Max/Min and Sup/Inf

Davide Pirino

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We already know that $\nexists x \in \mathbb{Q}$ such that $x^2 = 2$.

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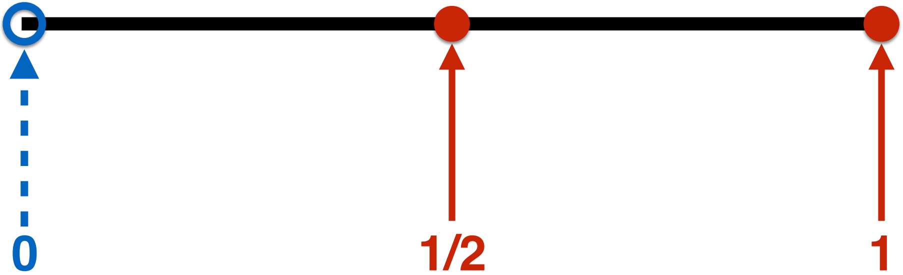
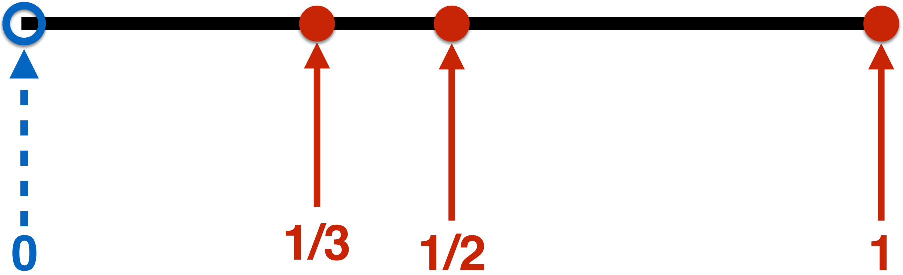
0
1


