3 - x_1}$ and hence $\alpha \in (0,1)$. Then I put, in the definition (19.3), $x = x_1$ and $y = x_3$ and I obtain

$$\begin{aligned}
f(\alpha x + (1-\alpha) y) &= f\left(\frac{x_3 - x_2}{x_3 - x_1} x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3\right) \\
&= f\left(\frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3\right) \\
&= f\left(\frac{\cancel{x_3 x_1} - x_2 x_1 + x_2 x_3 - \cancel{x_1 x_3}}{x_3 - x_1}\right) \\
&= f\left(x_2 \frac{x_3 - x_1}{x_3 - x_1}\right) = f(x_2).
\end{aligned} \tag{20.4}$$

Hence we can say that

$$f(x_2) = f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) = \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3)$$

which implies

$$\begin{aligned}
& (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3) \\
\Leftrightarrow\ & (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_3 - x_3 + x_2 - x_1) f(x_3) \\
\Leftrightarrow\ & (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_3 - x_1 - (x_3 - x_2)) f(x_3) \\
\Leftrightarrow\ & (x_3 - x_2) f(x_3) + (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_3 - x_1) f(x_3) \\
\Leftrightarrow\ & (x_3 - x_2) f(x_3) - (x_3 - x_2) f(x_1) \leq (x_3 - x_1) f(x_3) - (x_3 - x_1) f(x_2) \\
\Leftrightarrow\ & \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}
\end{aligned} \tag{20.5}$$

Now we do similar computations as before

$$
\begin{aligned}
&(x_3 - x_1)\ f(x_2) \le (x_3 - x_2)\ f(x_1) + (x_2 - x_1)\ f(x_3) \\
\Leftrightarrow\ &(x_3 - x_1)\ f(x_2) \le (x_1 - x_1 + x_3 - x_2)\ f(x_1) + (x_2 - x_1)\ f(x_3) \\
\Leftrightarrow\ &(x_3 - x_1)\ f(x_2) \le (x_3 - x_1 - (x_2 - x_1))\ f(x_1) + (x_2 - x_1)\ f(x_3) \\
\Leftrightarrow\ &(x_3 - x_1)\ f(x_2) - (x_3 - x_1)\ f(x_1) \le -(x_2 - x_1) f(x_1) + (x_2 - x_1)\ f(x_3) \\
\Leftrightarrow\ &(x_3 - x_1)\ (f(x_2) - f(x_1)) \le (x_2 - x_1)\ (f(x_3) - f(x_1)) \\
\Leftrightarrow\ &\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_1)}{x_3 - x_1}
\end{aligned}
\tag{20.6}
$$

so summing up

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_1)}{x_3 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

As each step is an equivalence, the argument reverses throughout. □

Theorem 20.2. Let f be a function which is differentiable on the open interval (a, b). Then f is convex on (a, b) if and only if f' is increasing on (a, b).

Proof. Consider four points on (a, b) such that $a < x_1 < x_2 < x_3 < x_4 < b$. By the property (19.1) of convex functions (used two times) we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2} \le \frac{f(x_4) - f(x_3)}{x_4 - x_3}.$$

Now let $x_2 \to x_1^+$ and $x_3 \to x_4^-$ obtaining (since f is differentiable!)

$$f'(x_1) \le f'(x_4),$$

from the arbitrariness of x_1 and x_4 we get that f' is increasing on (a, b). Now assume that f' is increasing on (a, b). Consider three points x_1, x_2 and x_3 with $x_1 < x_2 < x_3$. By the mean value theorem

$$\exists \alpha \in (x_1, x_2) : \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\alpha)$$

and

$$\exists \beta \in (x_2, x_3) : \frac{f(x_3) - f(x_2)}{x_3 - x_2} = f'(\beta).$$

Since $\alpha < \beta$ then $f'(\alpha) \le f'(\beta)$ and hence

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

which is condition (19.1), hence f is convex. □

Corollario 20.3. Let f be a function which is twice differentiable on the open interval (a, b). Then f is convex if and only if $f''(x) \ge 0$ for all $x \in (a, b)$.

The Second Derivative Test. Consider a function f which is twice differentiable on the open interval (a,b) with continuous second derivative. Consider a point $x_0 \in (a,b)$ such that $f'(x_0) = 0$.

1. If $f'(x_0) = 0$ and $f''(x_0) > 0$ then x_0 is a local minimum.
2. If $f'(x_0) = 0$ and $f''(x_0) < 0$ then x_0 is a local maximum.

Esercize 84. Find minima/maxima of the following function

$$f(x) = \frac{\ln(x)}{x}$$

Solution. The function is defined in $D = \{x \in \mathbb{R} \mid x > 0\}$. The first derivative is

$$f'(x) = \frac{1}{x^2} - \frac{\ln(x)}{x^2} = \frac{1-\ln(x)}{x^2}.$$

Hence $f'(x) = 0$ if and only if $x = e$. Besides since

$$f''(x) = -\frac{1}{x^3} - 2\frac{1-\ln(x)}{x^3} = -\frac{3-2\ln(x)}{x^3}$$

we have that

$$f''(e) = -\frac{3-2\ln(e)}{e^3} = -\frac{1}{e^3} < 0,$$

whence $x = e$ is a local maximum.

Esercize 85. Find minima/maxima of the following function

$$f(x) = \ln(1-\ln(x)) - \ln(x).$$

Solution. The domain of the function is $D = (0,e)$. The first derivative is

$$f'(x) = \frac{1}{1-\ln(x)}\left(-\frac{1}{x}\right) - \frac{1}{x} = -\frac{1}{x}\left(\frac{1}{1-\ln(x)} + 1\right) = -\frac{1}{x}\frac{2-\ln(x)}{1-\ln(x)} = \frac{1}{x}\frac{2-\ln(x)}{\ln(x)-1}$$

hence $f'(x) = 0$ has a unique solution $x = e^2$, but since $e^2 > e$ then the critical point is outside of the domain. The function has no minimum no maximum in $(0,e)$.

Esercize 86. Find minima/maxima of the following function

$$f(x) = \ln(1+\ln(x)) - \ln(x).$$

Solution. The domain of the function is $D = (1/e, +\infty)$. The first derivative is

$$f'(x) = \frac{1}{x(1+\ln(x))} - \frac{1}{x} = \frac{1-1-\ln(x)}{x(1+\ln(x))} = -\frac{\ln(x)}{x(1+\ln(x))},$$

so that $f'(x) = 0$ if and only if $x = 1$. The second derivative is

$$\begin{aligned} f''(x) &= -\frac{1}{x^2\,(1+\ln(x))} + \frac{\ln(x)}{x^2\,(1+\ln(x))^2}\,(1+\ln(x)+1) \\ &= \frac{-(1+\ln(x)) + \ln(x)\,(2+\ln(x))}{x^2\,(1+\ln(x))^2} \\ &= \frac{-1-\ln(x)+2\ln(x)+(\ln(x))^2}{x^2\,(1+\ln(x))^2} \\ &= \frac{(\ln(x))^2+\ln(x)-1}{x^2\,(1+\ln(x))^2}. \end{aligned} \tag{20.7}$$

It follows that $f''(x) > 0$ if and only if

$$(\ln(x))^2 + \ln(x) - 1 > 0$$

which, putting $t = \ln(x)$, gives

$$t^2 + t - 1 > 0$$

whose solutions are

$$t < -\frac{1+\sqrt{5}}{2} \text{ or } t > \frac{-1+\sqrt{5}}{2}$$

that is

$$\ln(x) < -\frac{1+\sqrt{5}}{2} \text{ or } \ln(x) > \frac{-1+\sqrt{5}}{2}$$

or, equivalently,

$$x < e^{-\frac{1+\sqrt{5}}{2}} \text{ or } x > e^{\frac{-1+\sqrt{5}}{2}}$$

Since $\sqrt{5} \approx 2.2361$ then $\sqrt{5} - 1 > 0$ and hence

$$e^{\frac{-1+\sqrt{5}}{2}} > 1,$$

while, trivially, $e^{-\frac{1+\sqrt{5}}{2}} < 1$ hence $x = 1$ is in the region in which $f''(x) < 0$, i.e.

$$e^{-\frac{1+\sqrt{5}}{2}} < 1 < e^{\frac{-1+\sqrt{5}}{2}}$$

so that $x = e$ is a maximum.

Exercise: an economic application. A monopolistic manufacturer determines that in order to sell x units of its product, the price per unit, in dollars, must be

$$p(x) = p_0 - x$$

where p_0 is a given positive constant. The manufacturer is also aware that the total cost of producing x units is given by $c(x) = c_0 + \alpha\, x$, where c_0 is a given positive constant. The total revenues are defined as the quantity sold times the price per unit, so the total revenues are

$$R(x) = x\,p(x) = x\,p_0 - x^2.$$

The profit is defined as the difference between total revenues and total cost, so the profit is

$$\Pi(x) = R(x) - c(x) = x\,p_0 - x^2 - c_0 - \alpha\,x = x\,(p_0 - \alpha) - x^2 - c_0.$$

Note that in order to be admissible the quantity of good produced and sold x must be such that $0 < x < p_0$.

Determine

1. How many units must the company produce and sell in order to maximize profit? Does this problem have an admissible solution for any value of p_0 and α?
2. What is the maximum profit?
3. Which is the maximum value for cost parameter c_0 that guarantees a positive maximum profit?
4. What price per unit must be charged in order to make this maximum profit?

Answers.

1. $\Pi'(x) = p_0 - \alpha - 2\,x$. Hence $\Pi'(x_0) = 0$ is solved by

   $$p_0 - \alpha - 2\,x_0 = 0 \Leftrightarrow x_0 = \frac{p_0 - \alpha}{2}$$

   We note that we find a positive solution if and only if $p_0 > \alpha$ and, of course, if this is the case we get $0 < x_0 < p_0$. Is this a maximum or a minimum for $\Pi(x)$? Since $\Pi''(x) = -2$ hence we have a maximum, in particular $\Pi(x)$ is increasing for $x \le x_0$ and decreasing for $x \ge x_0$.

2. The maximum profit is

   $$\Pi(x_0) = \frac{p_0 - \alpha}{2}\,(p_0 - \alpha) - \frac{(p_0 - \alpha)^2}{4} - c_0 = \frac{(p_0 - \alpha)^2}{4} - c_0$$

3. Of course

   $$c_0 \le \frac{(p_0 - \alpha)^2}{4}$$

   and hence if $c_0 > \frac{(p_0-\alpha)^2}{4}$ it is not convenient to start the production.

4. $$p(x_0) = p_0 - \frac{p_0 - \alpha}{2} = \frac{1}{2}\,(p_0 + \alpha)$$

Esercize 87. A producers knows that, at time n, the cost of producing x kilos of rice is given by

$$c_n(x) = \ln\left(\alpha_n + x^2\right),$$

with $\alpha_n > 0$ for all n. Simultaneously, she knows that the revenues for selling x kilos of rice do not depend on time and are given by

$$r(x) = \ln(x).$$

Assume that the producer is a profit-maximizer, i.e. she decides the production x by maximizing the total profit

$$\pi_n(x) = r(x) - c_n(x) = \ln(x) - \ln\left(\alpha_n + x^2\right).$$

Answer the following questions.

