The Ramsey Model

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Introduction

- Ramsey-Cass-Koopmans model: differs from the Solow model only because it explicitly models the consumer side and endogenizes savings.
- Beyond its use as a basic growth model, also a workhorse for many areas of macroeconomics.
- Infinite-horizon, continuous time.

Production Function, Technology

- Assume all firms have access to the same production function:
- Aggregate production function for the unique final good $Y(t)$ is:

$$Y(t) = F[K(t), L(t), A(t)] \tag{1}$$

- Assume capital $K(t)$ is the same as the final good of the economy. $L(t)$ denotes the demand for labor.
- $A(t)$ is technology.
- Major assumption: technology is **free**; it is publicly available as a non-excludable, non-rival good. In fact we will assume no technological progress, so A is constant.

Key Assumption

Assumption 1 **(Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale)** The production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0,$$
$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

Moreover, F exhibits constant returns to scale in K and L.

- Assume F exhibits *constant returns to scale* in K and L. I.e., it is *linearly homogeneous* (homogeneous of degree 1) in these two variables.

Review

Definition The function $g : \mathbb{R}^2 \to \mathbb{R}$ is homogeneous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if and only if

$$g(\lambda x, \lambda y) = \lambda^m g(x, y) \text{ for all } \lambda \in \mathbb{R}_+ .$$

Theorem **(Euler's Theorem)** Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y and is homogeneous of degree m in x and y. Then

$$\begin{aligned} mg(x, y, z) &= g_x(x, y)x + g_y(x, y)y \\ \text{for all } x &\in \mathbb{R}, y \in \mathbb{R} . \end{aligned}$$

Moreover, $g_x(x, y)$ and $g_y(x, y)$ are themselves homogeneous of degree $m - 1$ in x and y.

Market Structure and Endowments I

- We will assume that markets are competitive.
- Households own all of the labor $\bar{L}(t)$, which they supply inelastically.
- Households own the capital stock of the economy and rent it to firms.
- Denote the *rental price of capital* at time t be $R(t)$.

Market Structure and Endowments II

- The price of the final good is normalized to 1 *in all periods*
- Assume capital depreciates, with "exponential form," at the rate δ: out of 1 unit of capital this period, only $1-\delta$ is left for next period.
- This affects the interest rate (rate of return to savings) faced by the household.
- *Interest rate* faced by the household will be $r(t) = R(t) - \delta$.

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Firm Optimization I

- Only need to consider the problem of a *representative firm*:

$$\max_{L(t)\geq 0, K(t)\geq 0} F[K(t), L(t), A(t)] - w(t) L(t) - R(t) K(t).$$

- This is a static maximization problem.
  1. Firm is taking as given $w(t)$ and $R(t)$.
  2. Concave problem, since F is concave.

Firm Optimization II

- Since F is differentiable, first-order necessary conditions imply:

$$w(t) = F_L[K(t), L(t), A(t)],$$

and

$$R(t) = F_K[K(t), L(t), A(t)].$$

Firm Optimization III

Given RCS firms make no profits, and in particular,

$$Y(t) = w(t) L(t) + R(t) K(t).$$

- This follows immediately from Euler Theorem.

Second Useful Assumption

Assumption 2 **(Inada conditions)** F satisfies the Inada conditions

$$\begin{aligned}\lim_{K \to 0} F_K(\cdot) &= \infty \text{ and } \lim_{K \to \infty} F_K(\cdot) = 0 \text{ for all } L > 0 \text{ all } A \\ \lim_{L \to 0} F_L(\cdot) &= \infty \text{ and } \lim_{L \to \infty} F_L(\cdot) = 0 \text{ for all } K > 0 \text{ all } A.\end{aligned}$$

