Chapter 3 - Debt sustainability: primary balance, interest and growth

The purpose of this chapter is to present and discuss some stylized results on debt sustainability and growth developed in the last two **Sections 6** and **7**. Since these results rest on the architecture of Solow's growth model, if the reader is already familiar with it he may go directly to those last two sections. If he is not familiar, or has been given only a few cursory notions and formulas, he may first give a brief look at the last two sections just to taste the ground, but then, before returning to them to absorb them properly, he may find useful to go through **Sections 1-5** in order to gain, or renew, a sufficient working knowledge of this classic piece of theory of long-run economic dynamics.

1 - Preliminary definitions and assumptions

We summarize here some standard concepts, formulas and notations of dynamic economic analysis (a similar useful summary is found in **Jones & Vollrath 2013**, Appendix A).

Notation for time dynamics

Following customary notation, we use the *dot* for the time derivative and the *prime* for derivatives over other variables

$\dot{x}(t)$ is the derivative of the variable x over time, i.e. its time rate of *change*

$\ln x(t) \rightarrow \dot{(\ln x(t))} = \dfrac{\dot{x}(t)}{x(t)} = g_x$ is the *percentage* rate of change of x over time, i.e. its time rate of *growth* ($\gtreqless 0$, constant or variable)

$f'_x(\cdot)$ is the derivative of f over any non-time variable x

Some basic time variables

We define

$P(t)$ population $\rightarrow g_P = \dot{\ln P}$

$LF(t)$ labour force $\rightarrow g_{LF} = \dot{\ln LF}$. This is the number of people who are 'objectively' *capable* of working (broadly: healthy adults)

$LS(t)$ labour supply. Not all of the above need be *willing* to work at the existing conditions (broadly: wages)

$L(t)$ *natural employment* $\rightarrow g_L = \dot{\ln L}$. This is the *number of people* actually working, or, more precisely, the number of *natural physical units of labour* actually used in production.

$LD(t)$ labour demand. The amount (and types) of work *demanded* at the existing conditions may not coincide with actual employment. The difference $LD - L$ are *vacancies*: jobs on offer but not yet taken up.

$E(t)$ *effective employment* $= A(t)L(t)$. Multiplying $L(t)$ by $A(t)$ converts labour input *measured in natural units* into labour input *measured in effficiency* *(or effective) units,* and the growth rate of $A(t) = \overset{\cdot}{\ln A(t)} = g_A$ is known as *labour-augmenting technical progress*. Conceptually, such technical progress is defined as something that *over time* increases the physical productivity of labour input measured in natural units. Given a natural unit of labour input at time zero, over time - thanks to technical progress - that same natural unit becomes the same as $A(t)$ *effective* units. In other words, if we measure labour input in *efficiency* (or *effective*) units, and start at time zero with natural and efficiency units being the same, at time t one unit natural unit of labour has become equivalent to $A(t)$ effective units of labour. As a result the growth rate of effective employment g_E is equal to the growth rate of natural employment g_L plus the growth rate of the labour augmenting technical progress g_A

$\ln A(t)L(t) = \ln A(t) + \ln L(t)$

$\rightarrow \overset{\cdot}{(\ln A(t)L(t))} = \overset{\cdot}{(\ln A(t) + \ln L(t))} = \overset{\cdot}{\ln A(t)} + \overset{\cdot}{\ln L(t)} = g_A + g_L = g_E$

Equality assumptions (equal growth rates)

We assume

$g_P = g_{LF} = g_L$

This means assuming a constant share of the *labour force* in the *population* and a constant share of *employment* in the labour force. The second assumption means assuming a constant rate of unemployment. This includes the special case of full employment, more precisely defined as the 'natural' rate of unemployment (the vertical long-run aggregate supply in the $AD \times AS$ diagram of standard macroeconomics). Assuming a constant rate of unemployment means assuming that the $AD \times AS$ adjustment mechanism keeps all the time the economy at that (constant, possibly though not necessarily 'natural') level of unemployment. The rationale of these equality assumption is to put the theoretical analysis of growth in the perspective of the (very) long-run.

Figure 1 shows how to visualize the evolution of employment L, labour demand LD, labour supply LS, labour force LF, and population P, over time.

Constancy assumptions (of growth rates and saving ratio)

In general g_x, the rate of growth of a variable x over time, will not be constant. If it is constant the equation

$\dfrac{\dot{x}(t)}{x(t)} = g_x$ constant has the general solution $x(t) = ae^{gt}$. In other words

g_x constant $\Longleftrightarrow x(t) = ae^{gt} \rightarrow \ln x(t) = \ln ae^{gt} = \ln a + \ln e^{gt} = \ln a + gt$

We now assume constant growth rates for our variables and a constant saving ratio

$g_P = g_{LF} = g_L = n$

$g_A = \dfrac{\dot{A}(t)}{A(t)} = \lambda \rightarrow A(t) = e^{\lambda t}$ a constant (labour-augmenting) technical progress (with natural and effective units being the same at time zero)

$\rightarrow g_E = n + \lambda = g$

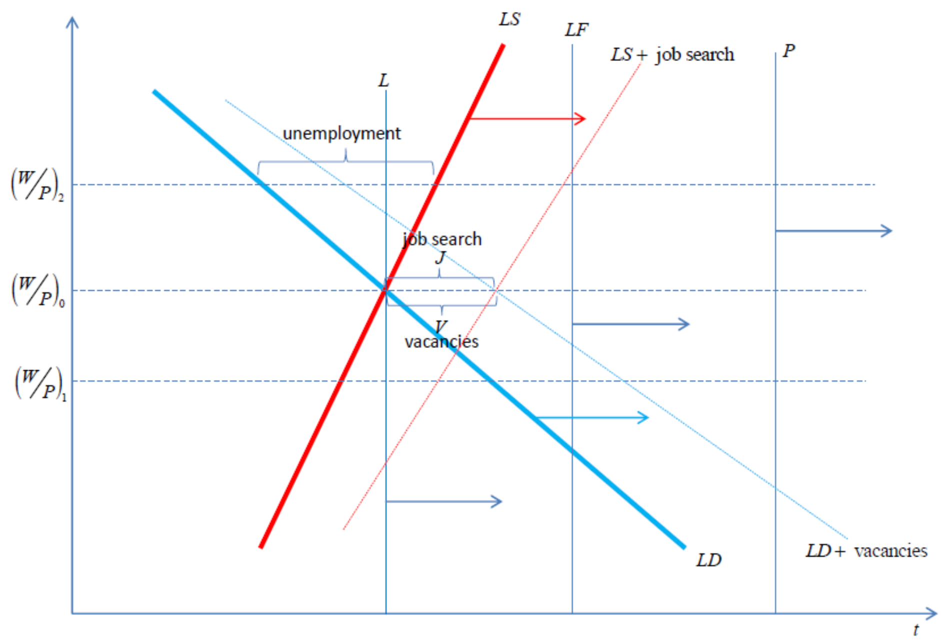

Fig. 1 Labour supply, demand, and employment, over time

$g_E = g$ is the constant growth rate of effective employment,
$s =$ saving ratio out of disposable income

The rationale of these constancy assumptions is different from that of the equality assumptions. They are used to describe how the economy would evolve under such constant growth rates, and how changes in the constant rates would affect its dynamics.

A digression on the special mathematical properties of the real number e and the exponential function e^x

1. Consider the real function $a^x, a \geq 0, x \in \mathbb{R}$ and its graph. The very special mathematical properties of the real number e are highlighted by any of the following facts:

The shape of the graph changes as a changes from $a > 1$ to $a = 1$ to $a < 1$. When the value of a becomes $e(> 1)$ the function $(a = e)^x$ acquires the peculiar interpretation of representing the path over x of a variable with a continuous rate of growth of 100% over x, with x standing normally for time t.

In addition we have $(a^x)'_x = (\ln a)a^x = a^x \iff a = e$. When $a = e$, and only then, the function $(a = e)^x$ is the *derivative of itself* (and thus all successive derivatives remain the same original function). For all values $a \neq e$ this remarkable and powerful property vanishes.

2. The above derivative property is obviously not accidental! It is embedded in the definition of the number e as the limit of the following sequence/function

Limit of a sequence

$\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e = \lim\limits_{n\to\infty} f(n)$ where $f(n) = \left(1 + \dfrac{1}{n}\right)^n$ for n positive integer

limit of a function

$\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x = e = \lim\limits_{x\to\infty} f(x)$ where $f(x) = \left(1 + \dfrac{1}{x}\right)^x$ defined $\forall x \neq 0$

3. Interpretation of e in terms of continuous (interest or growth) compounding

$$\to \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} \left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]^{rt} = e^{rt}$$

If we think of r as a rate of interest (or growth) per unit of time t (measured in any conventional unit, such as day, month, year, etc.), and of n as the number of times that r is computed in the given time unit, then we see that the above formula gives us the value that 1€ placed at time 0, and earning an interest r (or growing at a growth rate r, which is the same thing) compounded n times in the time unit, attains at time t.