0
1/2
1


0
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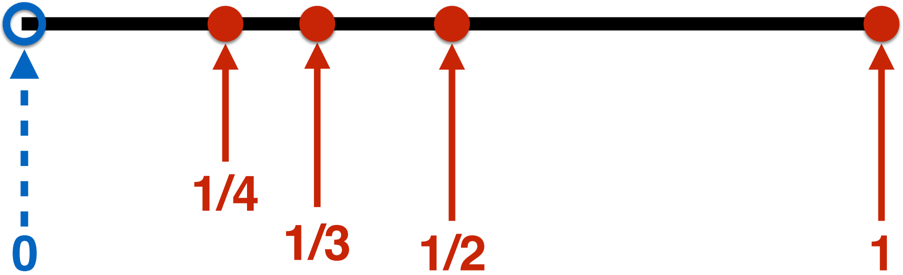
0
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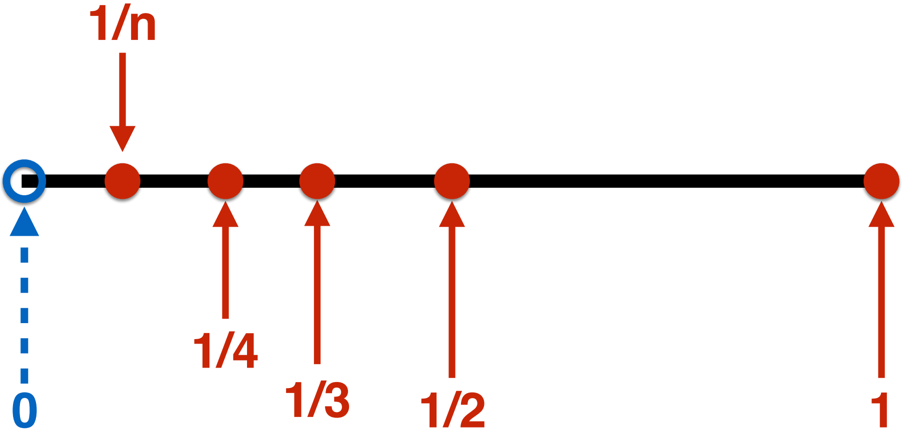
1/n
0
1/4
1/3
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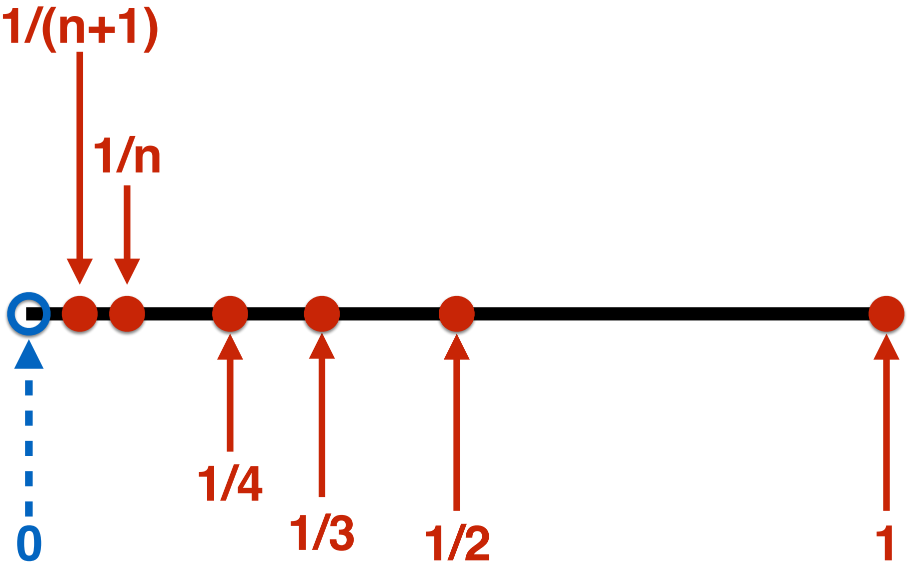
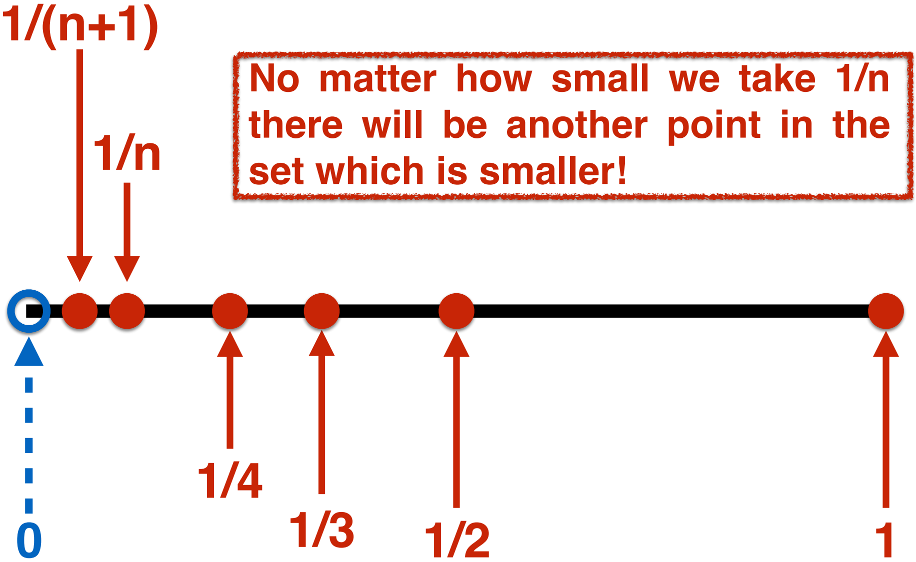
1


1/(n+1)
1/n
0
1/4
1/3
1/2
1


1/(n+1)
1/n
No matter how small we take 1/n there will be another point in the set which is smaller!
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$$E = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N},\ n > 0 \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$$

then $\nexists$ the minimum of E, nevertheless

$$E' = E \bigcup \{0\} = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$$

is such that 0 is the minimum point of E'. Similarly

$$F = \left\{ -\frac{1}{n} \;\middle|\; n \in \mathbb{N},\ n > 0 \right\} = \left\{ -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, \dots \right\}$$

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Remark

The set $E = \left\{ \frac{1}{n} \mid n \in \mathbb{N},\ n > 0 \right\}$ is clearly "limited from below", but...

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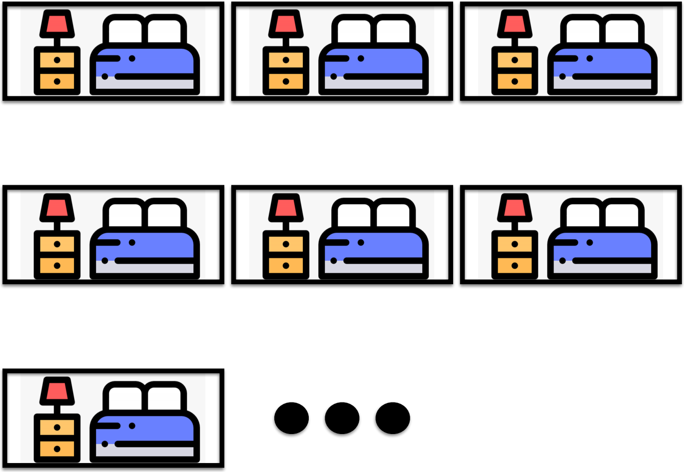
The definition of max and min appears to be too **tight**!