1. Determine, at each time n, the optimal amount x_n of kilos of rice that must be produced and the optimal profit.
2. Assume $\alpha_n = \frac{1}{4^n}$. Which is the first date (i.e. the first n) in which the producer faces a strictly positive optimal profit? Which is the total amount of rice produced from the initial time ($n = 0$) to infinity ($n = \infty$)?

Solution. 1) In order to determine the optimal amount of rice we have to find the maximum profit. For this purpose we compute the first derivative

$$\pi_n'(x) = \frac{1}{x} - \frac{2x}{\alpha_n + x^2} = \frac{\alpha_n + x^2 - 2x^2}{x\left(\alpha_n + x^2\right)} = \frac{\alpha_n - x^2}{x\left(\alpha_n + x^2\right)}$$

so $\pi_n'(x) = 0$ if and only if $x_n = \pm\sqrt{\alpha_n}$, nevertheless only $x_n = +\sqrt{\alpha_n}$ is admissible as a solution, since the amount of rice produced must be a positive quantity. The second derivative is given by

$$\pi_n''(x) = \frac{-2x}{x\left(\alpha_n + x^2\right)} - \frac{\alpha_n - x^2}{x^2\left(\alpha_n + x^2\right)^2}\left(\alpha_n + x^2 + 2x^2\right)$$

in particular

$$\pi_n''(x_n) = -\frac{2\sqrt{\alpha_n}}{2\alpha_n\sqrt{\alpha_n}} < 0$$

whence $x_n = \sqrt{\alpha_n}$ is a maximum. The optimal profit is

$$\begin{aligned}
\pi_n(x_n) &= \frac{1}{2}\ln(\alpha_n) - \ln(2\alpha_n) \\
&= \frac{1}{2}\ln(\alpha_n) - \ln(\alpha_n) - \ln(2) \\
&= -\frac{1}{2}\ln(\alpha_n) - \ln(2) \\
&= \ln\left(\frac{1}{\sqrt{\alpha_n}}\right) + \ln\left(\frac{1}{2}\right) \\
&= \ln\left(\frac{1}{2\sqrt{\alpha_n}}\right).
\end{aligned} \tag{20.8}$$

2) If $\alpha_n = 1/4^n$ then the optimal profit is

$$\pi_n(x_n) = \ln\left(\frac{2^n}{2}\right)$$

so for $n = 0$ the profit is $\pi_n(x_0) = \ln(1/2) < 0$, for $n = 1$ the profit is $\pi_n(x_1) = \ln(1) = 0$ and for $n = 2$ we get $\pi_n(x_1) = \ln(2) > 0$. So the first date is $n = 2$.

The total amount of rice produced is

$$\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2.$$

Esercize 88. Find the point on the graph of $y = \sqrt{x}$ nearest to the point $(4,0)$.

Solution. The distance of (x,y) from $(4,0)$ is

$$d(x,y) = \sqrt{(x-4)^2 + y^2}$$

but since the point must belong to $y = \sqrt{x}$ we have to find the minimum of

$$f(x) = \sqrt{(x-4)^2 + x}$$

for $x \geq 0$. Hence

$$f'(x) = \frac{1}{2\sqrt{(x-4)^2 + x}}\,(2\,(x-4)+1) = \frac{1}{2\sqrt{(x-4)^2 + x}}\,(2\,x-7) = 0 \Leftrightarrow x = \frac{7}{2}.$$

Now compute

$$f''(x) = -\frac{1}{4\left((x-4)^2 + x\right)^{3/2}}\,(2\,x-7)^2 + \frac{1}{2\sqrt{(x-4)^2 + x}}\,2 = \frac{1}{\sqrt{(x-4)^2 + x}}\left(1 - \frac{1}{4}\,(2\,x-7)^2\right)$$

whence

$$f''\left(\frac{7}{2}\right) > 0$$

hence $x = \frac{7}{2}$ is the minimum and the minimal distance is

$$\sqrt{\left(\frac{7}{2}-4\right)^2 + \frac{7}{2}}.$$

Esercize 89. Find the minimum distance between the point $(0,2)$ and the curve $g(x) = 4 - x^2$.

Solution. Compute the (squared) distance between the point $(0,2)$ and a generic point on the graph $\left(x, 4-x^2\right)$ as

$$f(x) = (x-0)^2 + \left(4-x^2-2\right)^2 = x^2 + \left(2-x^2\right)^2$$

whence

$$f'(x) = 2x + 2\left(2-x^2\right)\cdot(-2x)$$

that is

$$f'(x) = 2x - 4x\left(2-x^2\right) = x\left(2-4\left(2-x^2\right)\right) = x\left(4x^2-6\right).$$

Hence the solutions of $f'(x)=0$ are $x=0$ and $x=\pm\sqrt{\frac{3}{2}}$. Consider now the second derivative

$$f''(x) = 12x^2 - 6.$$

Hence $f''(0)<0$ so $x=0$ is not a minimum, while $f''\left(\pm\sqrt{\frac{3}{2}}\right)=12$, so both $x=\pm\sqrt{\frac{3}{2}}$ are minima and the minimum distance is

$$\sqrt{f\left(\pm\sqrt{\frac{3}{2}}\right)} = \sqrt{\frac{3}{2}+\left(2-\frac{3}{2}\right)^2} = \sqrt{\frac{7}{4}}.$$

Esercize 90. Find the minimum distance between the point $(0,0)$ and the curve $g(x)=\frac{1}{\sqrt{x}}$.

Solution. Compute the (squared) distance between the point $(0,0)$ and a generic point on the graph $\left(x, \frac{1}{\sqrt{x}}\right)$ as

$$f(x) = (x-0)^2 + \left(\frac{1}{\sqrt{x}}-0\right)^2 = x^2 + \frac{1}{x}$$

whence

$$f'(x) = 2x - \frac{1}{x^2}.$$

Hence $f'(x)=0$ is equivalent to

$$2x^3 - 1 = 0 \Leftrightarrow x^3 = \frac{1}{2} \Leftrightarrow x = \frac{1}{2^{1/3}}.$$

The second derivative is

$$f''(x) = 2 + \frac{2}{x^3}$$

Hence $f''\left(\frac{1}{2^{1/3}}\right)>0$ so $x=\frac{1}{2^{1/3}}$ is a minimum and the minimum distance is

$$\sqrt{f\left(\frac{1}{2^{1/3}}\right)} = \sqrt{\frac{1}{2^{2/3}}+2^{1/3}}$$

Definition In economics, the utility function $U(x)$ is defined as the satisfaction experienced by the consumer of a good x. In a rational choice framework every consumer decides to consume the amount of good x that maximizes the utility $U(x)$.

Esercize 91. A consumer is willing to buy a good x with utility $U(x) = u_0 \ln\left(x^2\right) - u_1 x$, with u_0 and u_1 positive contants. For which value of u_0 and u_1 will the consumer buy an amount of good larger than 1? For which value of u_0 and u_1 the optimal utility is positive?

Solution. The consumer has to maximize the utility $U(x)$. Hence we compute

$$U'(x) = \frac{2u_0}{x} - u_1 = 0$$

whose solution is

$$x_0 = \frac{2u_0}{u_1}.$$

Note that

$$U''(x) = -\frac{2u_0}{x^2} < 0,$$

so the function is concave everywhere, whence x_0 is a maximum. We get $x_0 > 1$ if and only if $u_1 < 2u_0$. The optimal utility is

$$U(x_0) = 2u_0 \ln\left(\frac{2u_0}{u_1}\right) - 2u_0 = 2u_0\left(\ln\left(\frac{2u_0}{u_1}\right) - 1\right)$$

hence we have $U(x_0) > 0$ if and only if

$$\ln\left(\frac{2u_0}{u_1}\right) > 1$$

that is

$$\frac{2u_0}{u_1} > e$$

or

$$u_1 < \frac{2u_0}{e}.$$

Esercize 92. Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximal.

Solution. Find x and y such that

$$x + y = 9 \Rightarrow y = 9 - x$$

and maximizes $F(x) = x\,y(x)^2 = x\,(9-x)^2$ with $x \in [0,9]$. Hence

$$F'(x) = 0 \Leftrightarrow -2x\,(9-x) + (9-x)^2 = 0 \Leftrightarrow (9-x)\,(9-x-2x) = 0 \Leftrightarrow (9-x)\,(9-3x) = 0$$

whose solution are

$$x_1 = 9, \quad x_2 = 3.$$

Now compute

$$F''(x) = -2(9-x) + 2x - 2(9-x) = -4(9-x) + 2x$$

. So $F''(9) = 18 > 0$ so $x = 9$ is the minimum. Since $F''(3) = -18$, hence $x = 3$ is the maximum and the two numbers are $x = 3$ and $y = 6$.

Esercize 93. Use Lagrange's theorem to prove that

$$\ln(x) \le x, \quad \forall x \in (0, \infty).$$

Solution. For $x \in (0,1)$ we have $\ln(x) < 0 < x$, so the inequality is obvious. For $x = 1$ the inequality that we want to prove is $0 \le 1$, which is true. Suppose $x > 1$. We apply Lagrange's theorem on the interval $[1, x]$ to the function $f(x) = \ln(x)$, obtaining

$$\frac{\ln(x) - \ln(1)}{x - 1} = \frac{1}{c}$$

with $c \in (1, x)$ and where we have used $(\ln(x))' = 1/x$. In particular $c > 1$. Hence we get

$$\ln(x) = \frac{x-1}{c} = \frac{x}{c} - \frac{1}{c} < \frac{x}{c} < x$$

where the last inequality follows exactly from $c > 1$.

Esercize 94. Use Lagrange's theorem to prove that

$$\sin(x) < x, \quad \forall x > 0.$$

Solution. For $x > \frac{\pi}{2} > 1$ we have immediately that

$$\sin(x) \le 1 < \frac{\pi}{2} < x.$$

Take now a $x \in \left(0, \frac{\pi}{2}\right)$. We apply Lagrange's theorem on the interval $[0, x]$ to the function $f(x) = \sin(x)$ obtaining

$$\frac{\sin(x) - \sin(0)}{x - 0} = \cos(c)$$

with $c \in (0, x)$. Now note that, since $0 < c < x$ we also have that $0 < c < \frac{\pi}{2}$ and hence $0 < \cos(c) < 1$, whence

$$\frac{\sin(x)}{x} = \cos(c) < 1,$$

which is what we wanted to prove.