- Important in ensuring the existence of *interior equilibria.*

Production Functions

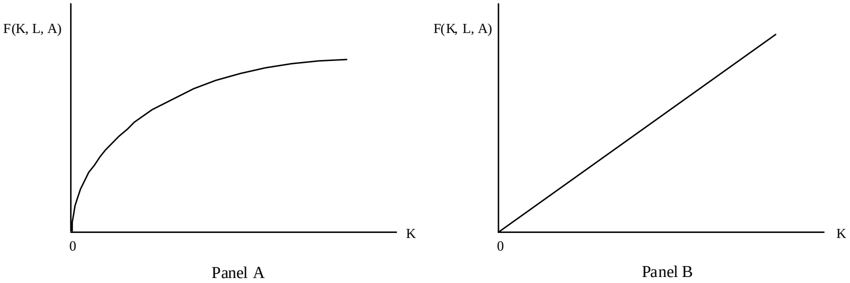

Figure: Production functions. Only in A the IC are satisfied.

Per Capita Production Function

- Per capita production function $f(\cdot)$

$$
\begin{aligned}
y(t) &\equiv \frac{Y(t)}{L(t)} \\
&= F\left[\frac{K(t)}{L(t)}, 1\right] \\
&\equiv f(k(t)),
\end{aligned}
$$

where

$$k(t) \equiv \frac{K(t)}{L(t)}.$$

- Competitive factor markets then imply:

$$R(t) = F_K[K(t), L(t)] = f'(k(t)).$$

Remember since F homogeneous of degree one, its derivatives are homogeneous of degree zero.

$$w(t) = F_L[K(t), L(t)] = f(k(t)) - k(t) f'(k(t)).$$

Preferences and Demographics I

- Representative household with instantaneous utility function

$$u\left(c\left(t\right)\right),$$

Assumption $u\left(c\right)$ is strictly increasing, concave, twice continuously differentiable satisfies the following Inada type assumptions:

$$\lim_{c\to 0} u'\left(c\right) = \infty \text{ and } \lim_{c\to\infty} u'\left(c\right) = 0.$$

- Suppose representative household represents set of identical households (normalized to 1).
- $L\left(0\right) = 1$ and

$$L\left(t\right) = \exp\left(nt\right).$$

Preferences and Demographics II

- All members of the household supply their labor inelastically.
- Objective function of each household at $t = 0$:

$$U(0) \equiv \int_0^{\infty} \exp(-(\rho - n)t)\, u(c(t))\, dt,$$

where

  - $c(t)$=consumption per capita at t,
  - ρ=time discount rate.
  - $$c(t) \equiv \frac{C(t)}{L(t)}$$

Preferences and Demographics III

- Utility of $u(c(t))$ per household member at time t, total utility of household $L(t)u(c(t)) = \exp(nt)u(c(t))$.

$$\rho > n.$$

- Ensures that discounted utility is finite.

Preferences, Technology and Demographics V

- Denote asset holdings of the representative household at time t by $\mathcal{A}(t)$. Then,

$$\dot{\mathcal{A}}(t) = r(t)\,\mathcal{A}(t) + w(t)\,L(t) - c(t)\,L(t)$$

- $r(t)$ is the rate of return on assets owned by the family , and $w(t)\,L(t)$ is the flow of labor income

Aside: Imagine to start from the usual budget constraint in discrete time

$$\mathcal{A}(t+1) - \mathcal{A}(t) = r(t)\mathcal{A}(t) + w(t)\,L(t) - c(t)\,L(t)$$

If we subdivide the unit time interval into $1/\Delta t$ subintervals of length Δt we can write:

$$\frac{\mathcal{A}(t+\Delta t) - \mathcal{A}(t)}{\Delta t} \simeq r(t)\mathcal{A}(t) + (w(t) - c(t))\,L(t)$$

as $\Delta t \to 0$ we get to the B.C. in continous time. Notice the $\simeq$ is used because it is assumed that $r(t)a(t) + w(t) - c(t)$ does not change in (t, t+Δt).