To define infimum and supremum of a set we need to define $\pm\infty$.

Hilbert's paradox of the Grand Hotel



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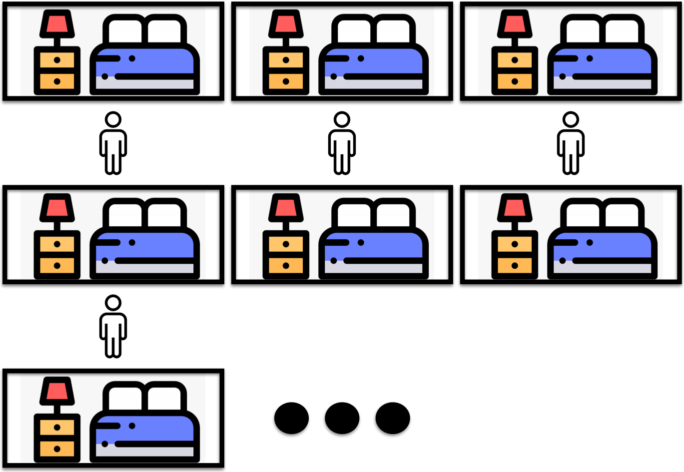

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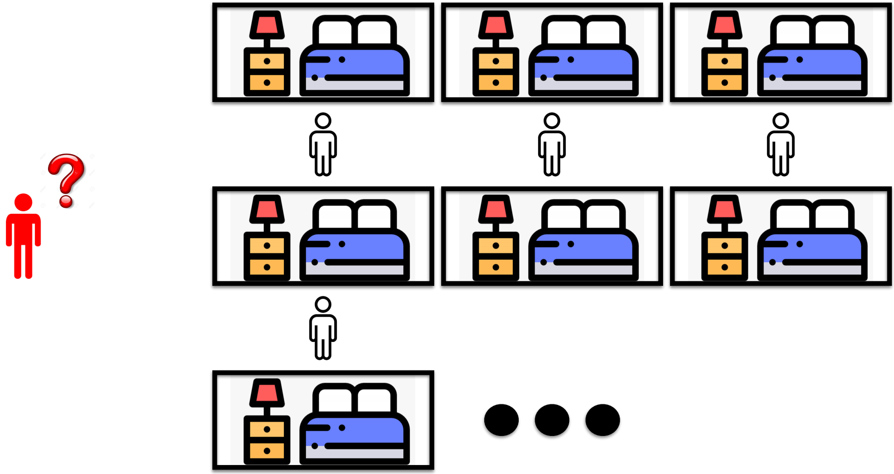

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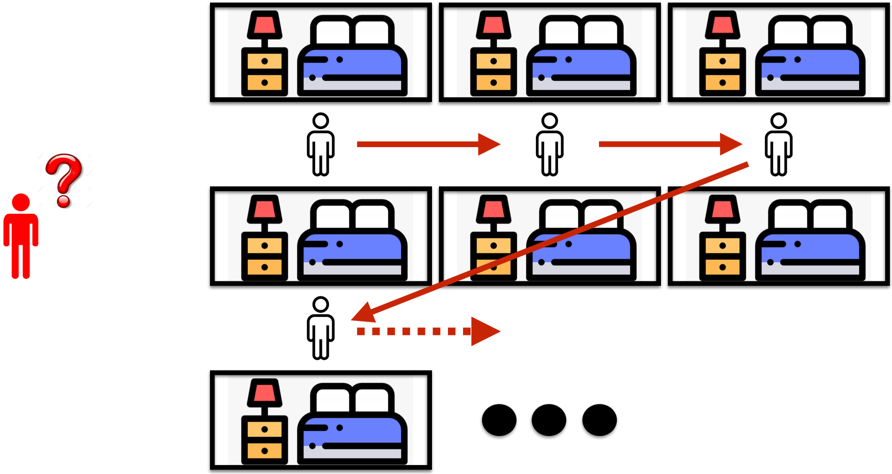