4. The number e is also obtained using the definition of the *real exponential function* as the following *convergent series* over the *real field* $\mathbb{R}$

$E(x) \equiv \exp(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = e^x$, absolutely convergent $\forall x \in \mathbb{R}$ and uniformly convergent over all bounded subsets of $\mathbb{R}$

$= \sum_{n=0}^{\infty} \dfrac{1}{n!} = e$ for $x = 1$

5. The (real) logaritmic function $\ln x$ is simply the inverse of the (real) exponential e^x

(for proofs of the above statements see **Chiang 2005,** Chapter10)

To get a feeling of the great power of the exponential function the mathematically inclined reader may be reminded that that function needs not be restricted to the real field. The *complex exponential function*, similarly defined as the *convergent series* over the *complex field* $\mathbb{C}$

$E(z) \equiv \exp(z) = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = e^z$

absolutely convergent $\forall z \in \mathbb{C}$ and uniformly convergent over all bounded subsets of $\mathbb{C}$, is the most important function in mathematics (**Rudin 1986**, Prologue).

2 - Constant returns to scale (CRS), marginal products, output elasticities, distribution

We start with a standard *CRS pf* (constant return to scale production function)

$Q = F(K, E)$

We recall the definition of *homogeneous functions* (**Chiang 2005**, Chapters 12.6 and 12.7)

$h(\gamma y_i) = \gamma^r h(y_i), \quad \forall \gamma \in \mathbb{R}$

A *CRS* function is a homogeneous function where $r = 1$

$h(\gamma y_i) = \gamma h(y_i)$

Applying this property to $Q = F(\cdot)$ above we can rewrite it in *intensive form*, i. e. choosing anyone of the variables, say E, as the *intensity variable*, and expressing the other in per unit of E

$CRS \rightarrow F(\dfrac{K}{E}, 1) = F(k, 1) = f(k) = \dfrac{1}{E} F(\cdot) = \dfrac{Q}{E} = q$, where k and q are capital and output per effective worker, respectively

$\rightarrow q = f(k)$

$Q = F(\cdot) = Ef(k)$

Marginal products & competitive rate of return on capital and real wage

$MPK = F'_K(\cdot) = (Ef(k))'_K = Ef'(k) \cdot k'_K = f'(k)$: competitive rate of return on capital (using $k = \dfrac{1}{E}K \rightarrow k'_K = \dfrac{1}{E}$)

$MPE = F'_E(\cdot) = (Ef(k))'_E = f(k) + Ef'(k) \cdot k'_E = f(k) - f'(k)k$: competitive real wage (using $k = \dfrac{1}{E}K \rightarrow k'_E = -\dfrac{K}{E^2} = -\dfrac{1}{E}k$)

Output elasticities and competitive shares

The Euler property of homogeneous functions states

$F'_K K + F'_E E = rF(\cdot) = F(\cdot) = Q$ under *CRS*

so that

$$\frac{F'_K K}{Q} + \frac{F'_E E}{Q} = 1$$

$\dfrac{F'_K K}{Q} = \varepsilon_{FK}(K,E)$ is the *output elasticity* with respect to K, and under *CRS* it coincides with the *competitive share* of capital in output (i. e. the share of *capital income* in output when capital earns its competitive rate of return $F'_K(\cdot)$)

$\dfrac{F'_E E}{Q} = \varepsilon_{FE}(K,E)$ is the output elasticity with respect to E, and also - under *CRS* - the competitive share of effective employment in output (i. e. the share of *labour income* in output when labour earns its competitive wage rate $F'_E(\cdot)$)

Using the above definitions of $MPK = F'_K(\cdot)$ and $MPE = F'_E(\cdot)$ we can express output elasticities and competitive shares in *intensive form*

Output/capital elasticity:

$$\frac{F'_K K}{Q} = \varepsilon_{FK}(K,E) = F'_K \frac{K/E}{Q/E} = f'(k)\frac{k}{f(k)} = \varepsilon_{fk}(k)$$

It turns out to be notationally convenient to use

$\alpha(k) \equiv \varepsilon_{FK}(K,E) = \varepsilon_{fk}(k)$ for the output/capital elasticity. Thus

$$\rightarrow MPK = f'(k) = \alpha(k)\frac{f}{k}$$

Output/labour elasticity:

$$F'_E \frac{E}{Q} \equiv \varepsilon_{FE}(K,E) = (f(k) - f'(k)k)\,\frac{1}{f(k)} = 1 - f'(k)\frac{k}{f(k)} = 1 - \alpha(k)$$

$$\rightarrow MPE = f(k) - f'(k)k = (1 - \alpha(k))f(k)$$

A remark on elasticities in homogeneous functions

It is a simple matter to show that the above handy output-elasticity formulas for our two-factor *CRS pf* are only a particular instance of another general useful property of *CRS pfs* (in fact of homogeneous *pfs* of any degree). Let $F(X_i)$ be a homogeneous function of degree r. Choose some X_s as the intensity variable, and write $f(x_i), \quad x_i = \dfrac{X_i}{X_s} = \forall i \neq s$ for the *intensive function* relative to X_s. Then it can be shown that:

$$\varepsilon_{FX_j}(X_i) = \varepsilon_{fx_j}(x_i), \forall j \neq s$$

$$\varepsilon_{FX_s}(X_i) = r - \textstyle\sum_j \varepsilon_{fx_j}(x_i)$$

The elasticities relative to all variables different from the intensity variable, calculated for the function in normal form, are the same as those calculated for the function in intensive form. The elasticity relative to the intensity variable, calculated for the function in normal form, is the complement to r of the sum

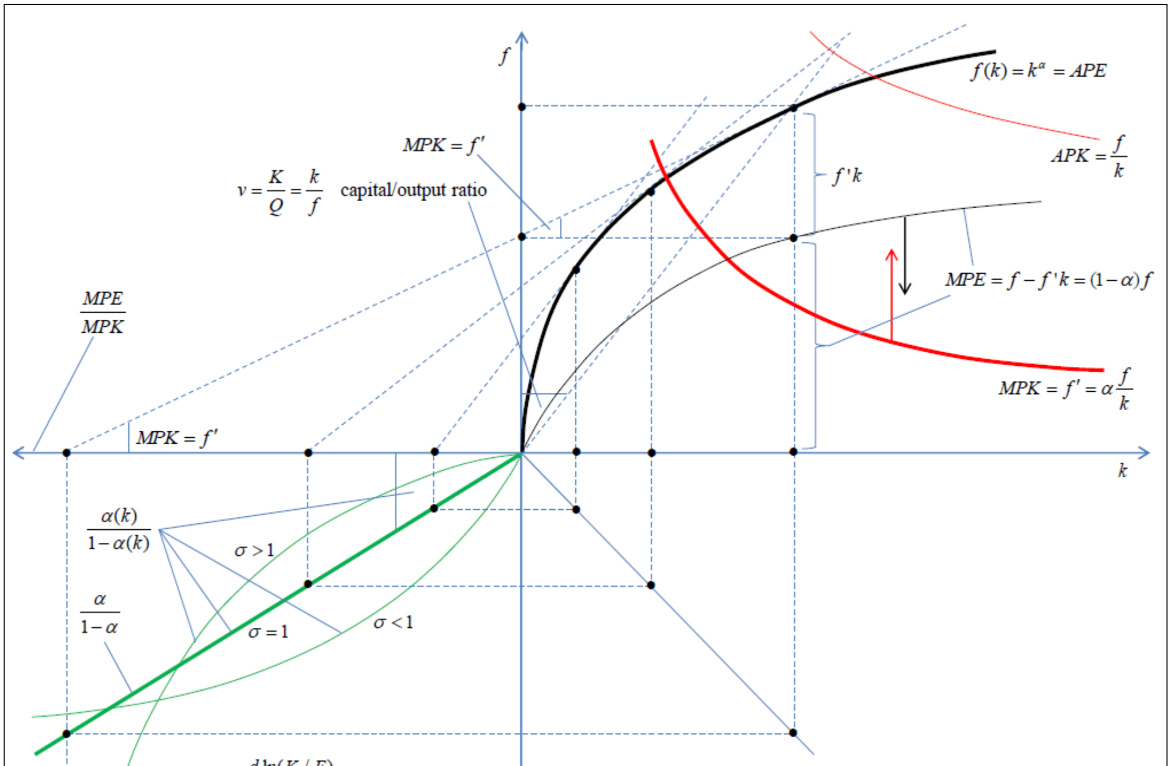

Fig 2 The neoclassical production function: CRS, marginal products, ouput elasticities, distribution

of all other intensive form elasticities (the proofs are simple but notationally cumbersome, and we leave them out).