- Defining per capita assets as

$$a(t) \equiv \frac{\mathcal{A}(t)}{L(t)},$$

we obtain:

$$\dot{a}(t) = (r(t) - n)\, a(t) + w(t) - c(t).$$

- Depreciation rate of δ implies

$$r(t) = R(t) - \delta.$$

The Budget Constraint I

- The differential equation

$$\dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t)$$

is a flow constraint.

- Not sufficient as a proper budget constraint unless we impose a lower bound on terminal assets: in finite-horizon with terminal time T we would simply impose $\mathcal{A}(T) \geq 0$ as a boundary condition.
- With infinite horizon: No-Ponzi Game Condition.

Technical Aside: Integration of $\dot{x} = y(t)x(t) + b(t)$
Multiply by both sides by $\exp\left(-\int_0^t y(v)\, dv\right)$,"integrating factor", ie write:

$(\dot{x} - y(t)x(t)) \exp\left(-\int_0^t y(v)\, dv\right) = b(t) \exp\left(-\int_0^t y(v)\, dv\right)$

on the LHS we have

$\dot{x} \exp\left(-\int_0^t y(v)\, dv\right) - y(t)\, x(t) \exp\left(-\int_0^t y(v)\, dv\right) = \frac{d\left(x(t)\exp\left(-\int_0^t y(v) dv\right)\right)}{dt}$

on the RHS we have

$b(t) \exp\left(-\int_0^t y(v)\, dv\right) = \frac{d}{dt} \int_0^t b(s) \exp\left(-\int_0^s y(v)\, dv\right) ds =$

So we can integrate both sides:

$x(t) \exp\left(-\int_0^t y(v)\, dv\right) = \int_0^t b(s) \exp\left(-\int_0^s y(v)\, dv\right) ds + C$,or:

$x(t) = \int_0^t b(s) \exp\left(\int_s^t y(v)\, dv\right) ds + C \exp\left(\int_0^t y(s)\, ds\right) dt$. If initial condition $x(0)=X_0$ then $C=X_0$

The Budget Constraint II

Integrating the BC: (IF(t)=exp $\left(\int_0^t (n - r(s))\, ds\right)$)

$\dot{a}(t) = (r(t) - n)\, a(t) + w(t) - c(t) \rightarrow a(t) = \int_0^t (w(s) - c(s)) \exp\left(\int_s^t (r(v) - n(v))\, dv\right) ds + a(0) \exp\left(\int_0^t (r(s) - n)\, ds\right)$

$a(t) \left(\exp - \left(\int_0^t (r(s) - n)\, ds\right)\right) =$

$\int_0^t (w(s) - c(s)) \left(\exp - \left(\int_0^s (r(v) - n(v))\, dv\right)\right) ds + a(0)$

The Budget Constraint III

- Infinite-horizon case: no-Ponzi-game condition,

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s)-n)\,ds\right) \geq 0.$$

PDV of financial wealth must be positive asymptotically.

The Budget Constraint V

- Take the limit of the BC as $t \rightarrow \infty$ and use the no-Ponzi-game condition to obtain

$$\begin{aligned} & \int_0^\infty (c(s) - w(s)) \exp\left(-\int_0^s (r(v) - n)\, dv\right) ds \\ = \; & a(0) \end{aligned}$$

- Thus no-Ponzi-game condition essentially ensures that the PD sum of expenditures minus labour income is no more than than initial wealth.

Household Maximization I

- To maximize $U(0) \equiv \int_0^\infty \exp(-(\rho - n)t)\, u(c(t))\, dt$ subject to $\dot{a}(t) = (r(t) - n)\, a(t) + w(t) - c(t)$, $a(0) = a_0$ and NPG $\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s) - n)\, ds\right) \geq 0$.
- Set up the current-value Hamiltonian:

$$\hat{H}(a, c, \mu) = u(c(t)) + \mu(t)\left[w(t) + (r(t) - n)\, a(t) - c(t)\right],$$

- *Necessary conditions:*

$$\begin{aligned} \hat{H}_c(a, c, \mu) &= u'(c(t)) - \mu(t) = 0 \\ \hat{H}_a(a, c, \mu) &= \mu(t)(r(t) - n) \\ &= -\dot{\mu}(t) + (\rho - n)\,\mu(t) \\ \lim_{t\to\infty}\left[\exp(-(\rho - n)t)\,\mu(t)\, a(t)\right] &= 0. \end{aligned}$$

  and the transition equation.
- Notice transversality condition is written in terms of the current-value costate variable.