Esercize 95. Use Cauchy's Theorem to prove that

$$0 < 1 - \cos(x) < \frac{x^2}{2}, \quad \forall x > 0.$$

Solution. We introduce the functions

$$f(x) = 1 - \cos(x), \quad g(x) = \frac{x^2}{2}.$$

By the Cauchy's theorem applied on the interval $[0, x]$ we know that there exists a $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{1 - \cos(x)}{\frac{x^2}{2}} = \frac{\sin(c)}{c}$$

nevertheless since $\frac{\sin(c)}{c} < 1$ for $c > 0$, whence

$$\frac{1 - \cos(x)}{\frac{x^2}{2}} < 1 \Rightarrow 1 - \cos(x) < \frac{x^2}{2}.$$

Esercize 96. Suppose that the number of mobile phones produced by a factory from time 0 to time t, is given by a function $f(t)$. Assume also that $f(0) = 0$ and that f verifies the hypotheses of the Lagrange's theorem. We can interpret $f'(t)$ as the instantaneous velocity of production (i.e. the number of mobile phones produced per unit of time). Assume that this velocity is bounded, that is

$$f'(t) \leq 7 \quad \forall t \geq 0.$$

Which is the maximum number of mobile phones produced at time t ?.

Solution. Using the Lagrange's theorem on $[0, t]$ applied to the function f we know that

$$\frac{f(t) - f(0)}{t - 0} = f'(c) \leq 7$$

whence $f(t) \leq 7t$.

Esercize 97. Does there exist a continuous and differentiable function $f(x)$ such that $f(0) = -1$ and $f(2) = 4$ and $f'(x) \leq 2$ for all x?

Solution. Since such a function would also verify the hypotheses of the Lagrange's theorem we would have

$$\frac{f(2) - f(0)}{2 - 0} = f'(c)$$

with $c \in (0, 2)$. This would imply

$$\frac{4 + 1}{2} = f'(c) \leq 2$$

which is impossible.

20.1 The Derivative of the Inverse Function

Theorem 20.4. Suppose that $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable in D. Assume that f is invertible and call $f^{(-1)} : I_f \subseteq \mathbb{R} \to \mathbb{R}$ the inverse function, where I_f denotes the image of f. Then $f^{(-1)}$ is differentiable in I_f and

$$\left[f^{(-1)}\right]'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}. \tag{20.9}$$

for all $y \in I_f$.

Proof. Take a point $x_0 \in D$ and call $y_0 = f(x_0)$, that is $x_0 = f^{-1}(y_0)$. Hence

$$\lim_{y \to y_0} \frac{f^{(-1)}(y) - f^{(-1)}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

whence

$$\left[f^{(-1)}\right]'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{(-1)}(y_0)\right)}. \quad \square$$

Definition 38. The function $\sin(x)$ is strictly monotonic and increasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so it can be inverted and the inverse is called $\arcsin(x)$ and it is defined in $[-1, 1]$ with values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Esercize 98. Compute the derivative of the $\arcsin(x)$.

Solution. We use the formula (19.9). Hence, for $x \in [-1, 1]$, we get

$$\frac{d \arcsin(x)}{dx} = \frac{1}{\cos(\arcsin(x))}.$$

Nevertheless since $\arcsin(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have

$$\cos(\arcsin(x)) = +\sqrt{1 - \sin^2(\arcsin(x))} = \sqrt{1 - x^2},$$

whence

$$\frac{d \arcsin(x)}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

Definition 39. The function $\cos(x)$ is strictly monotonic and decreasing in $[0, \pi]$ so it can be inverted and the inverse is called $\arccos(x)$ and it is defined in $[-1, 1]$ with values in $[0, \pi]$.

Esercize 99. Compute the derivative of the $\arccos(x)$.

Solution. We use the formula (19.9). Hence, for $x \in [-1, 1]$, we get

$$\frac{d \arccos(x)}{dx} = -\frac{1}{\sin(\arccos(x))}.$$

Nevertheless since $\arccos(x) \in [0, \pi]$ we have

$$\sin(\arccos(x)) = +\sqrt{1 - \cos^2(\arccos(x))} = \sqrt{1 - x^2},$$

whence

$$\frac{d \arccos(x)}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

Definition 40. The function $\tan(x)$ is strictly monotonic and increasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so it can be inverted and the inverse is called $\arctan(x)$ and it is defined in $\mathbb{R}$ with values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Esercize 100. Compute the derivative of the $\arctan(x)$.

Solution. We use the formula (19.9). Hence, for $x \in [-1, 1]$, we get

$$\frac{d \arctan(x)}{dx} = \frac{1}{\frac{1}{\cos^2(\arctan(x))}} = \cos^2(\arctan(x)).$$

Remember that

$$\cos^2(x) = \frac{1}{1+\tan^2(x)}$$

hence

$$\frac{d \arctan(x)}{dx} = \frac{1}{1+\tan^2(\arctan(x))} = \frac{1}{1+x^2}$$

Esercize 101. Compute the derivative of the $\ln x$ using formula (19.9).

Solution. We have

$$\frac{d \ln(x)}{dx} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

20.2 L'Hôpital's Rule

Theorem 20.5. Assume that $f(x)$ and $g(x)$ are real function continuous in A. Let x_0 be a limit point of A. Assume that f and g are differentiable in $A/\{x_0\}$ and $g(x_0) \neq 0$. If:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0,$$

or if

$$\lim_{x \to x_0} f(x) = \pm\infty, \quad \lim_{x \to x_0} g(x) = \pm\infty,$$

and if

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L.$$

Then

$$\exists \lim_{x \to x_0} \frac{f(x)}{g(x)} = L.$$

The rule is valid even if $x_0 = +\infty$ or $x_0 = -\infty$.

Proof. We will give the prove only for the case in which both f and g are infinitesimal, that is assume

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0.$$

By hypothesis for all $\epsilon > 0$ there exists δ_ϵ such that **for all** x such that $|x - x_0| < \delta_\epsilon$ we have:

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon.$$

Consider two points $x_1 < x_2$ in the interval $(x_o - \delta, x_o)$. We can apply the Cauchy's mean value theorem to f and g in $[x_1, x_2]$. This theorem says that $\exists\xi \in (x_1, x_2)$ such that

$$\frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \frac{f'(\xi)}{g'(\xi)}.$$

Note that $x_0 - \delta_\epsilon < x_1 < \xi < x_2 < x_0$ hence ξ is such that $|\xi - x_0| < \delta_\epsilon$. We thus obtain:

$$\left|\frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} - L\right| < \epsilon.$$

Now take the limit for $x_2 \to x_0^-$ and use the fact both f and g goes to zero

$$\left|\frac{f(x_1)}{g(x_1)} - L\right| < \epsilon.$$

Now for $\epsilon \to 0$ we have that (remember that $x_1 \in (x_o - \delta_\epsilon, x_0)$) $x_1 \to x_0^-$. Hence:

$$\exists \lim_{x\to x_0^-} \frac{f(x)}{g(x)} = L.$$

with an identical argument we arrive at:

$$\exists \lim_{x\to x_0^+} \frac{f(x)}{g(x)} = L.$$

i.e. the thesis. □

Esercize 102. Compute the limit:

$$\lim_{x\to 0} \frac{\sin x - x\,\cos x}{x^2\,\tan x}.$$

The limit is a $\frac{0}{0}$ form , we try to use l'hopital.

$$\lim_{x\to 0} \frac{\sin x - x\,\cos x}{x^2\,\tan x} \overset{?}{=} \lim_{x\to 0} \frac{\cos x - \cos x + x\,\sin x}{2\,x\,\tan x + x^2\,\frac{1}{\cos^2 x}} = \lim_{x\to 0} \frac{\sin x}{2\,\tan x + x\,\frac{1}{\cos^2 x}} = \frac{0}{0}.$$

Use L'Hopital again:

$$\lim_{x\to 0} \frac{\sin x}{2\,\tan x + x\,\frac{1}{\cos^2 x}} \stackrel{?}{=} \lim_{x\to 0} \frac{\cos x}{2\,\frac{1}{\cos^2 x} + \frac{1}{\cos^2 x} + 2\,x\,\frac{\sin x}{\cos^3 x}} = \frac{1}{3}. \tag{20.10}$$

Because the final limit exists we can say that:

$$\lim_{x\to 0} \frac{\sin x - x\,\cos x}{x^2\,\tan x} = \frac{1}{3}.$$

Esercize 103. Compute the limit:

$$\lim_{x\to 0} \frac{e^x - e^{\sin x}}{\tan x - x}.$$

The limit is a $\frac{0}{0}$ form , we try to use l'hopital.

$$\lim_{x\to 0} \frac{e^x - e^{\sin x}}{\tan x - x} \stackrel{?}{=} \lim_{x\to 0} \frac{e^x - e^{\sin x}\,\cos x}{\frac{1}{\cos^2 x} - 1} = \frac{0}{0}.$$

Use L'Hopital again:

$$\lim_{x\to 0} \frac{e^x - e^{\sin x}\,\cos x}{\frac{1}{\cos^2 x} - 1} \stackrel{?}{=} \frac{e^x - e^{\sin x}\,\cos^2 x + e^{\sin x}\,\sin x}{2\,\frac{\sin x}{\cos^3 x}} = \frac{0}{0}.$$

Use L'Hopital again:

$$\begin{aligned} \lim_{x\to 0} \frac{e^x - e^{\sin x}\,\cos^2 x + e^{\sin x}\,\sin x}{2\frac{\sin x}{\cos^3 x}} &\stackrel{?}{=} \lim_{x\to 0} \frac{e^x - e^{\sin x}\,\cos^3 x + 2\,\cos x\,\sin x\,e^{\sin x} + e^{\sin x}\,\cos x\,\sin x + e^{\sin x}\,\cos x}{2\,\frac{1}{\cos^2 x} + 6\,\frac{\sin^2 x}{\cos^4 x}} \\ &= \frac{1}{2}. \end{aligned}$$

Therefore:

$$\lim_{x\to 0} \frac{e^x - e^{\sin x}}{\tan x - x} = \frac{1}{2}.$$

Observation 8. Note that we cannot infer that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

using L'Hopital rule, because we need this identity to show that $\frac{d\sin x}{dx} = \cos x$. Similarly one could show that

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = n\,x^{n-1},$$

iteratively using L'Hopital rule. This procedure is totally wrong because we need the last identity to compute the derivative of x^n, which is used to apply Hopital. This kind of logical fallacy is called **begging the question** and is a form of circular reasoning that must be avoided.

Observation 9. The converse of L'Hopital theorem is not true. This is why we have to check that the limit of the ratio of the derivatives exists. Example:

$$\lim_{x\to+\infty}\frac{x+\sin x}{x}=1.$$

Nevertheless using Hopital one should arrive at:

$$\lim_{x\to\infty}\frac{1+\cos x}{1}.$$

that does not exists!