Figure 2 is a graphical representation of the model's relationships discussed so far. We suggest to use the figure as a kind of visual guide for reviewing the formal results and derivations.

A remark on non-competitive distribution

It is relatively simple to introduce a *non-competitive distribution between capital and labour* into the above theoretical framework. We start with

$Q = F(\cdot) = F'KK + F'EE$

$= MPK \cdot K + MPE \cdot E$

$= \varphi MPK \cdot K + \theta MPE \cdot E$

where the two coefficients φ and θ converte the competitive distribution into a non-competitive one

Dividing by Q we obtain

$1 = \varphi\alpha(k) + \theta(1 - \alpha(k))$

$\rightarrow \varphi = \dfrac{1}{\alpha(\cdot)} - \dfrac{1-\alpha(\cdot)}{\alpha(\cdot)}\theta$

The function $\varphi(\theta, \alpha(\cdot))$ is represented in **Figure 3**. The coefficient $\theta \geqslant 0$ applied to the MPE can be interpreted as a measure of market power by either one of the two factors. When $\theta = \varphi = 1$ neither capital nor labour have market power, and distribution is competitive. Values of $\theta \neq 1$ represent a situation of *non-competitive distribution*. Values below 1 mean market power by capital, and above 1 market power by labour. In the top-right corner of **Figure 2**, starting from the curves representing $MPK(k)$ (in red) and $MPE(k)$ (in black) respectively, the rising arrow (red) and descending arrow (black) are meant to show that when $MPK(k)$ shifts upwards (θ falls below 1 indicating a rise in capital's market power) $MPE(k)$ rotates downwards, according to the relationship $\varphi(\theta, \alpha(\cdot))$ which depends on technology. Such possibility, of a rate of return on capital higher than $MPK(k)$, is repeated also in **Figure 4,** in order to place it in the dynamic context represented therein.

3 - Cobb-Douglas (CD) production functions

The CD-CRS production function

Cobb-Douglas pfs are a special kind of homogeneous functions, extensively utilized in both theoretical and empirical work. The general expression is

$Q = F(X_i) = A \prod X_i^{\alpha_i}, \quad \sum \alpha_i = r,$ where $r \gtreqless 1$ means that the *pf* has *increasing*, *constant*, *decreasing* returns to scale. Taking our simple capital & labour case, the *CD pf* is

$Q = F(K, E) = AK^\alpha E^\beta$, with returns to scale equal to $\alpha + \beta$

Disregarding the productivity coefficient A without loss of generality, the proof of homogeneity is trivial:

$(\gamma K)^\alpha (\gamma E)^\beta = \gamma^\alpha K^\alpha \gamma^\beta E^\beta = \gamma^{\alpha+\beta} K^\alpha E^\beta = \gamma^{\alpha+\beta} F(K, E) \quad \blacksquare$

If $\beta = 1 - \alpha$ the *pf* is *CRS*. Writing it in intensive form (and disregarding again the coefficient A) we obtain the intensive *pf* shown in **Figure 2**

$Q = K^\alpha E^{1-\alpha} \rightarrow \dfrac{Q}{E} = \left(\dfrac{K}{E}\right)^\alpha \left(\dfrac{E}{E}\right)^{1-\alpha} = \dfrac{1}{E} K^\alpha E^{1-\alpha} = k^\alpha = q$ ouput per unit of effective emplyment. In short

$q = f(k) = k^\alpha$

The constancy of output elasticities and competitve shares

In the previous Section on the *CRS pf* $f(k)$ we have derived its output elasticities and competitive shares, denoting them as $\alpha(k)$ for capital and $1-\alpha(k)$ for labour. In those expressions $\alpha(k)$ is a generic function of k of any shape, including constancy as an 'accidental' special case. We can now easily see that

$\alpha(k) = \alpha = \text{constant} \Longleftrightarrow f(k) = Ak^\alpha$

Proof:

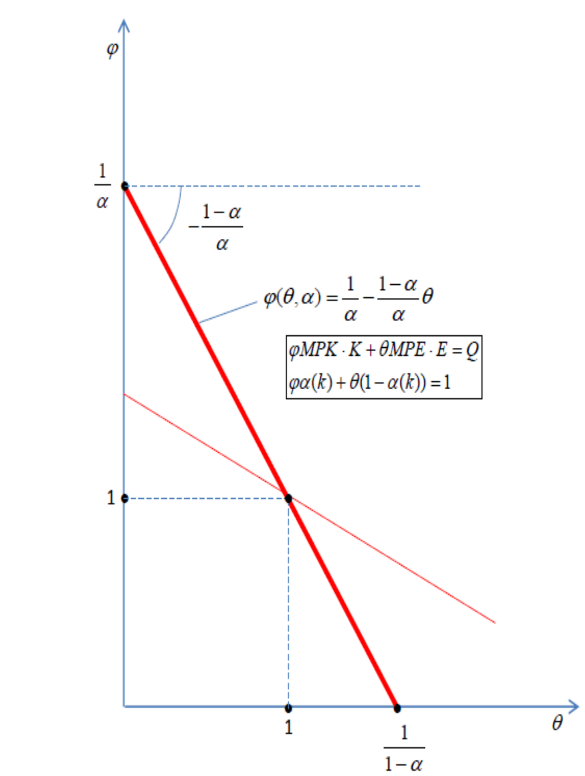

Fig 3 Non-competitive distribution

Suppose $f'(k)\dfrac{k}{f(k)} = \alpha =$ constant (see formulas in the previous Section). Then we see that it must be $f(k) = Ak^{\alpha}$

$\dfrac{f'}{f} = \alpha\dfrac{1}{k}$

$\int \dfrac{f'}{f}dk = \alpha \int \frac{1}{k}dk$

$\ln f = c + \alpha \ln k = c + \ln k^{\alpha}$

$f = e^{(c+\ln k^{\alpha})} = e^{c} \cdot e^{\ln k^{\alpha}} = e^{c}k^{\alpha} = Ak^{\alpha}$

Suppose $f(k) = Ak^{\alpha}$. Then we see that $f'(k)\dfrac{k}{f(k)} = \alpha$ ■

Warning. Let $\alpha(k) = f'(k)\dfrac{k}{f(k)}$ be the output/capital elasticity, as previously defined, with $\alpha(k)$ a generic function of k. We want to highlight that we are not allowed to write $f(k) = k^{\alpha(k)}$. We see this by writing $g(k) = k^{\alpha(k)}$ and checking that it doesen't yield the desired result $g'(k) = \alpha(k)\dfrac{g(k)}{k}$. By deriving $g(k)$ we obtain $g'(k) = \left(k^{\alpha(k)}\right)' = \alpha(k)k^{(\alpha(k)-1)} + \alpha'(k)\ln k \cdot k^{\alpha(k)} = \alpha(k)\dfrac{k^{\alpha(k)}}{k} + \alpha'(k)\ln k \cdot k^{\alpha(k)} \neq \alpha(k)\dfrac{g(k)}{k}$ ■

4 - Elasticity of substitution

Elasticity of substitution in general

Consider an arbitrary *pf* $Q = F(K, E)$. To any pair (K, E) there corresponds a pair $(MPE(K, E), MPK(K, E))$. Now take an *arbitrary isoquant*, drawn on the (K, E) space. Suppose we start at some point on the isoquant and then move along it, downwards and rightwards. As the pair (K, E) along the isoquant changes, also the pair $(MPE(K, E), MPK(K, E))$ changes. We thus obtain an *isoquant-based* relationship between the two pairs. Now look at the relationship going from (K, E) to $(MPE(K, E), MPK(K, E))$, and suppose that this relationship is a *function*, i. e. that every pair (K, E) maps into a corresponding pair $(MPE(K, E), MPK(K, E))$. Then define a new function, mapping the *ratio* (K/E) into the *ratio* $(MPE(K, E)/MPK(K, E))$, and assume it to be increasing. This is a reasonable assumption. Over a *given isoquant*, when the input ratio K/E increases it makes sense to assume that also the ratio $MPE(K, E)/MPK(K, E)$ increases. In other words, when the ratio of an input X over an input Y increases it makes sense to expect the ratio of the marginal productivity of Y over that of X to increase as well. Now *invert* this function, i. e. take the function that maps the ratio $MPE(K, E)/MPK(K, E)$ into the ratio K/E

$\dfrac{K}{E} = h\left(\dfrac{MPE(K, E)}{MPK(K, E)}\right)$

This function has a *particular economic meaning.* If we assume profit maximizing competitive firms, where inputs are paid their respective marginal products, then the function $h(\cdot)$ tells us by how much - at any point (K, E) on an isoquant - does the ratio of capital over labour change along the isoquant itself, in response to a change in the *inverted* ratio of their respective prices, i. e. in the ratio of the price of labour over that of capital. The *elasticity* of this function $h(\cdot)$, which as we know is its log derivative

$$\sigma_{KE}(\cdot) = h'(\cdot)\frac{MPE(\cdot)/MPK(\cdot)}{K/E} = \frac{d\ln(K/E)}{d\ln(MPE(\cdot)/MPK(\cdot))}$$

is called the *elasticity of substitution (ES).* It characterizes the above relationship expressed in % terms. The value of the *ES* at any point (K, E) on an isoquant says by how many % points does the ratio K/E change along the isoquant itself, when the ratio $MPE(\cdot)/MPK(\cdot)$ changes by 1% point.