Household Maximization II

- For any $\mu(t) > 0$, f is a strictly concave function of c and g is concave in a and c.
- The first necessary condition implies $\mu(t) > 0$ for all t.
- Therefore, $\hat{H}(t, a, c, \mu)$ is strictly jointly concave in a and c: necessary conditions are sufficient and the candidate solution is a unique optimum.
- Rearrange the second condition:

$$\frac{\dot{\mu}(t)}{\mu(t)} = -\left(r(t) - \rho\right),$$

- First necessary condition implies:

$$u'(c(t)) = \mu(t).$$

Household Maximization III

- Differentiate with respect to time and divide by $\mu(t)$,

$$\frac{u''(c(t))\,c(t)}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)}.$$

- Substituting into $\frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho)$, obtain the consumer Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho)$$

  where

$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t))\,c(t)}{u'(c(t))}$$

  is the elasticity of the marginal utility $u'(c(t))$.

- Consumption will grow over time when the discount rate is less than the rate of return on assets.

Household Maximization IV

- Speed at which consumption will grow is related to the elasticity of marginal utility of consumption, $\varepsilon_u(c(t))$.
- $\varepsilon_u(c(t))$ is the inverse of the *intertemporal elasticity of substitution*: governs willingness to substitute consumption over time.

Household Maximization VI

- Notice term $\exp\left(-\int_0^t r(s)\,ds\right)$ is a present-value factor: converts a unit of income at t to a unit of income at 0.
- Define an average interest rate between dates 0 and t

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds.$$

Household Maximization VII

- And the transversality condition,

$$\lim_{t\to\infty}\left[\exp\left(-\left(\bar{r}\left(t\right)-n\right)t\right)a\left(t\right)\right]=0.$$

- Integrate $\frac{\dot{c}(t)}{c(t)}=\frac{1}{\varepsilon_u(c(t))}\left(r\left(t\right)-\rho\right)$

$$c\left(t\right)=c\left(0\right)\exp\left(\int_0^t\frac{r\left(s\right)-\rho}{\varepsilon_u\left(c\left(s\right)\right)}ds\right)$$

Household Maximization VIII

- Special case where $\varepsilon_u\left(c\left(s\right)\right)$ is constant, $\varepsilon_u\left(c\left(s\right)\right) = \sigma$:

$$c\left(t\right) = c\left(0\right)\exp\left(\left(\frac{\bar{r}\left(t\right) - \rho}{\sigma}\right)t\right).$$

Definition of Equilibrium

Definition A competitive equilibrium of the Ramsey economy consists of paths $\left[c(t), k(t), w(t), R(t)\right]_{t=0}^{\infty}$, such that the representative household maximizes utility subject to its budget constraint given initial capital-labor ratio $k(0)$, and factor prices $\left[w(t), R(t)\right]_{t=0}^{\infty}$,the representative firm maximizes profits at these prices, and all markets clear. .

Equilibrium Prices

- Equilibrium prices given by w=f-f'k and R=f'.
- Capital market clearing:

$$a(t) = k(t).$$

- Thus market rate of return for consumers, $r(t)$, is given by

$$r(t) = f'(k(t)) - \delta.$$

- Substituting this into the consumer's problem, we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right)$$

Optimal Growth I

- In an economy that admits a representative household, optimal growth involves maximization of utility of representative household subject to technology and feasibility constraints:

$$\max_{[k(t),c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left(-\left(\rho - n\right) t\right) u\left(c\left(t\right)\right) dt,$$

subject to

$$\dot{k}\left(t\right) = f\left(k\left(t\right)\right) - (n+\delta)k\left(t\right) - c\left(t\right),$$

and $k\left(0\right) > 0$, and $\lim_{t\to\infty} k\left(t\right) \geq 0$.