Esercize 104. Compute

$$\lim_{x\to 0} x\,\ln(x)$$

Solution.

$$\lim_{x\to 0} x\,\ln(x)=\lim_{x\to 0}\frac{\ln(x)}{\frac{1}{x}}=\frac{\infty}{\infty}\overset{H}{=}\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0}-x=0.$$

Esercize 105. Compute

$$\lim_{x\to 0} x^x=0^0=???$$

Solution.

$$x^x=e^{\ln(x^x)}=e^{x\,\ln(x)}\to e^0=1$$

Esercize 106. Compute

$$\lim_{x\to 0} x^{\ln(1+x)}=0^0=???$$

Solution.

$$x^{\ln(1+x)}=e^{\ln\left(x^{\ln(1+x)}\right)}=e^{\ln(1+x)\,\ln(x)}$$

Consider that

$$\begin{aligned}\lim_{x\to 0^+}\ln(1+x)\,\ln(x) &= \lim_{x\to 0^+}\frac{\ln(1+x)}{\frac{1}{\ln x}}\\ &\overset{H}{=} \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{-\frac{1}{x\,(\ln x)^2}}\\ &= \lim_{x\to 0^+}-\frac{x\,(\ln x)^2}{1+x}=0 \qquad (20.11)\end{aligned}$$

whence

$$\lim_{x\to 0^+} x^{\ln(1+x)}=1.$$

Esercize 107. Compute
$$\lim_{x\to 0^+}\left(\ln\left(1+e^{-1/x}\right)\right)^x = 0^0 = ???$$

Solution.

$$\left(\ln\left(1+e^{-1/x}\right)\right)^x = e^{x\,\ln\left(\ln\left(1+e^{-1/x}\right)\right)}$$

$$\begin{aligned}
\lim_{x\to 0^+} x\,\ln\left(\ln\left(1+e^{-1/x}\right)\right) &= \lim_{x\to 0^+}\frac{\ln\left(\ln\left(1+e^{-1/x}\right)\right)}{\frac{1}{x}} \\
&\overset{H}{=} \lim_{x\to 0^+} -\frac{\left(\frac{1}{\ln\left(1+e^{-1/x}\right)}\right)\left(\frac{1}{1+e^{-1/x}}\right)\frac{e^{-1/x}}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to 0^+} -\left(\frac{1}{\ln\left(1+e^{-1/x}\right)}\right)\left(\frac{1}{1+e^{1/x}}\right) \\
&= \lim_{y\to 0^+} -\left(\frac{1}{\ln\left(1+y\right)}\right)\left(\frac{1}{1+1/y}\right) \\
&= \lim_{y\to 0^+} -\frac{1}{\ln\left(1+y\right)+\ln\left(1+y\right)^{1/y}} = -1 \qquad (20.12)
\end{aligned}$$

whence
$$\lim_{x\to 0^+}\left(\ln\left(1+e^{-1/x}\right)\right)^x = e^{-1} = \frac{1}{e}.$$

Esercize 108. Compute the limit
$$\lim_{x\to 0}\ln\left(\frac{1}{x}\right)^x$$

Solution. Note that
$$\lim_{x\to 0}\ln\left(\frac{1}{x}\right)^x = \lim_{x\to 0}\ln\left(x\right)^{-x} = \lim_{x\to 0} -x\,\ln\left(x\right) = 0.$$

Esercize 109. Compute the limit
$$\lim_{x\to 0}\left(\ln\left(\frac{1}{x}\right)\right)^x$$

Solution.

$$\begin{aligned}
\lim_{x\to 0^+}\left(\ln\left(\frac{1}{x}\right)\right)^x &= \lim_{x\to 0^+} e^{\ln\left(\ln\left(\frac{1}{x}\right)\right)^x} \\
&= \lim_{x\to 0^+} e^{x\ln\left(\ln\left(\frac{1}{x}\right)\right)} \\
&= \lim_{x\to 0^+} e^{\frac{\ln\left(\ln\left(\frac{1}{x}\right)\right)}{\left(\frac{1}{x}\right)}} \\
&= \lim_{x\to 0^+} e^{\frac{\left[\ln\left(\ln\left(\frac{1}{x}\right)\right)\right]'}{\left(\frac{1}{x}\right)'}} \\
&= \lim_{x\to 0^+} e^{\frac{\frac{\left(\ln\left(\frac{1}{x}\right)\right)'}{\left(\ln\left(\frac{1}{x}\right)\right)}}{-\frac{1}{x^2}}} \\
&= e^{-\lim_{x\to 0^+}\frac{\left(-\frac{1}{x}\right)}{\ln\left(\frac{1}{x}\right)}x^2} \\
&= e^{-\lim_{x\to 0^+}\frac{x}{\ln x}} \\
&= e^0 = 1.
\end{aligned}$$

Esercize 110. Compute the limit

$$\lim_{x\to\infty}\frac{5\,x+\sin(x)+\ln\left(\sqrt{x}\right)}{3\,x}$$

Solution. Blindly applying Hopital gives

$$\lim_{x\to\infty}\frac{5+\cos(x)+\frac{1}{\sqrt{x}}\,\frac{1}{2\sqrt{x}}}{3}$$

which does not exist. Nevertheless

$$\lim_{x\to\infty}\frac{5\,x}{3\,x}=\frac{5}{3}.$$

$$\lim_{x\to\infty}\frac{\sin(x)}{3\,x}=0.$$

$$\lim_{x\to\infty}\frac{\ln\left(\sqrt{x}\right)}{3\,x}\overset{H}{=}\lim_{x\to\infty}\frac{\frac{1}{\sqrt{x}}\,\frac{1}{2\sqrt{x}}}{3}=0,$$

whence

$$\lim_{x\to\infty}\frac{5\,x+\sin(x)+\ln\left(\sqrt{x}\right)}{3\,x}=\frac{5}{3}.$$

A strange case Consider the limit

$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Blindly applying L'Hopital's Rule repeatedly gives

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \cdots$$

But if we divide the numerator and denominator by e^x we get

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1+0}{1+0} = 1.$$

20.3 Taylor's Expansions

Theorem 20.6. Suppose that $f(x)$ is a function n-times differentiable in a neighbourhood of x_0 and assume that the n-th derivative is continuous in x_0. Then $f(x)$ can be written as:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + o\left[(x - x_0)^n\right].$$

where the "little-oh", $o\left[(x - x_0)^n\right]$, is a function such that:

$$\lim_{x\to x_0} \frac{o\left[(x - x_0)^n\right]}{(x - x_0)^n} = 0.$$

In other words we can approximate the function as polynomial plus an error that goes to zero faster than $(x - x_0)^n$.

Proof. Let consider the case $n = 1$, i.e. f is differentiable in a neighborhood of x_0 and the first derivative is continuous in x_0. Let $g(x)$ be defined by:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + g(x).$$

The fact that f is differentiable in x_0 implies that f is continuous in x_0 therefore:

$$\lim_{x\to x_0} g(x) = 0.$$

More over $g(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$ is differentiable in a neighborhood of x_0 as f does.

Now compute:

$$\lim_{x\to x_0} \frac{g(x)}{(x - x_0)} = \frac{0}{0} \overset{\mathrm{H}}{=} \lim_{x\to x_0} \frac{g'(x)}{1} = \lim_{x\to x_0} \left[f'(x) - f'(x_0)\right] = 0,$$

where the last identity follows from the continuity of the derivative. Therefore putting

$$o\left[(x - x_0)\right] \equiv g(x),$$

we proved the theorem for $n = 1$. For $n = 2$ we put:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + g(x).$$

Again:

$$\lim_{x \to 0} g(x) = 0.$$

and:

$$\begin{aligned} g'(x) &= f'(x) - f'(x_0) - f''(x_0)(x - x_0) \\ g''(x) &= f''(x) - f''(x_0). \end{aligned}$$

So $g'(x) \to 0$ and $g''(x) \to 0$ for the continuity of the first two derivatives. Therefore:

$$\begin{aligned} \lim_{x \to x_0} \frac{g(x)}{(x - x_0)^2} &= \frac{0}{0} \\ &\overset{\text{H}}{=} \lim_{x \to x_0} \frac{g'(x)}{2(x - x_0)} \\ &= \frac{0}{0} \\ &\overset{\text{H}}{=} \lim_{x \to x_0} \frac{g''(x)}{2} = 0. \end{aligned}$$

Therefore putting

$$o\left[(x - x_0)^2\right] \equiv g(x),$$

we proved the theorem for $n = 2$. For induction the theorem is proved for all n. □

Examples:

- $f(x) = \cos x$, $x_0 = 0$. Note that:

$$\begin{aligned} \cos'(0) &= -\sin(0) = 0 \\ \cos^{(2)}(0) &= -\cos(0) = -1 \\ \cos^{(3)}(0) &= \sin(0) = 0 \\ \cos^{(4)}(0) &= \cos(0) = 1 \\ &\vdots \quad \vdots \end{aligned}$$

As a consequence only the even terms appear (as it should be for an even function) and:

$$\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \ldots = \sum_{k=0}^{n} \frac{(-)^k}{(2k)!} x^{2k} + o\left[x^{2n}\right].$$

- $f(x) = \sin x$, $x_0 = 0$. Note that:

$$\begin{aligned}
\sin'(0) &= \cos(0) = 1 \\
\sin^{(2)}(0) &= -\sin(0) = 0 \\
\sin^{(3)}(0) &= -\cos(0) = -1 \\
\sin^{(4)}(0) &= \sin(0) = 0 \\
\vdots &\quad \vdots
\end{aligned}$$

As a consequence only the odd terms appear (as it should be for an odd function) and:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots = \sum_{k=0}^{n} \frac{(-)^k}{(2k+1)!} x^{2k+1} + o\left[x^{2n+1}\right].$$

- $f(x) = e^x$, $x_0 = 0$. We know that $f^{(k)}(x) = e^x$ for all k. Therefore:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots = \sum_{k=0}^{n} \frac{1}{k!} x^k + o[x^n].$$

- $f(x) = \ln(1+x)$, $x_0 = 0$. Note that

$$\begin{aligned}
f'(x) &= \frac{1}{1+x} \\
f''(x) &= -\frac{1}{(1+x)^2} \\
f'''(x) &= 2\frac{1}{(1+x)^3} \\
f''''(x) &= -6\frac{1}{(1+x)^4} \\
\vdots &\quad \vdots
\end{aligned}$$

whence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots. + \frac{(-1)^{n+1}}{n} x^n + o(x^n).$$

Esercize 111. Compute:

$$\lim_{x\to 0} \frac{\sin^2 x - x^2 \cos x}{x^4}.$$

This limit can be computed with L'Hopital rule or Taylor formula. Use Taylor:

$$\sin x = x - \frac{x^3}{6} + o\left(x^3\right).$$

Note that I don't need other terms because in the numerator we have the second power of the $\sin x$ that must be compared with x^4. Similarly:

$$\cos x = 1 - \frac{x^2}{2} + o\left(x^4\right).$$

Hence:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 x - x^2\,\cos x}{x^4} &= \lim_{x\to 0}\frac{\left\{x-\frac{x^3}{6}+o\left(x^3\right)\right\}^2 - x^2\left\{1-\frac{x^2}{2}+o\left(x^4\right)\right\}}{x^4}\\ &= \lim_{x\to 0}\frac{x^2-2\,\frac{x^4}{6}+o\left(x^4\right)-x^2+\frac{x^4}{2}+o\left(x^4\right)}{x^4}.\end{aligned}$$

Note that every term that goes to zero faster than x^4 has been included in $o\left(x^4\right)$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 x - x^2\,\cos x}{x^4} &= \lim_{x\to 0}\frac{-\frac{x^4}{3}+\frac{x^4}{2}+o\left(x^4\right)}{x^4}\\ &= \lim_{x\to 0}\left(-\frac{1}{3}+\frac{1}{2}+\frac{o\left(x^4\right)}{x^4}\right)=\frac{1}{6}.\end{aligned}$$

Esercize 112. Compute the limit:

$$\lim_{x\to\infty} x\left(e^{-\frac{1}{x}}-1\right).$$

Change variable and use Taylor formula:

$$\begin{aligned}\lim_{x\to\infty} x\left(e^{-\frac{1}{x}}-1\right) &= \lim_{y\to 0}\frac{1}{y}\left(e^{-y}-1\right)\\ &= \frac{1-y+o\left(y\right)-1}{y}=\lim_{y\to 0}\left(-1+\frac{o\left(y\right)}{y}\right)=-1.\end{aligned}$$

Esercize 113. Compute an approximation for $\ln(2)$ by using a Taylor's expansion of $\ln(1+x)$ around $x=0$ at the fifth order.