Now, it can be shown (**Chiang 2005**) that if the *pf* is homogeneous, of any degree, then the function $h(\cdot)$ is scale-ivariant, i. e. it does not depend on which isoquant we choose, and so is - of course - its *ES* (this is intuitive, because the isoquants of homogeneous functions are radial expansions of one another). More generally, with respect to their *ES*, homogeneous functions (of any degree) can be partitioned into two subsets, those with *non-constant ES*, and those with *constant ES.* Those with constant *ES* are called *CES pfs*, and can in turn be distinguished according to whether the constant *ES* is $\gtreqless 1$. The special case *ES=1* coincides with the set of *Cobb-Douglas (CD) pfs.*

The elasticity of substitution of a *CRS* pf

Consider our *CRS pf* expressed in intensive form

$$Q = F(K, E) \to q = f(k)$$

The function $h(\cdot)$ relative to this *pf* can be immediately obtained using the formulas derived in **Section 2** above:

$$MPE(k) = (1 - \alpha(k)f(k)$$

$$MPK(k) = \alpha(k)\frac{f(k)}{k}$$

$$\frac{MPE(k)}{MPK(k)} = \frac{1-\alpha(k)}{\alpha(k)}k$$

$$\to k = h\left(\frac{MPE(K,E)}{MPK(K,E)}\right) = \frac{\alpha(k)}{1-\alpha(k)} \cdot \frac{MPE(k)}{MPK(k)}$$

This is the formula for the curves in green shown in the bottom-left corner of **Figure 2**. In the *CD* case we have a constant α, and the $h(\cdot)$ function is represented by the straight thick green line. The elasticity of this straight line, which rotates towards the horizontal axis when α rises, is the *ES* of the *pf* drawn in the top-right corner, and it is by construction always *unity*, independently of what the value of α happens to be. The thin green curves are instead meant to represent cases of *pfs* where the term $\alpha(k)$ changes over k in such a way as to produce functions $h(\cdot)$ with constant elasticity $\neq 1$. The *ES* of such *pfs* are precisely the constant non-unitary elasticities of these thin curves, one higher than 1 and the other lower.

5 - Input-output dynamics

Growth rates and elasticities

Given a function $f(x_i)$, let each x_i change by some % points. Then the % change in f is equal to the sum of the % changes in each x_i, times their respective elasticities:

$f(x_i) \to df = \sum_i f_i'(\cdot)dx_i$

$\dfrac{df}{f} = \sum_i f_i'(\cdot)\dfrac{dx_i}{f}\dfrac{x_i}{x_i} = \sum_i f_i'(\cdot)\dfrac{x_i}{f}\dfrac{dx_i}{x_i} = \sum_i \varepsilon_{fx_i}(\cdot)\dfrac{dx_i}{x_i}$

Let now all x_i be functions of some independent variable, say time t. Then we can rewrite the previous expressions *in terms of growth rates* (recall our notational convention of using the *dot* for time derivatives, and g_x for the *growth rate* over time of a variable x):

$f(t) = f(x_i(t))$

$\dot{f} = \sum_i f_i'\dot{x}_i$

$\dfrac{\dot{f}}{f} = \sum_i f_i'\dfrac{\dot{x}_i}{f} = \sum_i f_i'\dfrac{\dot{x}_i}{f}\dfrac{x_i}{x_i} = \sum_i f_i'\dfrac{x_i}{f}\dfrac{\dot{x}_i}{x_i} = \sum_i \varepsilon_{fx_i}(\cdot)\dfrac{\dot{x}_i}{x_i}$. In short

$g_f = \sum_i \varepsilon_{fx_i}(\cdot)g_{x_i}$

The growth rate of $f(x_i(t))$ is equal to the sum of the growth rates of the $x_i(t)$ times their respective elasticities

Dynamics of output, capital and labour.

Consider our *pf* as moving over time: $Q(t) = F(K(t), E(t))$. Then

$\dot{Q} = F_K'\dot{K} + F_E'\dot{E}$

$\dfrac{\dot{Q}}{Q} = F_K'\dfrac{K}{Q}\dfrac{\dot{K}}{K} + F_E'\dfrac{E}{Q}\dfrac{\dot{E}}{E}$

As anticipated in **Section 1**, in our very long-run perspective we assume a constant growth rate of effective emplyment, $g_E = g$. In short the above becomes

$g_Q = \varepsilon_{FK}(\cdot)g_K + \varepsilon_{FE}(\cdot)g$

Under *CRS* this becomes (see **Section 2**)

$g_Q = \alpha(k)g_K + (1-\alpha(k))g =$

$= \alpha(k)(g_K - g) + g$

$= \alpha(k)g_k + g =$ growth rate of output

$g_{Q/E} = g_q = g_Q - g = \alpha(k)g_k =$ growth rate of output $q = \dfrac{Q}{E} =$ output per effective worker (per unit of effective employment)

$g_{Q/L} = g_Q - g_L = g_Q - n = g_Q - (g - \lambda)$

$= (g_Q - g) + \lambda = g_q + \lambda = \alpha(k)g_k + \lambda =$ growth rate of output per head (per natural worker = per unit of natural employment)

Steady state k

A constant k

The equation of motion of k is

$\dot{k} = \dot{\left(\frac{K}{E}\right)} = \frac{\dot{K}E - K\dot{E}}{E^2} = \frac{\dot{K}}{E} - gk$

$\rightarrow \frac{\dot{K}}{E} = \dot{k} + gk$

$\frac{\dot{K}}{E}$ is *actual* investment per unit of E, while

gk is the *required* investment per unit of E, needed to keep $K/E = k$ constant, because

$\dot{k} = \dot{\left(\frac{K}{E}\right)} = \frac{\dot{K}E - K\dot{E}}{E^2} = \frac{\dot{K}}{E} - gk = 0 \Longrightarrow \frac{\dot{K}}{E} = gk$

Fundamental Solow equation

Assume a given aggregate propensity to save s out of *GDP*, $S = sQ$.

Since in long-run economic dynamics *actual* saving = *actual* investment we have

$S = sQ = \dot{K}$

$s\frac{Q}{E} = \frac{\dot{K}}{E}$

$= sf(k) = \dot{k} + gk$

$\rightarrow \dot{k} = sf(k) - gk = \dot{k}(k)$

This is the fundamental *dynamic Solow equation*.

In order to have a *long-run, steady state (ss) equilibrium*, the level of k, here denoted by ssk, must be constant, so that

$\dot{k} = 0 \rightarrow sf(k) = gk$

$\rightarrow \frac{k}{f(k)} = \frac{K/E}{Q/E} = \frac{K}{Q} = \frac{s}{g}$ *ss* capita/output ratio

For any given propensity to save, the ssk level of k is obtained where the saving-investment curve $sf(k)$ intersects the growth curve gk, and where this happens the capital/output ratio is equal to the ratio of the *saving rate* over the *growth rate*. By way of example, if these two rates were the same the *ss* capital/ouptut ratio would be equal to unity. But of course we know that in practice the saving rate is usually quite higher than the (real) growth rate.

The dynamic relationships formally derived above are represented graphically in **Figure 4.**

Sustainable consumption per head

As for consumption,

$C = Q - \dot{K} = (1-s)Q$ is *current* aggretate consumption, and

$c_E = \frac{C}{E} = \frac{Q}{E} - \frac{\dot{K}}{E} = (1-s)\frac{Q}{E} = (1-s)f(k) = f(k) - (\dot{k} + gk)$ is *current* consumption per effective worker

(recall $s\frac{Q}{E} = sf(k) = \frac{\dot{K}}{E} = \dot{k} + gk$).