Optimal Growth II

- Again set up the current-value Hamiltonian:

$$\hat{H}(k, c, \mu) = u(c(t)) + \mu(t)\left[f(k(t)) - (n+\delta)k(t) - c(t)\right],$$

- Candidate solution from the *Maximum Principle*:

$$\begin{aligned}
\hat{H}_c(k, c, \mu) &= 0 = u'(c(t)) - \mu(t), \\
\hat{H}_k(k, c, \mu) &= -\dot{\mu}(t) + (\rho - n)\mu(t) \\
&= \mu(t)\left(f'(k(t)) - \delta - n\right), \\
\lim_{t\to\infty}\left[\exp(-(\rho - n)t)\mu(t)k(t)\right] &= 0.
\end{aligned}$$

- *Sufficiency Theorem* $\Rightarrow$ u f strictly concave,$\mu(t)$ positive: unique solution.

Optimal Growth III

- Repeating the same steps as before, these imply

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right),$$

which is identical to the Euler equation in the market economy.

- Thus the competitive equilibrium is a Pareto optimum and the Pareto allocation can be decentralized as a competitive equilibrium.

Proposition In the neoclassical growth model described above, with standard assumptions on the production function, the equilibrium is Pareto optimal and coincides with the optimal growth path maximizing the utility of the representative household.

Steady-State Equilibrium I

- Steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant, thus:

$$\dot{c}\left(t\right) = 0.$$

- From (**??**), as long as $f\left(k^*\right) > 0$, *irrespective* of the exact utility function, we must have a capital-labor ratio k^* such that

$$f'\left(k^*\right) = \rho + \delta,$$

- Pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.
- *Modified golden rule*: level of the capital stock that does not maximize steady-state consumption (as in the Solow model), because earlier consumption is preferred to later consumption.

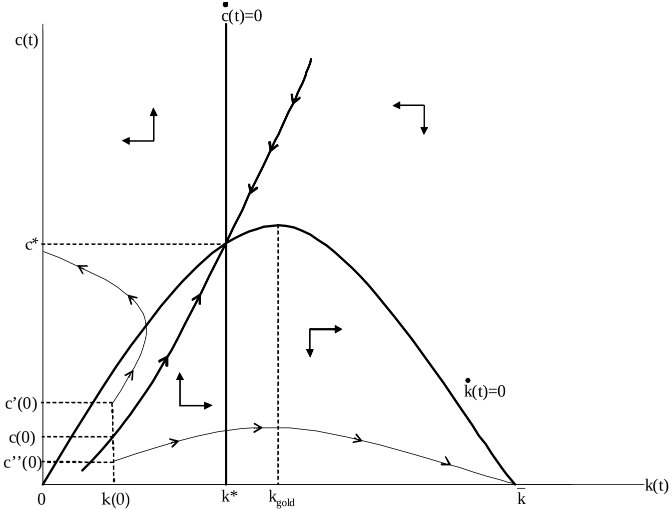

Figure: Steady state in the baseline neoclassical growth model

Steady-State Equilibrium II

- Given k^*, steady-state consumption level:

$$c^* = f(k^*) - (n + \delta)k^*,$$

- Given ρ higher than n, a steady state where the capital-labor ratio and thus output are constant necessarily satisfies the transversality condition.

Proposition In the neoclassical growth model described above the steady-state equilibrium capital-labor ratio, k^*, is is independent of the utility function.

- Parameterize the production function as follows

$$f(k) = A\tilde{f}(k),$$

Steady-State Equilibrium III

- Since $f(k)$ satisfies the regularity conditions imposed above, so does $\tilde{f}(k)$.