Solution. For example

$$\ln(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+O\left(x^6\right)$$

or more generally

$$\ln(1+x)=\sum_{k=1}^{n}(-1)^{k+1}\frac{x^k}{k}+o\left(x^{n+1}\right)$$

so that

$$\ln(2)\approx 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}=0.7833$$

Esercize 114. We start at year $t = 0$ with a unit of money. Our capital at year t is given by

$$K(t) = (1+r)^t$$

where $0 < r < 1$. By using a Taylor's expansion of $\ln(1+r)$ around $r = 0$ at the first order, establish how long it takes to double the capital. Assume $\ln(2) \approx 0.693$

Solution. We have to solve

$$K(t) = 2 \Leftrightarrow (1+r)^t = 2 \Leftrightarrow t = \log_{(1+r)}(2)$$

Since

$$\log_{(1+r)}(2) = \frac{\ln(2)}{\ln(1+r)}$$

and by Taylor's expansion

$$\ln(1+r) \approx r$$

Hence

$$t = \frac{\ln(2)}{r} \approx \frac{0.693}{r}.$$

Observation 10. Note that, similarly to what happens with L'Hopital's rule, we cannot infer that:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

using Taylor formula, because we need this identity to show that $\frac{d\sin x}{dx} = \cos x$ and to compute the Taylor expansion. The issue is not so problematic because in both cases we obviously obtain a consistent result.

21 Exercizes from past mid-term exams

Esercize 115. By knowing that $\arctan(1) = \pi/4$ use a Taylor expansion of the $f(x) = \arctan(x)$ function around 0 to the seventh order to find an approximation of π.

Solution

$$f'(x) = \frac{1}{\sqrt{1+x^2}}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$f'''(x) = \frac{8x^2}{(x^2+1)^3} - \frac{2}{(x^2+1)^2}$$

$$f''''(x) = \frac{24x}{(x^2+1)^3} - \frac{48x^3}{(x^2+1)^4}$$

so that

$$\arctan(x) = x - \frac{2}{3!}x^3 + o\left(x^3\right) = x - \frac{1}{3}x^3 + o\left(x^3\right)$$

or more generally

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots + \frac{(-1)^n}{2n+1}x^{2n+1} + o\left(x^{2n+1}\right)$$

whence

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \Rightarrow \pi \approx 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right)$$

it's a very bad approximation though.

Esercize 116. Write the Taylor polinomial of degree 4 for $f(x) = e^{x-1}$ in $x = 1$ and use it to find an approximation of e.

Solution

$$f'(x) = f''(x) = f'''(x) = f''''(x) = e^{x-1}$$

and

$$f'(1) = f''(1) = f'''(1) = f''''(1) = 1$$

so that

$$e^{x-1} = 1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + o\left[(x-1)^4\right]$$

so I use the previous formula with $x = 1$ to get

$$e^{2-1} = e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{48+12+4+1}{24} = \frac{65}{24} = 2.7083.$$

Esercize 117. Can we apply Lagranges Theorem to $f(x) = 2x + |x+1|$ in $[-1,1]$?

Solution

Recall the theorem:

Theorem 21.1. If f is a real continuous function on $[a,b]$ which is differentiable in (a,b) then there is a point $\xi \in (a,b)$ at which

$$f(b) - f(a) = (b-a)\, f'(\xi)\,.$$

in our case $a = -1$ and $b = -1$. The function is continuous everywhere on $\mathbb{R}$. The function is not differentiable in $x = -1$, so the function is continuous in $[-1,1]$ and differentiable in $(-1,1)$ hence we can apply the theorem in $[-1,1]$.

Esercize 118. Evaluate

$$\lim_{n\to\infty}\left(\frac{3n+5}{3n}\right)^{2n}$$

Solution.

$$\begin{aligned}\left(\frac{3n+5}{3n}\right)^{2n} &= \left(1+\frac{5}{3n}\right)^{2n}\\ &= \left(1+\frac{10}{3\cdot 2n}\right)^{2n}\end{aligned} \tag{21.1}$$

So

$$\lim_{n\to\infty}\left(\frac{3n+5}{3n}\right)^{2n} = \lim_{n\to\infty}\left(1+\frac{10}{3\cdot 2n}\right)^{2n} = \lim_{m\to\infty}\left(1+\frac{10}{3\cdot m}\right)^{m} = e^{\frac{10}{3}}.$$

Esercize 119. Domain,sign, limits, asymptotes, maximum and minimum points and graph of $f(x) = \frac{1-\ln(x)}{\ln(x)}$.

Solution.

Domain: $\{x > 0, x \neq 1\}$.

Sign. The numerator is positive if $1 - \ln(x) > 0$ that is $\ln(x) < 1$ hence $x < e$. The denominator is positive if $\ln(x) > 0$ hence $x > 1$. Whence $f < 0$ if $0 < x < 1$ or $x > e$ and $f > 0$ if $1 < x < e$.

$$\lim_{x\to 0^+}\frac{1-\ln(x)}{\ln(x)} = \frac{+\infty}{-\infty} \overset{H}{=} \lim_{x\to 0^+}\frac{-\frac{1}{x}}{\frac{1}{x}} = -1$$

$$\lim_{x\to 1^-} \frac{1-\ln(x)}{\ln(x)} = -\infty$$

$$\lim_{x\to 1^+} \frac{1-\ln(x)}{\ln(x)} = +\infty$$

$$\lim_{x\to \infty} \frac{1-\ln(x)}{\ln(x)} = \frac{-\infty}{\infty} \overset{H}{=} \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} = -1$$

Asymptot: one vertical at $x = 1$ and one horizontal at $y = -1$.

Max and min:

$$f'(x) = -\frac{1}{x\ln x} - \frac{1-\ln x}{x(\ln x)^2} = -\left(\frac{\ln x + 1 - \ln x}{x(\ln x)^2}\right) = -\frac{1}{x(\ln x)^2}$$

hence no max no min.

Esercize 120. Write the Taylor polynomial of degree 4 for $f(x) = \ln(1+x)$ around $x = 0$ and use it to find an approximation of $\ln(2)$.

Solution.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o\left(x^4\right)$$

whence

$$\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{12-6+4-3}{12} = \frac{7}{12}$$

Esercize 121. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n} \cos\left(\frac{1}{n}\right)$.

Solution.

$$\left|\frac{(-1)^n}{n} \cos\left(\frac{1}{n}\right)\right| = \left|\frac{1}{n} \cos\left(\frac{1}{n}\right)\right| \le \frac{1}{n} \to 0.$$

Esercize 122. Given the set $B = \left\{\frac{1}{5n} \mid n \in \mathbb{N}\right\} \cup \{0\}$ establish if it is a closed set or not. Motivate your answer.

Solution

A set is closed if every limit point of the set is a point of the set. Let's compute the limit points of B.

$$B = \left\{0, \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, ...\right\}$$

The only limit point is 0, since, no matter how I take ε small I can find a $x \in B$, $x \neq 0$, such that

$$|x - 0| = |x| < \varepsilon.$$

All the other points are isolated points. Take a generic element $x_n = \frac{1}{5n}$. The points of B that are closest to x_n are x_{n-1} and x_{n+1}. The two distances are

$$x_{n-1} - x_n = \frac{1}{5(n-1)} - \frac{1}{5n} = \frac{n - n + 1}{5n(n-1)} = \frac{1}{5n(n-1)}$$

and

$$x_n - x_{n+1} = \frac{1}{5n} - \frac{1}{5(n+1)} = \frac{n+1-n}{5n(n+1)} = \frac{1}{5n(n+1)} < \frac{1}{5n(n-1)} = x_{n-1} - x_n$$

so any neighborhood of x_n with radius smaller than $x_n - x_{n+1}$ does not contain points of B. In conclusion the only limit point is 0 and since $0 \in B$ we have that B is closed.

Esercize 123. Study the function $f(x) = \frac{\ln x}{1-\ln x}$ domain, sign, limits, asymptotes, maximum and minimum points and graph.

Solution.

Domain: $D = \{x > 0 \wedge x \neq e\}$.

Sign: $\ln x > 0$ if $x > 1$ and $1 - \ln x > 0$ if $\ln x < 1$ that is $x < e$. Whence f is positive on $(1, e)$ and negative otherwise.

Limits:

$$\lim_{x\to 0^+} \frac{\ln x}{1 - \ln x} = \lim_{x\to 0^+} \frac{1}{\frac{1}{\ln x} - 1} = -1$$

$$\lim_{x\to e^-} \frac{\ln x}{1 - \ln x} = +\infty$$

$$\lim_{x\to e^+} \frac{\ln x}{1 - \ln x} = -\infty$$

$$\lim_{x\to +\infty} \frac{\ln x}{1 - \ln x} = -1$$

Max and min:

$$f'(x) = \frac{1}{x(\log(x) - 1)^2}$$

hence $f'(x) > 0$ so no max no min.

Esercize 124. Can we apply Weierstrass Theorem and Rolles Theorem to $f(x) = |x - 1|$ in $[-2, 2]$? Motivate your answer.

Solution.

For the Weierstrass' Theorem we only require the function to be continuous on a closed limited set. Since $f(x)$ is continuous on $[-2, 2]$ the we can apply Weierstrass and so the function attains a maximum and a minimum. For the Rolle's Theorem we require the function to be continuous on $[-2, 2]$ and differentiable in $(-2, 2)$. The function $f(x) = |x - 1|$ is not differentiable for $x = 1$ and so we cannot apply Rolle.

Esercize 125. Let $f(x)$ be a function such that

$$\lim_{x \to 0} f(x) = 0.$$

Define a new function $g(x)$ in this way

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Which is $\lim_{x \to 0} g(x)$? Motivate your answer.

Solution. We now that

$$\forall \varepsilon > 0 \exists \delta > 0 : 0 < |x| < \delta \Rightarrow |f(x)| < \varepsilon.$$

but if x is such that $0 < |x| < \delta$ then $x \neq 0$ and so $f(x) = g(x)$ and so

$$\forall \varepsilon > 0 \exists \delta > 0 : 0 < |x| < \delta \Rightarrow |g(x)| < \varepsilon.$$

which means that $\lim_{x \to 0} g(x) = 0$.