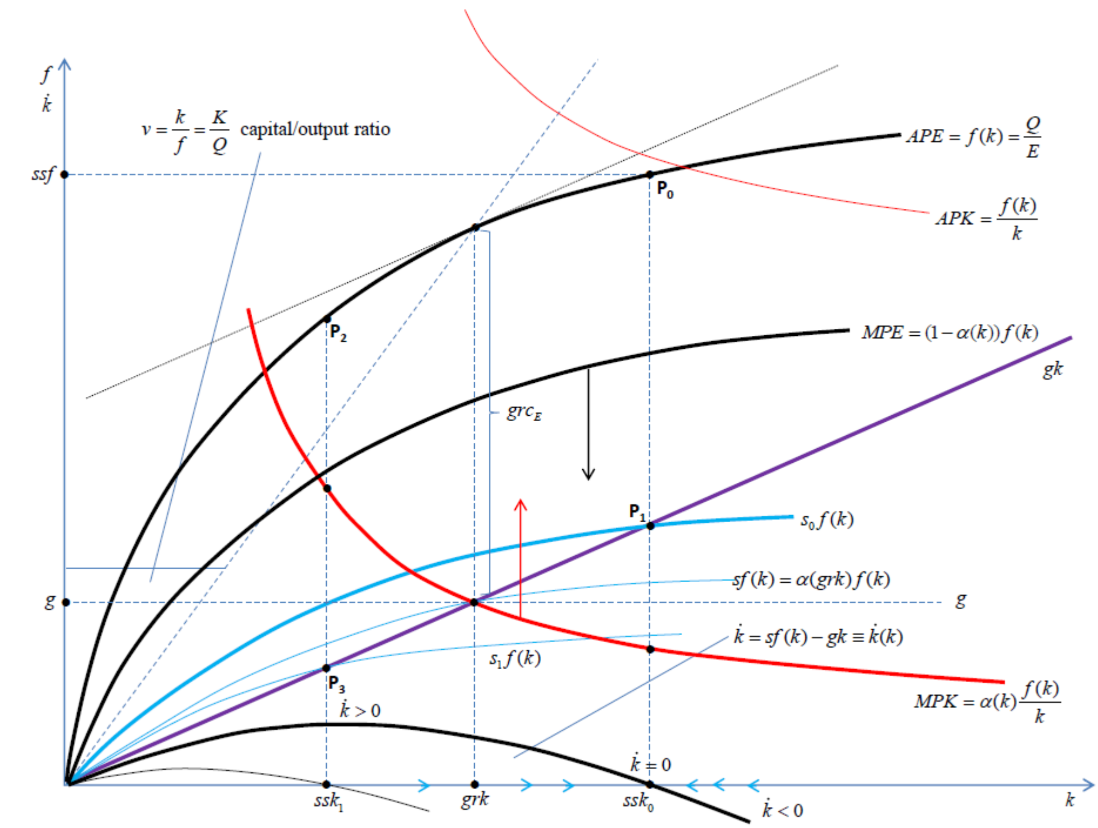

Fig. 4 Steady state, adjustment dynamics , distribution

In **Figure 4** this is the vertical distance between the $f(k)$ and $sf(k)$ curves. Clearly, for any given s consumption per head rises with k, but of course not any level of c_E is sustainable! Only those levels of consumption per head that satisfy the *steady state (long-run) equilibrium condition* are sustainable. We denote them by ssc_E. Since in ss $\dot{k} = 0$, the set of such feasible ss levels of consumption per head is defined by the condition

$ssc_E = f(k) - gk$

i. e. by the vertical distance between the $f(k)$ and gk curves.

Notice that for any constant level c_E of consumption per *effective* worker, the level $c_L = \dfrac{C}{L}$ of consumtion per *actual* worker is not constant, but increases at a rate of growth equal to the rate of technical progress g_A.

The golden rule and dynamic efficiency

We now ask: what is the saving propensity that maximizes the *steady state (long-run) consumption per head*? The answer to this question is the so-called *golden rule* of accumulation.

Since the ssc_E is the vertical distance between the $f(k)$ and gk curves, we see in the figure that such distance is maximized where the slopes of the two curves coincide. Formally

$\max_k ssc_E = f(k) - gk \rightarrow (f(k) - gk)'_k = 0$

$\rightarrow f'(k) = g$

Since $f'(k) = \alpha(k)\dfrac{f(k)}{k}$, the condition can be restated as

$\alpha(k)f(k) = gk$

In the figure, the particular k satisfying the condition is denoted by grk (golden rule k) and can be graphically determined in two different but equivalent ways: either by finding the k level where the curves $f(k)$ and gk have the same slope, or by finding the intersection between the $MPK(k) = f'(k) = \alpha(k)\dfrac{f(k)}{k}$ curve and the gk straight line. For brevity we shall denote this maximum ssc_E by grc_E.

But since the ss condition $\dot{k} = 0$ implies $sf(k) = gk$, there is another interesting way to characterize the condition for maximizing ssc_E. The condition is equivalent to requiring that s be equal to $\alpha(gkr)$: the intersection between the saving-investment curve $sf(k)$ and the growth line gk takes the economy to the ssc_E maximizing $grk \Longleftrightarrow s = \alpha(grk)$. When α is constant (the CD case) this becomes more simply $s = \alpha$.

In short, maximizing ss consumption per effective worker requires (i) that the competitive long-run (real) rate of return on capital be equal to the long-run (real) rate of growth. A *different but fully equivalent requirement* is (ii) that the aggregate saving propensity s (out of $Q = GDP$) be equal to the competitive share of profits in GDP, here denoted by $\alpha(\cdot)$. Stated in slightly different terms it requires that an amount equal to all profits that would be earned by capital under competitive conditions be invested, and an amount equal to all wages that would similarily be earned under competitive conditions be consumed.

The *golden rule* of accumulation is also known as the rule ensuring *dynamic efficiency.* We can use the figure to show what this means. Suppose the saving-investment propensity takes the economy to the steady state ssk_0. *Ss* consumption per head ssc_E is the vertical distance $\mathbf{P}_0\mathbf{P}_1$. We can see that this economy *saves-invests too much.* By reducing saving-investment today to the lower level required by the golden rule, k would decrease towards grk, the present generations would *increase* their consumption per head, and also the future generations would enjoy a consumption per head *higher* than before *for ever after.* In other words, moving from ssk_0 to grk through a reduction in the saving-investment propensity amounts to an *actual Pareto improvement* because it benefits *all generations.* This is the reason why a *ss* like ssk_0, lying to the right of grk, is said to be *dynamically inefficient.* Now suppose the saving-investment propensity takes the economy to the steady state ssk_1. *Ss* consumption per head ssc_E is the vertical distance $\mathbf{P}_2\mathbf{P}_3$. This economy saves-invests *too little.* But in order to move from ssk_1 to grk the presnt generations need to *increase* their saving-investment. Their consumption per head would *decrease*, and only later generations would enjoy a consumption per head higher than before for ever after. In this case the economy can still 'buy' a *higher consumption per head for all future generations*, but only at the cost of a *lower consumption per head by the present generations.* The move from ssk_1 to grk does point towards greater efficiency, but it doesn't amount to to an actual Pareto improvement because it benefits some generations at the cost of others. However, and under an admittedly highly questionable logical twist, it can still be classified as a *potential Pareto improvement*, because the future generations enjoying a higher consumption are many more (indeed they are infinite, so to speak), while the present generations enjoying a lower consumption are only a few (a finite number, so to speak), so that - in the abstarct - it would *certainly* be possible for the future generations to *compensate* the present ones, and still retain some *net benefit.* The logical weakness of this proposition lies in the fact that in an intertemporal context *backward compensation*, though conceivable in the abstarct, is *impossible even in principle*: later generations could never *actually* compensate earlier ones, even if they wanted! The impact *over time* on consumption per head caused by the *instantaneous* changes in the saving-investment propensity just described is represented here in **Figure 5.** In reading this figure the reader is reminded that we are considering consumption per *effective worker* c_E, and that a constant c_E corresponds to a consumption per *natural worker* c_L increasing at the constant rate of the (labour-augmenting) technical progress $g_A = \lambda$:

$$c_E(t) = \frac{C(t)}{A(t)L(t)} = \frac{c_L(t)}{A(t)} \rightarrow c_L(t) = A(t)c_E(t) = e^{\lambda t}c_E(t)$$

At this point we must ask ourselves whether, and if so to what extent, these results do provide any *substantial insight* into the working of a capitalist economy. In particular, the question comes to mind in view of, and possibly