Proposition Consider the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption 4′, and suppose that $f(k) = A\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(A, \rho, n, \delta)$ and the steady-state level of consumption per capita by $c^*(A, \rho, n, \delta)$ when the underlying parameters are A, ρ, n and δ. Then we have

$$
\begin{aligned}
\frac{\partial k^*(\cdot)}{\partial A} &> 0, \ \frac{\partial k^*(\cdot)}{\partial \rho} < 0, \ \frac{\partial k^*(\cdot)}{\partial n} = 0 \text{ and } \frac{\partial k^*(\cdot)}{\partial \delta} < 0 \\
\frac{\partial c^*(\cdot)}{\partial A} &> 0, \ \frac{\partial c^*(\cdot)}{\partial \rho} < 0, \ \frac{\partial c^*(\cdot)}{\partial n} < 0 \text{ and } \frac{\partial c^*(\cdot)}{\partial \delta} < 0.
\end{aligned}
$$

Steady-State Equilibrium IV

- The discount factor affects the rate of capital accumulation.
- Loosely, lower discount rate implies greater patience and thus greater savings.
- Without technological progress, the steady-state saving rate can be computed as
$$s^* = \frac{(\delta + n)\, k^*}{f\,(k^*)}.$$
- Rate of population growth has no impact on the steady state capital-labor ratio, which contrasts with the basic Solow model.
  - result depends on the way in which intertemporal discounting takes place.
- k^* and thus c^* do *not* depend on the instantaneous utility function $u\,(\cdot)$.
  - form of the utility function only affects the transitional dynamics.

Transitional Dynamics I

- Equilibrium is determined by two differential equations:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t) \tag{2}$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right). \tag{3}$$

- Moreover, we have an initial condition $k(0) > 0$, and a boundary condition at infinity, (which comes from rewriting the TVC: do as an exercise)

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t \left(f'(k(s)) - \delta - n\right) ds\right)\right] = 0.$$

Transitional Dynamics II

- *Saddle-path stability*:
  - consumption level is the control variable, so $c(0)$ is free: has to adjust to satisfy transversality condition: if there were more than one $c(0)$ satisfying the necessary conditions equilibrium would be indeterminate. However we know solution is unique. So if we find one equilibrium satisfying NC we are done.
- See Figure.

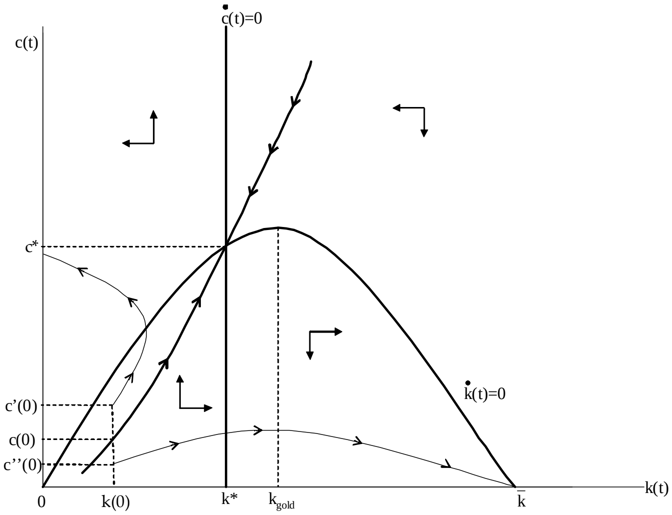

Figure: Transitional dynamics in the baseline neoclassical growth model

k_{gold} maximizes ss consumption
$c^* = f(k^*) - (n+\delta)k^* \rightarrow$f'$(k_{gold}) = n + \delta$.
f'(k*) $= \rho + \delta$ as we know $n < \rho$ and f' is a decreasing function we see k$^* < k_{gold}$.