22 Training for the Exam

22.1 Domains

Esercize 126. Find the domain of the following function

$$f(x) = \ln(\ln(x)).$$

Solution. The function $\ln(\ln(x))$ is the composition of two functions

$$x \longrightarrow \ln(x) \longrightarrow \ln(\ln(x)).$$

This composition is defined for all x such that $\ln(x) > 0$ and hence the domain is

$$D = (1, +\infty).$$

Esercize 127. Find the domain of the following function

$$f(x) = \ln(\ln(\ln(x))).$$

Solution. The function $\ln(\ln(\ln(x)))$ is the composition of three functions

$$x \longrightarrow \ln(x) \longrightarrow \ln(\ln(x)) \longrightarrow \ln(\ln(\ln(x))).$$

This composition is defined for all x such that $\ln(\ln(x)) > 0$ hence it must be that $\ln(x) > 1$ that is $x > e$ where e is the Neper number. So the domain is

$$D = (e, +\infty).$$

Esercize 128. Find the domain of the following function

$$f(x) = e^{\frac{\sqrt{x}}{x-2}}.$$

Solution. The function $e^{\frac{\sqrt{x}}{x-2}}$ is defined whenever the argument of the exponential function is defined (remember that the exponential function is defined everywhere), hence it must be that $x > 0$ in order to have $\sqrt{x}$ defined and moreover it must be $x \neq 2$ in order to have the fraction $1/(x-2)$ defined. Hence the domain is

$$D = (0,2) \bigcup (2, \infty).$$

Esercize 129. Find the domain of the following function

$$f(x) = \frac{\sqrt{x} + \sqrt{1-x}}{\sqrt{x-2}}.$$

Solution. The function $\frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{x-2}}$ is defined in all x such that the three functions $\sqrt{x}$, $\sqrt{1-x}$ and $1/\sqrt{x-2}$ are defined. Let's analyze them separately. The function $\sqrt{x}$ is defined for $x \geq 0$. The function $\sqrt{1-x}$ is defined for $x \leq 1$. The function $1/\sqrt{x-2}$ is defined for $x > 2$ (note that I am not writing $x \geq 2$ since the denominator must be different from zero). Hence the original function $\frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{x-2}}$ is defined for all x such that $x \geq 0$ and $x \leq 1$ and $x > 2$. So the domain of the function $\frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{x-2}}$ is empty.

22.2 Limits

Recall the operations with infinity and the Figure 10 that lists the most common indeterminate

$$\begin{aligned}
a + \infty = +\infty + a &= +\infty, && a \neq -\infty \\
a - \infty = -\infty + a &= -\infty, && a \neq +\infty \\
a \cdot (\pm\infty) = \pm\infty \cdot a &= \pm\infty, && a \in (0, +\infty] \\
a \cdot (\pm\infty) = \pm\infty \cdot a &= \mp\infty, && a \in [-\infty, 0) \\
\frac{a}{\pm\infty} &= 0, && a \in \mathbb{R} \\
\frac{\pm\infty}{a} &= \pm\infty, && a \in (0, +\infty) \\
\frac{\pm\infty}{a} &= \mp\infty, && a \in (-\infty, 0)
\end{aligned}$$

forms and the transformations for applying l'Hopital's rule.

Indeterminate form	Conditions	Transformation to 0/0	Transformation to ∞/∞
0/0	$\lim_{x\to c} f(x) = 0,\ \lim_{x\to c} g(x) = 0$	–	$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{1/g(x)}{1/f(x)}$
∞/∞	$\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{1/g(x)}{1/f(x)}$	–
$0 \times \infty$	$\lim_{x\to c} f(x) = 0,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} \frac{f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} \frac{g(x)}{1/f(x)}$
$\infty - \infty$	$\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} \frac{1/g(x) - 1/f(x)}{1/(f(x)g(x))}$	$\lim_{x\to c}(f(x) - g(x)) = \ln \lim_{x\to c} \frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to c} f(x) = 0^+, \lim_{x\to c} g(x) = 0$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{\ln f(x)}{1/g(x)}$
1^∞	$\lim_{x\to c} f(x) = 1,\ \lim_{x\to c} g(x) = \infty$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{\ln f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{g(x)}{1/\ln f(x)}$
∞^0	$\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = 0$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{\ln f(x)}{1/g(x)}$

Fig. 10: Common indeterminate forms and the transformations for applying l'Hopital's rule

Esercize 130. Compute the limit

$$\lim_{x\to\infty}\left(\frac{x-1}{x^2}\right)^{x^2}.$$

Solution. Consider that

$$\left(\frac{x-1}{x^2}\right)^{x^2}=e^{\ln\left(\left(\frac{x-1}{x^2}\right)^{x^2}\right)}=e^{x^2\,\ln\left(\left(\frac{x-1}{x^2}\right)\right)}$$

Let's study the asymptotic behaviour of the argument of the logarithm that appears above

$$\lim_{x\to+\infty}\frac{x-1}{x^2}=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{x}=\frac{1}{+\infty}=0$$

hence (remember that $\lim_{y\to0^+}\ln(y)=-\infty$)

$$\lim_{x\to+\infty}\ln\left(\frac{x-1}{x^2}\right)=-\infty$$

whence

$$\lim_{x\to+\infty}x^2\,\ln\left(\frac{x-1}{x^2}\right)=(+\infty)\times(-\infty)=-\infty$$

so that

$$\lim_{x\to\infty}\left(\frac{x-1}{x^2}\right)^{x^2}=\lim_{x\to\infty}e^{x^2\,\ln\left(\left(\frac{x-1}{x^2}\right)\right)}=e^{-\infty}=0.$$

Esercize 131. Compute the limit

$$\lim_{x\to0}\left(1+\frac{\sin x^2}{x}\right)^{1/x}.$$

Solution. Consider that

$$\lim_{x\to0}\frac{\sin\left(x^2\right)}{x}=\lim_{x\to0}\frac{\sin\left(x^2\right)}{x^2}x$$

but

$$\lim_{x\to0}\frac{\sin\left(x^2\right)}{x^2}=\lim_{y\to0}\frac{\sin(y)}{y}=1$$

so that

$$\lim_{x\to0}\frac{\sin\left(x^2\right)}{x}=\lim_{x\to0}\frac{\sin\left(x^2\right)}{x^2}x=0.$$

Hence the limit

$$\lim_{x\to0}\left(1+\frac{\sin\left(x^2\right)}{x}\right)^{1/x}$$

is a 1^∞ indeterminate form. Re-write the limit as

$$\lim_{x\to0}\left(1+\frac{\sin\left(x^2\right)}{x}\right)^{1/x}=\lim_{x\to0}e^{\ln\left(\left(1+\frac{\sin\left(x^2\right)}{x}\right)^{1/x}\right)}=\lim_{x\to0}e^{\frac{1}{x}\,\ln\left(1+\frac{\sin x^2}{x}\right)}.$$

Now consider the exponent and apply Hopital's rule

$$\lim_{x\to0}\frac{1}{x}\,\ln\left(1+\frac{\sin\left(x^2\right)}{x}\right)=\frac{0}{0}\overset{H}{=}\lim_{x\to0}\frac{1}{1+\frac{\sin x^2}{x}}\left(\underbrace{\frac{2\,x\,\cos x^2}{x}}_{\to2}-\underbrace{\frac{\sin\left(x^2\right)}{x^2}}_{\to1}\right)=1$$

so that

$$\lim_{x\to 0}\left(1+\frac{\sin\left(x^2\right)}{x}\right)^{1/x} = e.$$

Esercize 132. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x^2 & \text{if } x \neq 0 \end{cases}$$

compute $\lim_{x\to 0} f(x)$.

Solution. Remember that in the definition of the limit of a function

$$\lim_{x\to x_0} f(x) = L \Leftrightarrow \forall\epsilon > 0 \exists\delta > 0 \text{ such that } \forall x \text{ such that } 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

since it is required that $|x - x_0| > 0$ the value of the function in x_0, that is $f(x_0)$, does not enter in the definition. In other words, what really matters is the behaviour of the function around x_0, irrespectively of the value of the function in x_0. So consider the function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x^2 & \text{if } x \neq 0 \end{cases}$$

the idea is that, for all $x \neq 0$, if x is very close to 0 then also $f(x) = x^2$ is very close to zero, so the limit

$$\lim_{x\to 0} f(x)$$

is exactly 0. Let's try to verify this claim using the definition

$$\forall\epsilon > 0 \exists\delta > 0 \text{ such that } \forall x \text{ such that } 0 < |x| < \delta \Rightarrow \left|x^2\right| < \varepsilon,$$

which is true. In fact it is enough to take, $\forall\epsilon > 0$, any $\delta < \sqrt{\epsilon}$ so that if $0 < |x| < \delta$ we have $0 < x^2 < \delta^2 < \epsilon$.

Esercize 133. Compute the limit

$$\lim_{x\to 0} \ln\left(\left|\frac{\sin x}{x}\right|\right)$$

Solution. First remember the notable limit

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

which, by continuity of the absolute value, implies

$$\lim_{x\to 0}\left|\frac{\sin x}{x}\right| = |1| = 1$$

which, by continuity of the logarithm, implies

$$\lim_{x\to 0} \ln\left(\left|\frac{\sin x}{x}\right|\right) = \ln(1) = 0.$$

Esercize 134. Compute the limit

$$\lim_{x\to 0^+} x^{\sin x}$$

Solution. Note that

$$x^{\sin x} = e^{\ln\left(x^{\sin x}\right)} = e^{\sin x\,\ln x}$$

now consider that

$$\lim_{x\to 0^+} \sin x\,\ln x$$

is a $0\times(-\infty)$ indeterminate form. Re-write $\sin x\ln x$ as

$$\lim_{x\to 0^+}\sin x\,\ln x = \lim_{x\to 0^+}\frac{\sin x}{\frac{1}{\ln x}} = \frac{0}{0} \overset{H}{=} \lim_{x\to 0^+}\frac{\cos x}{-\frac{1}{x\,(\ln x)^2}} = \lim_{x\to 0^+} -\left(x\,\cos x\,(\ln x)^2\right).$$

Again we have a $0\times\infty$ indeterminate form....

$$\lim_{x\to 0^+} x\,(\ln x)^2 = \lim_{x\to 0^+}\frac{(\ln x)^2}{\frac{1}{x}} \overset{H}{=} \lim_{x\to 0^+}\frac{\frac{2\,\ln x}{x}}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+}\frac{2\,\ln x}{\left(-\frac{1}{x}\right)} \overset{H}{=} \lim_{x\to 0^+}\frac{\frac{2}{x}}{\frac{1}{x^2}} = \lim_{x\to 0^+} 2\,x = 0$$

and so

$$\lim_{x\to 0^+}\sin x\,\ln x = 0$$

and finally

$$\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} e^{\sin x\,\ln x} = e^{\lim_{x\to 0^+}\sin x\,\ln x} = 1.$$

22.3 Series

Esercize 135. Establish if

$$\sum_{n=1}^{\infty} 2(\sqrt{n}-\sqrt{n-1}) - \frac{1}{\sqrt{n}}$$

converges or not.