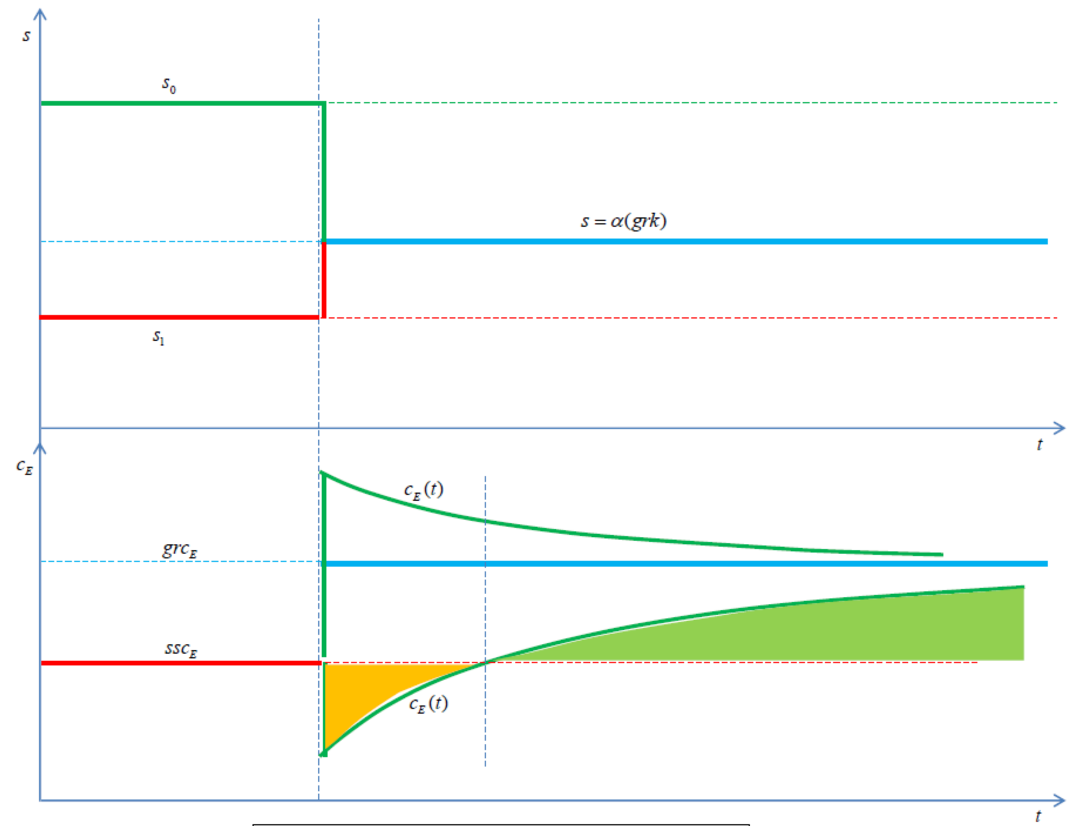

Fig. 5 Dynamic inefficiency: transition to the Golden rule

stimulated by, the recent weighty study by Thomas Piketty (**Piketty 2014**). We leave it to the thoughtful reader to reflect on his own on this *general* question. For our part, elaborating on **Solow 2000**, pp. 53-70, we use the highly stylized theoretical architecture developed so far to analize the problem of *debt sustainability*, and we shall see that - at least as far as this particular analytical exercise is concerned - that architecture does provide some economically meaningful results.

6 - The government budget and the Debt/GDP ratio

Some definitions

Using a stilized formulation of the government budget equation we obtain, after a number of substitutions and rearrangements, the dynamic equation of the ratio of public debt to *GDP.* First some definitions

$Q = GDP$ is *real* output

g is the growth rate of Q

P is the (aggregate) price level

G is *real* public expenditure on goods and services, and γ is the share of G in Q, so that

$G = \gamma Q$

B is nominal government debt, which, divided by P, becomes

$b = \dfrac{B}{P}$ the *real* government debt, which in turn, divided by Q, becomes

$\beta = \dfrac{b}{Q}$ the government debt as a percentage of output

$\dot{B}(t)$ is the government nominal deficit, i.e. the *rate of change* of B per unit of time

R is the nominal rate of interest on B, and

$\pi = \dfrac{\dot{P}}{P}$ is the rate of inflation, so that

$r = R - \pi$ is the *real* interest on B, with $r = R$ when $\pi = 0$

T is total nominal tax revenues

$\tau = \dfrac{T}{PQ + RB}$ is the average ratio of tax revenues over incomes, so that $T = \tau(PQ + RB)$

Total and primary deficits

The *total deficit* (or surplus, when negative) is the excess of total public expenditures over total public revenues. The *primary deficit* is the same less the net interest payments on the net outstanding debt. The total interest bill may in turn be $\gtreqless 0$ depending on whether the net outstanding debt is $\gtreqless 0$, and on the possible differences in returns between assets and liabilities. Using the above notations the aggregate equations are

$$\begin{aligned} PG + RB - T &= \dot{B} \\ P\gamma Q + RB - \tau PQ - \tau RB &= \dot{B} \\ (\gamma - \tau)PQ + (1-\tau)RB &= \dot{B} \end{aligned}$$

The last equation is the *total nominal deficit*, which is more conveniently expressed in real terms by dividing it by P

$$(\gamma - \tau)Q + (1-\tau)Rb = \frac{\dot{B}}{P}, \text{ where } b = \frac{B}{P} \tag{1}$$

The *primary deficit* is the first term in the left side of the equation

From the definitions of *nominal and real debt* $B = bP$ we obtain the relationship between the *nominal* and *real total deficits*, either directly

$$\begin{aligned} \dot{B} &= \dot{b}P + b\dot{P} \\ \dot{b} &= \frac{\dot{B}}{P} - b\frac{\dot{P}}{P} = \frac{\dot{B}}{P} - b\pi, \text{ where } \pi = \frac{\dot{P}}{P} \\ \frac{\dot{B}}{P} &= \dot{b} + b\pi \end{aligned} \tag{2}$$

or by applying the standard formula of the time rate of change (time derivative) of a ratio

$$\begin{aligned} \dot{\left(\frac{x}{y}\right)} &= \frac{\dot{x}y - x\dot{y}}{y^2} = \frac{\dot{x}}{y} - \frac{x}{y}g_y \rightarrow \frac{\dot{x}}{y} = \dot{\left(\frac{x}{y}\right)} + \frac{x}{y}g_y \\ \dot{\left(\frac{B}{P}\right)} &= \dot{b} = \frac{\dot{B}P - B\dot{P}}{P^2} = \frac{\dot{B}}{P} - \pi b \end{aligned}$$

Nominal and real total deficits are *different economic and statistical concepts*, which coincide if and only if there is *zero inflation.* The *total real deficit* is equal to the *total nominal deficit* less the loss of value of the *debt* stock due to inflation (by reducing the real value of a financial asset inflation damages the *creditor,* who owns of the asset, and favours the *debtor*, who owes it). Again, the relationship is more conveniently expressed in terms of *ratios to GDP*

$$\frac{\dot{b}}{Q} = \frac{\dot{B}}{PQ} - \beta\pi, \text{ where } \beta = \frac{b}{Q}$$

because then, using the previous standard formula for the rate of change of a ratio we obtain a useful expression relating the *real* total deficit ratio to the *debt* ratio and *real growth*, and the *nominal* total deficit ratio to the *debt* ratio and *nominal growth*

$$
\begin{aligned}
\left(\dot{\frac{b}{Q}}\right) &= \dot{\beta} = \frac{\dot{b}Q - b\dot{Q}}{Q^2} = \frac{\dot{b}}{Q} - \frac{b}{Q}\frac{\dot{Q}}{Q} = \frac{\dot{b}}{Q} - g\beta \\
&\rightarrow \quad \frac{\dot{b}}{Q} = \dot{\beta} + g\beta \\
&\rightarrow \quad \frac{\dot{B}}{PQ} = \dot{\beta} + (g+\pi)\beta
\end{aligned} \tag{3}
$$

7 - The dynamics of the debt/GDP ratio

Basic assumptions

In order to focus on the most basic relationships we make three convenient assumptions: (**i**) a *closed* economy, (**ii**) *no debt monetization*: public debt is held entirely by the (private) economy in the form of interest bearing securities, and (**iii**) an *exogenous* real rate of interest r. A general discussion of whether, and to what extent, the real interest may be treated as exogenous in an economy integrated into a liberalized world capital market lies outside the scope of these lectures. In the long-run model at hand, of a closed economy, the real interest r is usually identified with the *competitive rate of return on capital*, in which case it is clearly not exogenous, but a *decreasing function* - depending on the properties of the production function - of capital per effective worker $\frac{K}{E} = k$, or equivalently of the capital/output ratio $\frac{K}{Q} = v$ (see **Figure 2**). However this identification is *not a necessary feature* of the model, which can easily accomodate 1) an exogenous rate of return on capital, determined for instance by non-competitive conditions or by the *world market*, or 2) more generally, a separation between a *financial interest* charged by lenders of capital (to business and governments) and the *real rate of return* on business investment.