Transitional Dynamics: Sufficiency

- Why is the stable arm unique?
  1. Mangasarian
  2. To build intuition: Graphical Analysis

-
  - if $c\left(0\right)$ started below it, say $c''\left(0\right)$, consumption would reach zero, thus capital would accumulate continuously until a level of capital (reached with zero consumption) $\bar{k} > k_{gold}$. This would violate the transversality condition $\lim_{t\to\infty}\left[k\left(t\right)\exp\left(-\int_0^t\left(f'\left(k\left(s\right)\right)-\delta-n\right)ds\right)\right]=0$,as $f'\left(\bar{k}\right)-\delta-n<0$.
  - if $c\left(0\right)$ started above this stable arm, say at $c'\left(0\right)$, the capital stock would reach 0 in finite time, while consumption would remain positive. But this would violate feasibility.
  - These arguments require a little care, since remember: necessary conditions do not apply at the boundary.

Technological Change and the Neoclassical Model

Extend the production function to:

$$Y(t) = F[K(t), A(t)L(t)],$$

where

$$A(t) = \exp(gt)A(0).$$

Technological Change II

- Define

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{A(t)L(t)} \\ &= F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f(k(t)), \end{aligned}$$

where

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}.$$

- Also need to impose a further assumption on preferences in order to ensure balanced growth.

Technological Change III

- Balanced growth requires that per capita consumption and output grow at a constant rate. $\tilde{c}(t) \equiv C(t)/L(t)$ .Euler equation

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{1}{\varepsilon_u(\tilde{c}(t))}(r(t) - \rho).$$

Technological Change IV

- If $r(t) \longrightarrow r^*$ (which is implied by $k(t) \longrightarrow k^*$), then $\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} \longrightarrow g_c$ is only possible if $\varepsilon_u(\tilde{c}(t)) \longrightarrow \varepsilon_u$, i.e., if the elasticity of marginal utility of consumption is asymptotically constant.
- Thus balanced growth is only consistent with utility functions that have asymptotically constant elasticity of marginal utility of consumption.

Example: CRRA Utility I

- Recall the Arrow-Pratt coefficient of relative risk aversion for a twice-continuously differentiable concave utility function $U(c)$ is

$$\mathcal{R} = -\frac{U''(c)\,c}{U'(c)}.$$

- Constant relative risk aversion (CRRA) utility function satisfies the property that $\mathcal{R}$ is constant.
- The family of CRRA utility functions is given by

$$U(c) = \begin{cases} \frac{c^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma \neq 1 \text{ and } \sigma \geq 0 \\ \ln c & \text{if } \sigma = 1 \end{cases},$$

with the coefficient of relative risk aversion given by σ. (Verify)

Technological Change V

- When $\sigma = 0$, these represent linear preferences, when $\sigma = 1$, we have log preferences, and as $\sigma \to \infty$, infinitely risk-averse, and infinitely unwilling to substitute consumption over time.
- Assume that the economy admits a representative household with CRRA preferences

$$\int_0^\infty \exp\left(-(\rho - n)t\right) \frac{\tilde{c}\left(t\right)^{1-\sigma} - 1}{1-\sigma} dt,$$

Technological Change VI

- Refer to this model, with labor-augmenting technological change and CRRA preference as the *canonical model*
- Euler equation takes the simpler form:

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{1}{\sigma}\left(r(t) - \rho\right).$$

- Steady-state equilibrium first: since with technological progress there will be growth in per capita income, $\tilde{c}(t)$ will grow.

Technological Change VII

- Instead define

$$\begin{aligned} c(t) &\equiv \frac{C(t)}{A(t)L(t)} \\ &\equiv \frac{\tilde{c}(t)}{A(t)}. \end{aligned}$$

- This normalized consumption level will remain constant along the BGP:remember c=C/LA=$\tilde{c}$/A

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} &\equiv \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} - g \\ &= \frac{1}{\sigma}\left(r(t) - \rho - \sigma g\right). \end{aligned}$$

Technological Change VIII

- For the accumulation of capital stock:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta) k(t),$$

where $k(t) \equiv K(t) / A(t) L(t)$.