Remember that $\sum_n \frac{1}{n^p}$ *converges* $\Leftrightarrow p > 1$.

Solution. Consider that

$$\left(2\sqrt{n} - 2\sqrt{n-1} - \frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}}\left(2\,n - 2\,\sqrt{n}\,\sqrt{n-1} - 1\right) = \frac{1}{\sqrt{n}}\left(\sqrt{n}-\sqrt{n-1}\right)^2,$$

hence

$$\begin{aligned}
\sum_{k=1}^{n}\left(2\sqrt{k} - 2\sqrt{k-1} - \frac{1}{\sqrt{k}}\right) &= \sum_{k=1}^{n}\frac{1}{\sqrt{k}}\left(\sqrt{k}-\sqrt{k-1}\right)^2 \\
&= \sum_{k=1}^{n}\frac{1}{\sqrt{k}}\left(\frac{(\sqrt{k}-\sqrt{k-1})(\sqrt{k}+\sqrt{k-1})}{(\sqrt{k}+\sqrt{k-1})}\right)^2 \\
&= \sum_{k=1}^{n}\frac{1}{\sqrt{k}\left(\sqrt{k}+\sqrt{k-1}\right)^2} \sim \sum_{k=1}^{n}\frac{1}{\sqrt{k}\,k} < \infty.
\end{aligned}$$

Esercize 136. For which values of x the series

$$\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n} \tag{22.1}$$

converges? For which values of x it diverges?

Solution. The series trivially converges if $x = 0$. Assume now $x \neq 0$. Apply the ratio criterion

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!\, |x|^{n+1}}{(n+1)^{n+1}} \frac{n^n}{n!\, |x|^n} = |x| \left(\frac{n}{n+1} \right)^n = |x| \left(\frac{1}{\frac{n+1}{n}} \right)^n = \frac{|x|}{\left(1 + \frac{1}{n}\right)^n}.$$

Hence the series absolute converges (and hence converges) if $|x| < e$. Nevertheless if $|x| \geq e$ we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|}{\left(1 + \frac{1}{n}\right)^n} \geq \frac{e}{\left(1 + \frac{1}{n}\right)^n} \geq 1.$$

Therefore $|a_n|$ is increasing and it cannot happen that $|a_n| \to 0$, and hence it cannot happen that $a_n \to 0$, hence the necessary condition is not satisfied. Summarizing

$$\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n} < \infty \Leftrightarrow x \in (-e, e).$$

22.4 Taylor's expansions

Esercize 137. Using a Taylor's expansion of $\ln(1+x)$ around $x = 0$ truncated at the third order compute an approximation for the number $\ln(2)$.

Solution. We want to use the formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + o\left((x - x_0)^4\right)$$

using $x_0 = 0$, with $f(x) = \ln(1+x)$ and neglecting the error term $o\left((x - x_0)^4\right)$. So first not that $f(0) = \ln(1) = 0$ and then compute the derivatives

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1,$$

$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f'(0) = -1,$$

$$f'''(x) = 2 \frac{1}{(1+x)^3} \Rightarrow f'(0) = 2,$$

whence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{2}{3!} x^3 + o\left(x^4\right).$$

Neglecting the error term $o\left(x^4\right)$ and computing the formula above for $x = 1$ we get

$$\ln(2) \approx 1 - \frac{1}{2} + \frac{1}{3} = \frac{6 - 3 + 2}{6} = \frac{5}{6}.$$

22.5 Graphs of functions

Esercize 138. Find the domain, vertical asymptotes, horizontal asymptotes and the intersection with the $x = 0$ and $y = 0$ axes, plus study the sign, monotonicity, maxima, minima, concavity and convexity of the function

$$f(x) = x^2 e^{-x}.$$

Solution.

- **Domain**. The function $x^2 e^{-x}$ is the product of the function x^2 with e^{-x} and they are both defined everywhere on the real line, so the domain D of the function is $D = \mathbb{R}$.

- **Asymptotes**. There are no vertical asymptotes since the function has no critical points. Now consider the limit

$$\lim_{x\to+\infty} x^2 e^{-x} = \infty \cdot 0 = \lim_{x\to+\infty} \frac{x^2}{\frac{1}{e^{-x}}} = \frac{\infty}{\infty} \overset{H}{=} \lim_{x\to+\infty} \frac{2x}{e^x} = \frac{\infty}{\infty} \overset{H}{=} \lim_{x\to+\infty} \frac{2}{e^x} = 0.$$

$$\lim_{x\to-\infty} x^2 e^{-x} = \lim_{x\to+\infty} x^2 e^{x} = (+\infty)\cdot(+\infty) = +\infty.$$

  so $x = 0$ is an horizontal asymptote for $x \to +\infty$.

- **Intersection with** $x = 0$. If $x = 0$ we get $f(0) = 0$.

- **Intersection with** $y = 0$. The equation

$$x^2 e^{-x} = 0$$

  is equivalent to

$$x^2 = 0$$

  and this is because $e^{-x} > 0$ for all x. Hence the function intersects the axis $y = 0$ only in $x = 0$.

- **Sign**. Since, trivially, $x^2 \geq 0$ and $e^{-x} \geq 0$ we have that $f(x) \geq 0$ for all x.

- **Monotonicity**. Compute the first derivative

$$f'(x) = 2x e^{-x} - x^2 e^{-x} = e^{-x} x (2 - x).$$

  Hence

$$f'(x) \geq 0 \Leftrightarrow x(2-x) \geq 0$$

  whence $f'(x) \geq 0$ if $x \in [0, 2]$, so the function is decreasing in $(-\infty, 0]$, increasing in $[0, 2]$ and decreasing in $[2, \infty)$

- **Maxima and minima**. Since $f'(0) = 0$ and $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $0 < x < 2$ we have that $x = 0$ is a minimum. Similarly since $f'(2) = 0$ and $f'(x) > 0$ for $0 < x < 2$ and $f'(x) < 0$ for $x > 2$ we have that $x = 2$ is a maximum.

- **Concavity and convexity**. Consider the second derivative

$$f''(x) = -e^{-x}x\,(2-x) + e^{-x}\,(2-x) - e^{-x}x = e^{-x}(x^2 - 4x + 2).$$

Consider the roots of the polynomial $x^2 - 4x + 2$

$$x_{1,2} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

so $f''(x) > 0$, and hence f is convex, for $x \in \left(-\infty, 2-\sqrt{2}\right)$ or $x \in \left(2+\sqrt{2}, \infty\right)$. Viceversa $f''(x) < 0$, and hence f is concave, if $x \in \left(2-\sqrt{2}, 2+\sqrt{2}\right)$.

- **Graph.** See Figure 11.

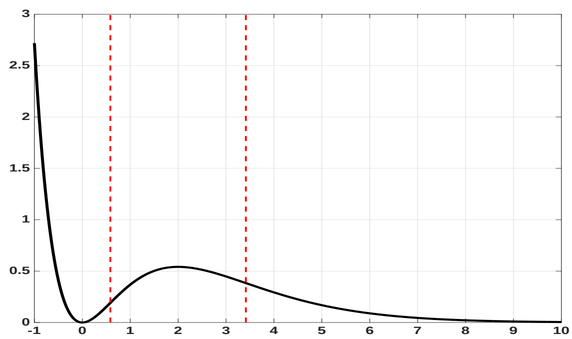


Fig. 11: The red dotted lines indicate the position of the points $2+\sqrt{2}$ and $2-\sqrt{2}$.

Esercize 139. Find the domain, vertical asymptotes, horizontal asymptotes and the intersection with the $x = 0$ and $y = 0$ axes, plus study the sign, monotonicity, maxima, minima, concavity and convexity of the function

$$f(x) = \frac{x - x^3}{1 + x^2}.$$

Solution.

- **Domain**. Since the numerator is a polynomial and the denominator is $1 + x^2 > 0$ for all x then the domain D of the function is $D = \mathbb{R}$.

- **Asymptotes**. There are no vertical asymptotes since the function has no critical points. Now consider the limit

$$\lim_{x\to+\infty} \frac{x - x^3}{1 + x^2} = \lim_{x\to+\infty} \frac{x^3\left(\frac{1}{x^2} - 1\right)}{x^2\left(\frac{1}{x^4} + 1\right)} = -\infty$$

$$\lim_{x\to-\infty}\frac{x-x^3}{1+x^2}=\lim_{x\to-\infty}\frac{x^3\left(\frac{1}{x^2}-1\right)}{x^2\left(\frac{1}{x^4}+1\right)}=+\infty$$

so there are no horizontal asymptotes.

- **Intersection with** $x=0$. If $x=0$ we get $f(0)=0$.
- **Intersection with** $y=0$. The equation

$$\frac{x-x^3}{1+x^2}=0$$

is equivalent to

$$x-x^3=x\left(1-x^2\right)=0$$

and this is because $1+x^2>0$ for all x. Hence the function intersects the axis $y=0$ in $x=0$ and $x=\pm1$.

- **Sign**. Since, trivially, $1+x^2\geq 0$ we have that $f(x)\geq 0$ for all x such that $x\left(1-x^2\right)\geq 0$ hence for all x such that $x\in(-\infty,-1]$ or $x\in[0,1]$.
- **Monotonicity**. Compute the first derivative

$$f'(x)=-\frac{x^4+4x^2-1}{\left(x^2+1\right)^2}$$

Hence

$$f'(x)\geq 0\Leftrightarrow x^4+4x^2-1\leq 0.$$

Put $x^2=t$ and find the solution of

$$t^2+4t-1=0$$

which are

$$t_{1,2}=-2\pm\sqrt{5}.$$

Since $t=x^2\geq 0$ only the solution $-2+\sqrt{5}$ is acceptable. So the equation

$$x^4+4x^2-1=0$$

has the two real solutions $-\sqrt{-2+\sqrt{5}}$ and $+\sqrt{-2+\sqrt{5}}$. So the function is decreasing in $\left(-\infty,-\sqrt{-2+\sqrt{5}}\right]$, increasing in $\left[-\sqrt{-2+\sqrt{5}},+\sqrt{-2+\sqrt{5}}\right]$ and decreasing in $\left[+\sqrt{-2+\sqrt{5}},\infty\right)$

- **Maxima and minima**. Since $f'\left(-\sqrt{-2+\sqrt{5}}\right)=0$ and $f'(x)<0$ for $x<-\sqrt{-2+\sqrt{5}}$ and $f'(x)>0$ for $-\sqrt{-2+\sqrt{5}}<x<\sqrt{-2+\sqrt{5}}$ we have that $x=-\sqrt{-2+\sqrt{5}}$ is a minimum. Similarly since $f'\left(\sqrt{-2+\sqrt{5}}\right)=0$ and $f'(x)>0$ for $-\sqrt{-2+\sqrt{5}}<x<\sqrt{-2+\sqrt{5}}$ and $f'(x)<0$ for $x>\sqrt{-2+\sqrt{5}}$ we have that $x=\sqrt{-2+\sqrt{5}}$ is a maximum.