Inflation and the actual tax burden on real interest

Since we want to concentrate on real, not nominal, variables we also need to clarify the difference between gross real interest and net (of taxes) real interest. Specifically, assuming that the tax on interest income is a tax rate applied to nominal interest, we want to see what is what kind of tax wedge between gross and net real interest is inserted by such a tax, for any given level of inflation. The following are the relevant eqs

$$
\begin{aligned}
\tau Rb &= \tau(r+\pi)b \\
rb - \tau(r+\pi)b &= [r - \tau(r+\pi)]b \\
&= [(1-\tau)r - \tau\pi]b
\end{aligned} \tag{4}
$$

Given an inflation π, and a tax rate τ on $R = r + \pi$, the net real interest received by the holders of public debt (and paid by the debtor) is $(1-\tau)r - \tau\pi$.

Predictably, with a positive inflation, the tax burden on real interest is higher than τ, by an amount precisely equal to the second order term $\tau\pi$. For any given τ, the higher is inflation, the higher is the tax burden, because inflation raises the excess of the tax base R above r. With this our analysis expressed in real variables becomes inflation-independent. We do not need to assume zero inflation and $R = r$. However, for simplicity we may leave out the term $\tau\pi$ because we assume either that it is negligible, or that the tax rate applies to real interest, or that inflation is zero.

The dynamic equations

We procede with the following steps

Substitute $R = r + \pi$ and (2) into (1)

$$\begin{aligned}(\gamma - \tau)Q + (1-\tau)(r+\pi)b &= \frac{\dot{B}}{P} = \dot{b} + b\pi \\ (\gamma - \tau)Q + ((1-\tau)r - \tau\pi)b &= \dot{b}\end{aligned}$$

Divide by Q

$$(\gamma - \tau) + ((1-\tau)r - \tau\pi)\beta = \frac{\dot{b}}{Q} \tag{5}$$

Substitute (3) into (5)

$$\begin{aligned}(\gamma - \tau) + ((1-\tau)r - \tau\pi)\beta &= \frac{\dot{b}}{Q} = \dot{\beta} + g\beta \\ (\gamma - \tau) + (1-\tau)r\beta &= \frac{\dot{b}}{Q} = \dot{\beta} + g\beta, \text{ when } \pi = 0\end{aligned}$$

This yields the equation of the dynamics of β as a function of β itself and the parameters γ, τ, g, r, π

$$(\gamma - \tau) + ((1-\tau)r - \tau\pi)\beta = \dot{\beta} + g\beta$$

$$\begin{aligned}\dot{\beta} &= (\gamma - \tau) - [g - ((1-\tau)r - \tau\pi)]\beta \to \dot{\beta}(\beta; \gamma, \tau, g, r, \pi) \qquad (6) \\ \dot{\beta} &= (\gamma - \tau) - (g - (1-\tau)r)\beta \to \dot{\beta}(\beta; \gamma, \tau, g, r), \text{ with } \pi = 0\end{aligned}$$

giving the steady state *debt/GDP* ratio β_{ss} as a function of the parameters

$$\begin{aligned}\dot{\beta} &= 0 \to \beta = \frac{(\gamma - \tau)}{[g - ((1-\tau)r - \tau\pi)]} = \beta_{ss}(\gamma, \tau, g, r, \pi) \qquad (7) \\ \dot{\beta} &= 0 \to \beta = \frac{(\gamma - \tau)}{g - (1-\tau)r} = \beta_{ss}(\gamma, \tau, g, r)\end{aligned}$$

The numerator $\gamma - \tau \gtreqless 0$ is the *primary deficit/GDP ratio*, the denominator $g - ((1-\tau)r - \tau\pi) \gtreqless 0$ is the difference *real growth rate* minus *net of tax real*

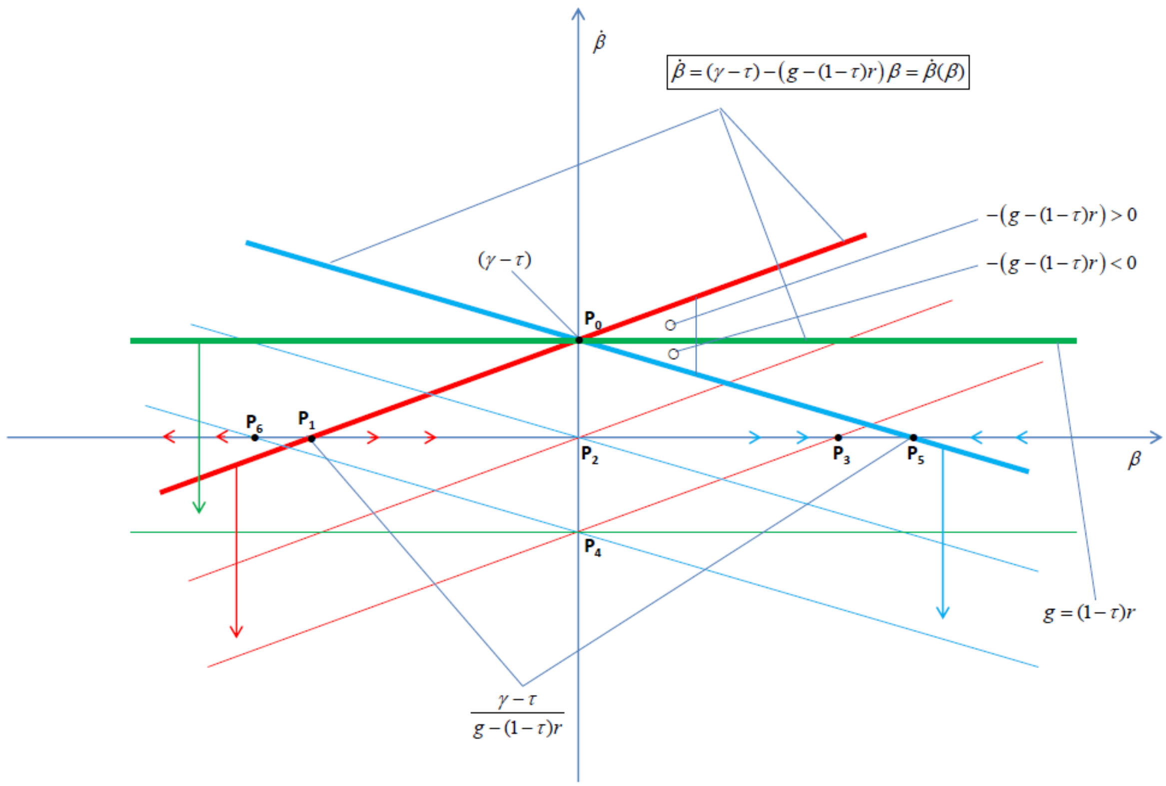

Fig 6 The dynamics of the Debt/GDP ratio

interest (we've already explained above, in eq (4) the economic rationale of the second order term $\tau\pi$). As already noted above the simpler expression in (4) applies when either $\pi = 0$, or the tax rate τ is calculated on *real* interest, or $\tau\pi$ is negligible. Thus the formulas do not - strictly - depend on assuming zero inflation. *Using real growth and real interest they describe the dynamics of the debt/GDP ratio for whatever level of (constant) inflation.* The numerator and denominator are the two factors affecting the *dynamics* of the *debt/GDP ratio*. This dynamics is graphically illustrated in the *phase diagram* of **Figure 6,** with β on the horizontal axis and $\dot{\beta}$, the time rate of change of β, on the vertical one (on phase diagrams the reader is referred to **Chiang 2005**, Chapters 15.6 (one variable) and 19.5 (two variables))

The instability of the *debt/GDP* ratio when net interest exceeds real growth

Consider first the *thick red line*, which corresponds to what may be regarded as the most likely case $g < (1-\tau)r$, a *net of tax real interest higher than the real rate of growth of GDP*. The slope of the red line, equal to $-(g-(1-\tau)r) > 0$, is positive, and at $\beta = 0$ the line intersects the vertical $\dot{\beta}$ axis at point $\mathbf{P_0}$, a positive height $\gamma - \tau > 0$. We see that under such conditions - a positive primary deficit and a net real interest higher than real growth - the steady state level β_{ss} where the red line intersects the horizontal axis is *negative*, at point $\mathbf{P_1}$, and is *unstable*. In words, with a net interest higher than real growth a primary deficit can only be sustained if the government holds a positive net financial position with respect to the economy, i.e. if the net *debt/GDP* ratio β is negative. Given a primary deficit, in order for β to remain constant the government needs to supplement taxes with an extra net interest revenue, which brings the total real deficit down to zero. If β lies to the left of $\mathbf{P_1}$ it will keep decreasing indefinitely. If β lies to the right of $\mathbf{P_1}$ it will keep rising indefinitely (changing from negative to positive at some point in time), at a speed that depends on the height of the primary deficit $\gamma - \tau$ and the excess of the net interest over real growth $-(g-(1-\tau)r)$. The economics of this dynamics is perfectly intuitive. The *other two - thin - red lines* serve to point out two other economically intuitive facts. Given a net interest higher than real growth, the intersection at $\mathbf{P_2}$ means that a zero *debt/GDP* ratio is sustainable only with a zero primary balance. A positive *debt/GDP* ratio (here at $\mathbf{P_3}$) is *sustainable* only with a *primary surplus* (here at $\mathbf{P_4}$), whose level depends - of course - on the excess of net interest over real growth (i.e. on the slope of the line going through $\mathbf{P_3}$).