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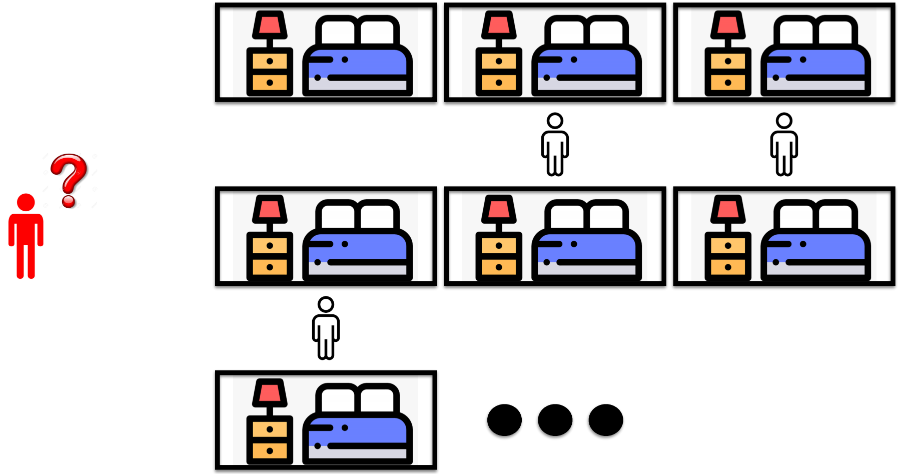

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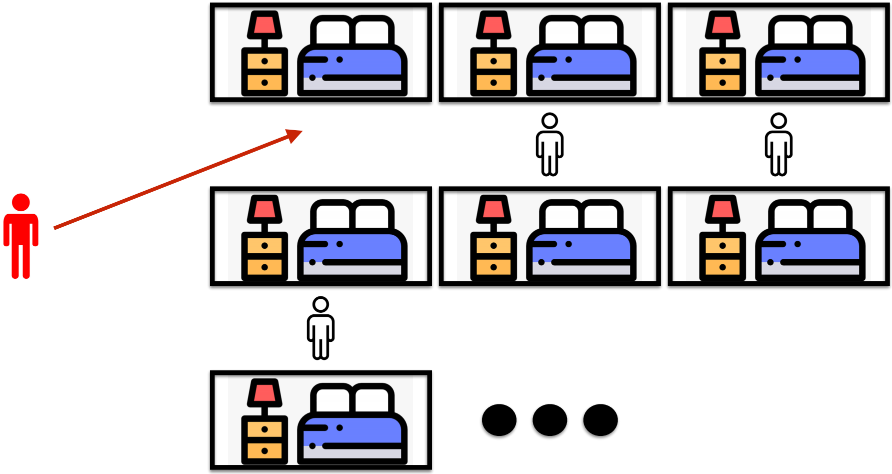

THE PARADOX

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There is no room for a new guest!

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All rooms are occupied!

There is no room for a new guest!

Are **not equivalent** if the number of rooms is infinite!!

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Indeterminate forms

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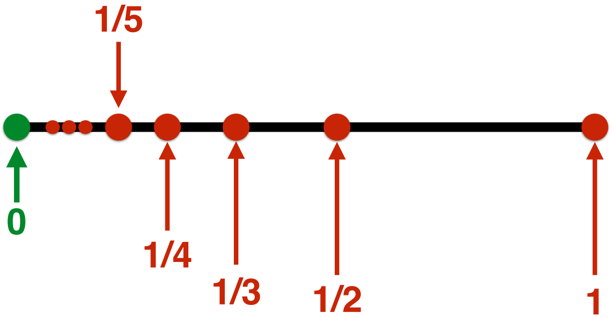
Any point 0<x<1 cannot be neither a lower bound nor an upper bound!



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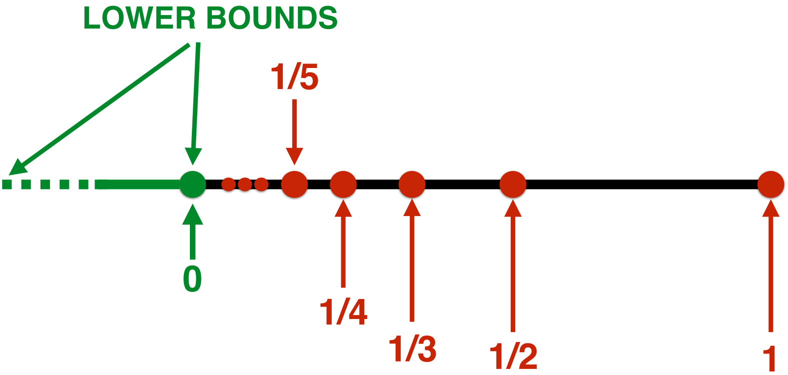