- **Concavity and convexity**. Consider the second derivative
$$f''(x) = \frac{4x\left(x^2-3\right)}{\left(x^2+1\right)^3}$$
so $f''(x) > 0$, and hence f is convex, for $x \in \left(-\sqrt{3}, 0\right)$ or $x \in \left(\sqrt{3}, \infty\right)$. Viceversa $f''(x) < 0$, and hence f is concave, if $x \in \left(0, \sqrt{3}\right)$.

- **Graph.** See Figure 12.

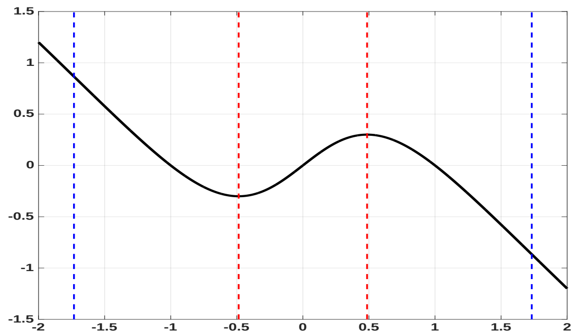


Fig. 12: The red dotted lines indicate the position of the points $-\sqrt{-2+\sqrt{5}}$ and $\sqrt{-2+\sqrt{5}}$. The blue dotted lines indicate the position of $-\sqrt{3}$ and $\sqrt{3}$.

Esercize 140. Find the domain, vertical asymptotes, horizontal asymptotes and the intersection with the $x = 0$ and $y = 0$ axes, plus study the sign, monotonicity, maxima, minima, concavity and convexity of the function
$$f(x) = 2x + \ln\left(\frac{1-x}{1+x}\right).$$

Solution.

- **Domain**. The domain of
$$f(x) = 2\,x + \ln\left(\frac{1-x}{1+x}\right)$$
coincides with the domain of $\ln\left(\frac{1-x}{1+x}\right)$. The logarithmic function $\ln(y)$ is defined if and only if $y > 0$ so we need to impose that $\frac{1-x}{1+x} > 0$ which implies $x \in (-1, 1)$ so the domain is $D = (-1, 1)$.

- **Asymptotes**. There are two possible vertical asymptotes. Consider the limits in the critical points $x = -1$ and $x = 1$. If $x \to -1^+$ then $\frac{1-x}{1+x} \to +\infty$ and hence
$$\lim_{x\to-1^+}\left(2\,x + \ln\left(\frac{1-x}{1+x}\right)\right) = +\infty$$
so $x = -1$ is a vertical asymptote. If $x \to 1^-$ then $\frac{1-x}{1+x} \to 0^+$ and hence
$$\lim_{x\to1^-}\left(2\,x + \ln\left(\frac{1-x}{1+x}\right)\right) = -\infty$$

so $x = 1$ is a vertical asymptote. We cannot look for horizontal asymptotes given that the domain of f is bounded.

- **Intersection with** $x = 0$. If $x = 0$ we get $f(0) = 0$.
- **Intersection with** $y = 0$. The equation

$$2\,x + \ln\left(\frac{1-x}{1+x}\right) = 0 \tag{22.2}$$

has at least the solution $x = 0$. From the sign of the derivative we can establish if this solution is unique or note, so let's move forward.

- **Sign**. We cannot say anything on the inequality

$$2\,x + \ln\left(\frac{1-x}{1+x}\right) \geq 0,$$

again we have to use the sign of the derivative to say more.

- **Monotonicity**. Compute the first derivative

$$f'(x) = -\frac{2x^2}{1-x^2}$$

so $f'(x) < 0$ for all x in the domain of the function, except for $x \neq 0$. This means that the function is **strictly** decreasing in its domain. Since $f(0) = 0$ this means that $x = 0$ is the unique solution of the equation (21.3) and, besides, that $f(x) > 0$ for $x < 0$ and $f(x) < 0$ for $x > 0$.

- **Maxima and minima**. Since f is **strictly** decreasing in its domain there are no maxima and no minima.
- **Concavity and convexity**. Consider the second derivative

$$f''(x) = -\frac{4x}{\left(x^2-1\right)^2}$$

so $f''(x) > 0$, and hence f is convex, for $x < 0$. Viceversa $f''(x) < 0$, and hence f is concave, if $x > 0$.

- **Graph.** See Figure 13.

Esercize 141. Find the domain, vertical asymptotes, horizontal asymptotes and the intersection with the $x = 0$ and $y = 0$ axes, plus study the sign, monotonicity, maxima, minima, concavity and convexity of the function

$$f(x) = \frac{\sqrt{x}}{1+\ln(x)}$$

Solution.

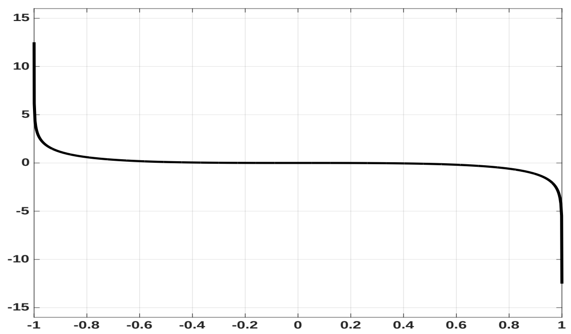


Fig. 13: The graph of $f(x) = 2x + \ln\left(\frac{1-x}{1+x}\right)$.

- **Domain**. The domain of

$$f(x) = \frac{\sqrt{x}}{1+\ln(x)}$$

is determined by the conditions $x > 0$ (in order to have $\sqrt{x}$ and $\ln(x)$ defined) and $\ln(x) \neq -1$ (in order to have the denominator different from zero), which is equivalent to $x \neq e^{-1} = 1/e$. So the domain is $D = (0, 1/e) \cup (1/e, \infty)$.

- **Asymptotes**. There is one possible vertical asymptote at the critical point $x = 1/e$ and one at the critical point $x = 0$. If $x \to (1/e)^+$ then $1 + \ln(x) \to 0^+$ and hence

$$\lim_{x\to(1/e)^+} \frac{\sqrt{x}}{1+\ln(x)} = +\infty$$

while if $x \to (1/e)^-$ then $1 + \ln(x) \to 0^-$ and hence

$$\lim_{x\to(1/e)^-} \frac{\sqrt{x}}{1+\ln(x)} = -\infty.$$

Hence $x = 1/e$ is a vertical asymptote. Nevertheless since

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{1+\ln(x)} = \frac{0}{-\infty} = 0.$$

then $x = 0$ it is not a vertical asymptote. Now consider

$$\lim_{x\to\infty} \frac{\sqrt{x}}{1+\ln(x)} = \frac{\infty}{\infty} \overset{H}{=} \lim_{x\to\infty} \frac{x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1}{2}\sqrt{x} = +\infty,$$

so there are no horizontal asymptotes.

- **Intersection with** $x = 0$. The point $x = 0$ is outside of the domain.

- **Intersection with** $y = 0$. The equation

$$\frac{\sqrt{x}}{1+\ln(x)} = 0 \tag{22.3}$$

has no solution, since the numerator is zero at $x = 0$ but the denominator is not defined at $x = 0$. Nevertheless we already know that the function $\frac{\sqrt{x}}{1+\ln(x)}$ approaches zero as $x \to 0^+$.

- **Sign**. Since $\sqrt{x} \geq 0$ always, the sign of

$$\frac{\sqrt{x}}{1+\ln(x)},$$

is equivalent to the sign of $1+\ln(x)$. Hence $f(x) \geq 0$ if $x \in (1/e, \infty)$ and $f(x) \leq 0$ if $x \in (0, 1/e)$.

- **Monotonicity**. Compute the first derivative

$$f'(x) = \frac{\log(x) - 1}{2\sqrt{x}(\log(x) + 1)^2}$$

so $f'(x) < 0$ for all $x < e$ (f is decreasing) and $f'(x) > 0$ for all $x > e$ (f is increasing).

- **Maxima and minima**. By the considerations above $x = e$ is a minimum and there are no maxima.

- **Concavity and convexity**. Consider the second derivative

$$f''(x) = \frac{7 - \log(x)(\log(x) + 2)}{4x^{3/2}(\log(x) + 1)^3}.$$

In order to find the zeros of $f''(x)$ we have to solve the equation

$$7 - \log(x)(\log(x) + 2) = 0,$$

which, putting $y = \log(x)$, is equivalent to

$$7 - y\,(y + 2) = 0$$

whose solutions are

$$y_{1,2} = -1 \pm 2\sqrt{2}$$

hence the solutions of $7 - \log(x)(\log(x) + 2) = 0$ are

$$x_{1,2} = e^{y_{1,2}} = e^{-1\pm 2\sqrt{2}}.$$

Hence the numerator of $f''(x)$ is positive for $x \in \left(e^{-1-2\sqrt{2}}, e^{-1+2\sqrt{2}}\right)$ while the denominator is positive for $(\log(x) + 1)^3 > 0$ which is equivalent to $\log(x) + 1 > 0$, that is for $x > 1/e$. So combining the sign of the numerator and of the denominator we get $f''(x) > 0$ for $x \in \left(0, e^{-1-2\sqrt{2}}\right)$, $f''(x) < 0$ for $x \in \left(e^{-1-2\sqrt{2}}, 1/e\right)$, $f''(x) > 0$ for $x \in \left(1/e, e^{-1+2\sqrt{2}}\right)$ and $f''(x) < 0$ for $x \in \left(e^{-1+2\sqrt{2}}, \infty\right)$.

- **Graph.** See Figure 13.

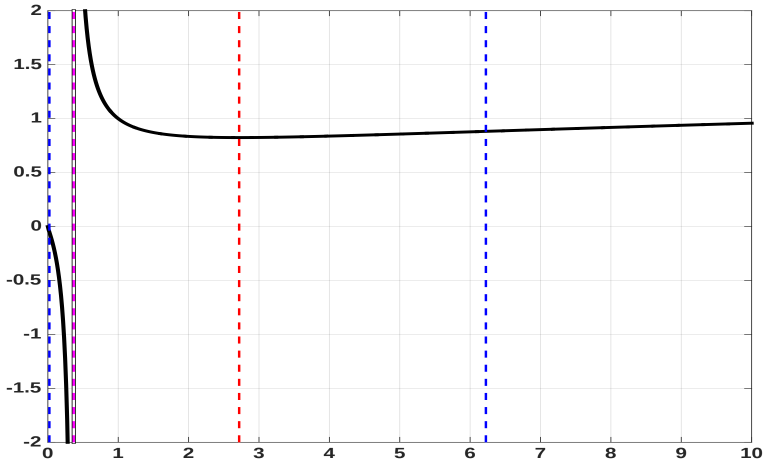


Fig. 14: Blue lines represents the position of the points $e^{-1\pm 2\sqrt{2}}$. The magenta line is the vertical asymptote $x = 1/e$ while the red line is the position of the minimum $x = e$.