- Transversality condition, in turn, can be expressed as

$$\lim_{t\to\infty} \left\{ k(t) \exp\left(-\int_0^t \left[f'(k(s)) - g - \delta - n \right] ds \right) \right\} = 0. \quad (4)$$

- By homogeneity of degree zero of f' equilibrium $r(t)$ is still given by:

$$r(t) = f'(k(t)) - \delta$$

Technological Change IX

- Since in steady state $c(t)$ must remain constant:

$$r(t) = \rho + \sigma g$$

or

$$f'(k^*) = \rho + \delta + \sigma g,$$

- Pins down the steady-state value of the normalized capital ratio k^* uniquely.
- Normalized consumption level is then given by

$$c^* = f(k^*) - (n + g + \delta) k^*,$$

- Per capita consumption grows at the rate g.

Technological Change X

- Because there is growth, to make sure that the transversality condition is in fact substituted :

$$\lim_{t \to \infty} \left\{ k(t) \exp\left(-\int_0^t [\rho - (1-\sigma) g - n] \, ds \right) \right\} = 0,$$

$$\rho - n > (1 - \sigma) g.$$

- Remarks:
  - Strengthens Assumption $4'$ when $\sigma < 1$.
  - Alternatively, recall in steady state $r = \rho + \sigma g$ and the growth rate of output is $g + n$.
  - Therefore, equivalent to requiring that $r > g + n$.

Technological Change XI

Proposition Consider the neoclassical growth model with labor augmenting technological progress at the rate g and preferences given by (**??**). Suppose that Assumptions 1, 2, assumptions on utility above hold and $\rho - n > (1-\sigma) g$. Then there exists a unique balanced growth path with a normalized capital to effective labor ratio of k^*, given by (**??**), and output per capita and consumption per capita grow at the rate g.

- Steady-state capital-labor ratio no longer independent of preferences, depends on σ.
  - Positive growth in output per capita, and thus in consumption per capita.
  - With upward-sloping consumption profile, willingness to substitute consumption today for consumption tomorrow determines accumulation and thus equilibrium effective capital-labor ratio.

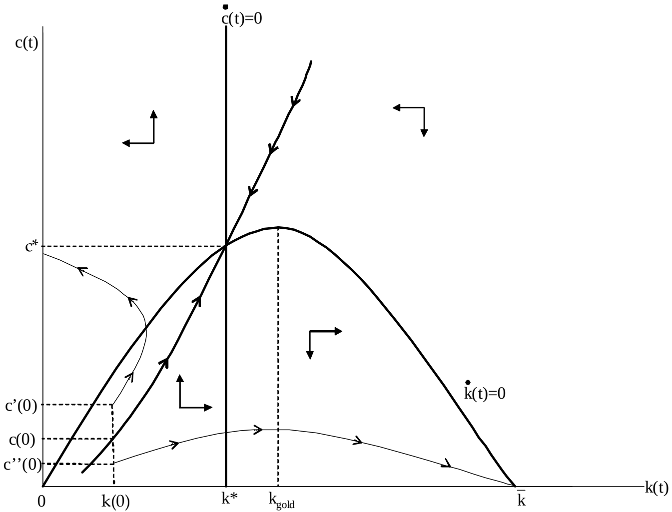

Figure: Transitional dynamics with technological change.

Technological Change XII

- Steady-state effective capital-labor ratio, k^*, is determined endogenously, but steady-state growth rate of the economy is given exogenously and equal to g.

Proposition Consider the neoclassical growth model with labor augmenting technological progress at the rate g and preferences given by (**??**). Suppose that Assumptions 1, 2, assumptions on utility above hold and $\rho - n > (1 - \sigma) g$. Then there exists a unique equilibrium path of normalized capital and consumption, $(k(t), c(t))$ converging to the unique steady-state (k^*, c^*) with k^* given by (**??**). Moreover, if $k(0) < k^*$, then $k(t) \uparrow k^*$ and $c(t) \uparrow c^*$, whereas if $k(0) > k^*$, then $c(t) \downarrow k^*$ and $c(t) \downarrow c^*$.