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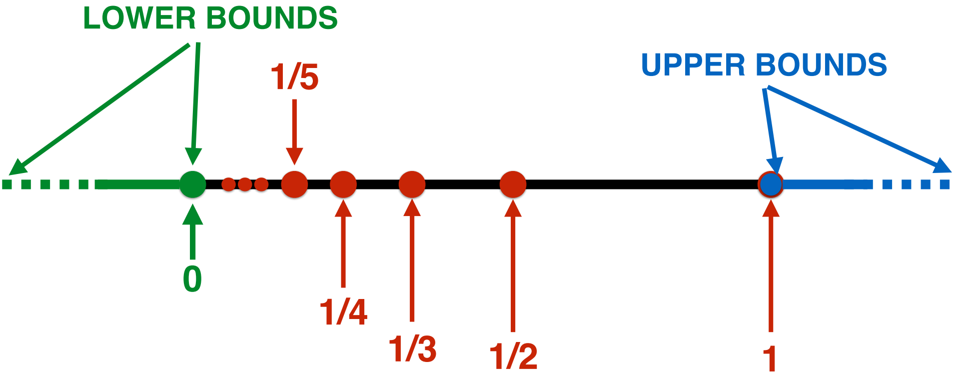

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$E = \left\{\frac{1}{n} \;\middle|\; n \in \mathbb{N},\ n > 0\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$ *then*

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$$U_E = \{u \in \mathbb{Q} \mid u \geq 1\}, \quad L_E = \{\ell \in \mathbb{Q} \mid \ell \leq 0\}$$

Example

$E = \left\{\frac{1}{n^2} \;\middle|\; n \in \mathbb{N},\ n > 0\right\} = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\right\}$ *then*

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Definition

Let $E \subseteq \mathbb{Q}$. Recall that

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Example

Consider the set

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Theorem

Let $E \subseteq \mathbb{Q}$ be a subset of $\mathbb{Q}$. The following statements hold

- *If E has a maximum point M then $\sup_{\mathbb{Q}}(E) = M$.*

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Although they could be $+\infty$ or $-\infty$ ...

Exercize

Set	max	min	sup	inf

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Exercize

Set	max	min	sup	inf
$\{n \in \mathbb{N} \mid n \leq 10\}$	10			

Exercize

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Sup/Inf and Max/Min: some exercises

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$\left\{1 - \frac{1}{n^3} \mid n \in \mathbb{N},\ n > 0\right\}$	$\nexists$	0	1	0
$\{q \in \mathbb{Q} \mid 0 \leq q < 2\}$				

Exercize

Set	max	min	sup	inf
$\{n \in \mathbb{N} \mid n \leq 10\}$	10	0	10	0
$\{q \in \mathbb{Q} \mid q > -1\}$	$\nexists$	$\nexists$	$+\infty$	-1
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Sup/Inf and Max/Min: some exercises

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Can we fill them?

Natural, $\mathbb{N}$

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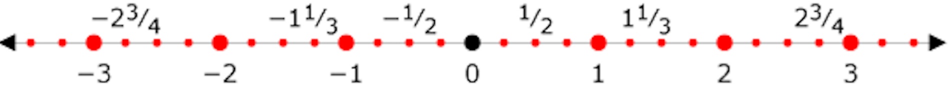

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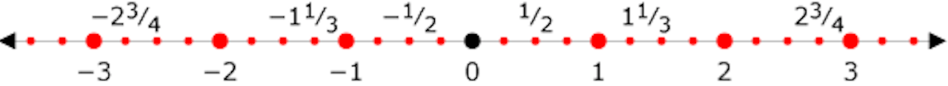
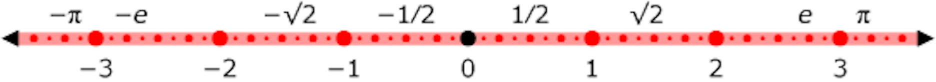


Real, $\mathbb{R}$ Fill in all the numbers to make a continuous line.



Rational numbers

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The set $\mathbb{I}$ is called the set of irrational numbers and contains all the number whose decimal representation is **neither finite nor periodic**!

Irrational numbers: $\sqrt{2}$

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$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885038753432764157273501384623091229702049\ldots$$

We will never know the true value of $\sqrt{2}$! We can only construct algorithms to compute which are the next digits ...

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Example

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