From the point of view of stability, the excess of net interest over real growth $-(g-(1-\tau)r)$ implies *always insatbility.* If β is at points such as $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$ it stays there, but as long as $\gamma - \tau$ remains unchanged, if it's to the *left*, it keeps *decreasing* and if it's to the *right* it keeps *increasing.* Suppose it's to the right. Then if the government wants the ratio β to stop increasing it must reduce the value of $\gamma - \tau$. The red line shifts downwards, so that its intersection with the horizontal axis moves to the right. The intersection point and the actual level of β move both to the right. As the intersection point draws closer to β its speed of increase of slows down, untill it stops altogether if and when the former catches up with the latter.

The stability of the *debt/GDP* ratio when real growth exceeds net interest

The *thick blue line* represents the opposite case. Here net interest is less than real growth. The line has negative slope equal to $-(g-(1-\tau)r) < 0$. A primary deficit at $\mathbf{P_0}$ is sustainable, in the sense that, whatever the initial level of the *debt/GDP* ratio, this tends to the *ss* level at $\mathbf{P_5}$. The reader can easily work out the economic interpretation of the *thin blue lines* crossing the horizontal axis at points $\mathbf{P_2}$ and $\mathbf{P_6}$, respectively. In particular, with real growth higher than net interest a primary surplus, say, at $\mathbf{P_4}$ would cause a positive debt to first vainsh, and then to become negative (a *public credit*) and tend to the *ss* level $\mathbf{P_6}$.

The case of *net interest equal to real growth* is represented by the *horizontal green lines*. Their interpretation in terms of primary deficit and the dyanamics of β is is left to the reader.

8 - The 'burden of the debt'

The excess of real growth over net interest allows the government to run a permanent primary deficit. It allows for instance to run a positive γ with a zero τ, i.e. to provide G without covering its cost with taxation. At first sight this looks close to a refutation of the saying 'There is no such thing as a free lunch' (made popular in the economics profession by **Milton Friedman)** because people may enjoy some G without having to pay for it, but on closer inspection it is not. As shown in **Figure 6**, such policy generates in steady state a (stable) positive *debt/GDP* ratio. If we regard *the interest bill on public debt* as a *redistributive social cost, or burden*, on the economy, then any government budget policy generating a positive β, with or without taxation, would involve some such social cost. The redistributive burden of public debt is a complex issue, whose in-depth treatment lies outside the scope of this chapter. We therefore limit ourselves to three remarks giving a summary idea of the nature of the problem.

First, interest payments on public debt are treated as transfers, as if they were negative taxes. It is a fact that interest payments on public debt are by general convention rergarded not as an income derived from market wealth creation but as a transfer of income from the government to debt holders. In other words, just as *taxes* are regarded as *transfers* from the economy to the government (i.e. from the private (market) economy to the public (non-market) economy), public debt interest payments are regarded as transfers in the opposite direction. The rationale for this lies precisely in the separation between the market and non-market parts of the economy. In the market part, interest payments on private debt are viewed as income from market wealth creation because it is assumed that - at least in principle - such payments are covered by the additional market wealth (profits) created by the productive investments financed by the private loans underlying private debt. And loans for market consumption are similarily likened to investment loans because they also may be thought of as creating market wealth, albeit indirectly, through their positive impact on market demand. On the contrary, in the non-market field interest payments on public debt do not - by definition - come out of market wealth creation. By definition, they can only be covered by taxes, or by further public indebtment, because public production, even when it has an exclusive or predominantly production-enhancing role, is by definition not sold in the market. However, in the case of production-enhancing public expenditures, these increase market output, and this increases automatically tax revenues, so that, to an extent, such expenditures may be regarded as self-financing, though always through taxation and not through the market. As mentioned above, a proper treatment of production-enhancing public expenditures and their implications for assessing the merits and demerits of deficit financing lies outside the scope of this chapter.

Second, being transfers, like (negative) taxes, interest payments on public debt perform in general a redistribution of income (wealth) relative to the income (wealth) distribution that would prevail without them. Suppose person A, with income equal to 100, owns 10% of debt, while person B, with income equal to 50, owns 5%, then the transfer interest payments on this 15% of debt exercise a regressive impact on income (re) distribution from B to A because they increase A′s income proportionally more than B′s. If this state of affairs is general, in the sense that whenever somebody is wealthier than somebody else the former owns a larger share of debt than the latter, then we may state the general proposition that the existence of public debt exercises always a regressive impact on income distribution, and that this regression increases with the size of the debt and/or the interest rate (**Dernburg & Dernburg 1969**, p. 124).

Third, Domar's aggregate measure of the debt burden. In his classic **1944** paper **Evsey Domar** uses as an aggregate measure of this redistributive social cost or burden of the debt the ratio of the gross interest bill on public debt, over *GDP* plus the the gross interest bill itself. Using the notations introduced so far, **Domar**'s ratio in nominal terms is given by

$$\tau = \frac{RB}{PQ + RB} \tag{8}$$

which we denote by τ because it can also be interpreted as the average tax rate on all incomes required to service the debt. But what **Domar** 'wants to know is whether the redistributive burden of national debt will grow larger or diminish' over time (**Dernburg & Dernburg 1969**, p. 124). The first step in the analysis is to assume that the government borrows a *constant fraction of GDP*. To avoid confusion we see here the importance of the distinction, carefully introduced above (see eq (3)), between nominal and real deficits. Let us first choose the *nominal deficit*. Dividing both numerator and denominator in (8), first by P, then by Q, and then by β, the measure of the debt burden becomes a function of the *debt/GDP* ratio β and the *nominal interest* R

$$\begin{aligned} \tau &= \frac{RB}{PQ + RB} = \frac{Rb}{Q + Rb} = \frac{R\beta}{1 + R\beta} = \frac{R}{\dfrac{1}{\beta} + R} \\ &= \tau(\beta, R) \end{aligned}$$

Assuming that the government borrows a constant *nominal deficit/GDP* fraction $\dfrac{\dot{B}}{PQ} = H$ yields, from (3), the following elementary dynamic eq for β

$$\dot{\beta} + (g + \pi)\beta = H$$

whose general solution is

$$\beta(t) = Ae^{-(g+\pi)t} + \frac{H}{g+\pi}$$

with initial value equal to $\beta(0) = A + \dfrac{H}{g+\pi}$, and *ss* value at $t \to \infty$ equal to $\beta_{ssH} = \dfrac{H}{g+\pi}$.

But for conformity with our previous *primary deficit* study of the *debt/GDP* dynamics we prefer to work with the *real deficit* and the *real interest*. We thus rewrite the debt burden measure τ using the real interest

$$\begin{aligned}\tau &= \frac{rB}{PQ+rB} = \frac{rb}{Q+rb} = \frac{r\beta}{1+r\beta} = \frac{r}{\frac{1}{\beta}+r} \qquad (9)\\ &= \tau(\beta, r)\end{aligned}$$

and assume a constant *real deficit/GDP* fraction $\dfrac{\dot{b}}{Q} = \eta$, obtaining from (3) the formally similar eq and solutions for β

$$\begin{aligned}\dot{\beta} + g\beta &= \eta\\ \beta(t) &= Ae^{-gt} + \frac{\eta}{g}\\ \beta(0) &= A + \frac{\eta}{g}\\ \beta_{ss\eta} &= \frac{\eta}{g}\end{aligned}$$

For the sake of completeness we may easily check that the *same nominal and real deficit/GDP ratios* $H = \eta$ yield the same *ss* values $\beta_{ssH} = \beta_{ss\eta} \Longleftrightarrow \pi = 0$ because

$$\frac{H}{g+\pi} = \frac{\eta}{g} \to H = \eta(1 + \frac{\pi}{g})$$

To avoid having to distinguish between nominal and real deficits we may assume for simplicity a *zero inflation*.

Now, substituting $\beta_{ss\eta} = \dfrac{\eta}{g}$ into (8) we obtain the *ss* measure of the debt burden associated to a constant real deficit/GDP fraction

$$\tau_{ss} = \frac{r}{\frac{g}{\eta} + r}$$

As expected, the *ss* debt burden increases with η and r and decreases with g (**Dernburg & Dernburg 1969**, p. 126). All these relationships are graphically synthesized in **Figure 7**.

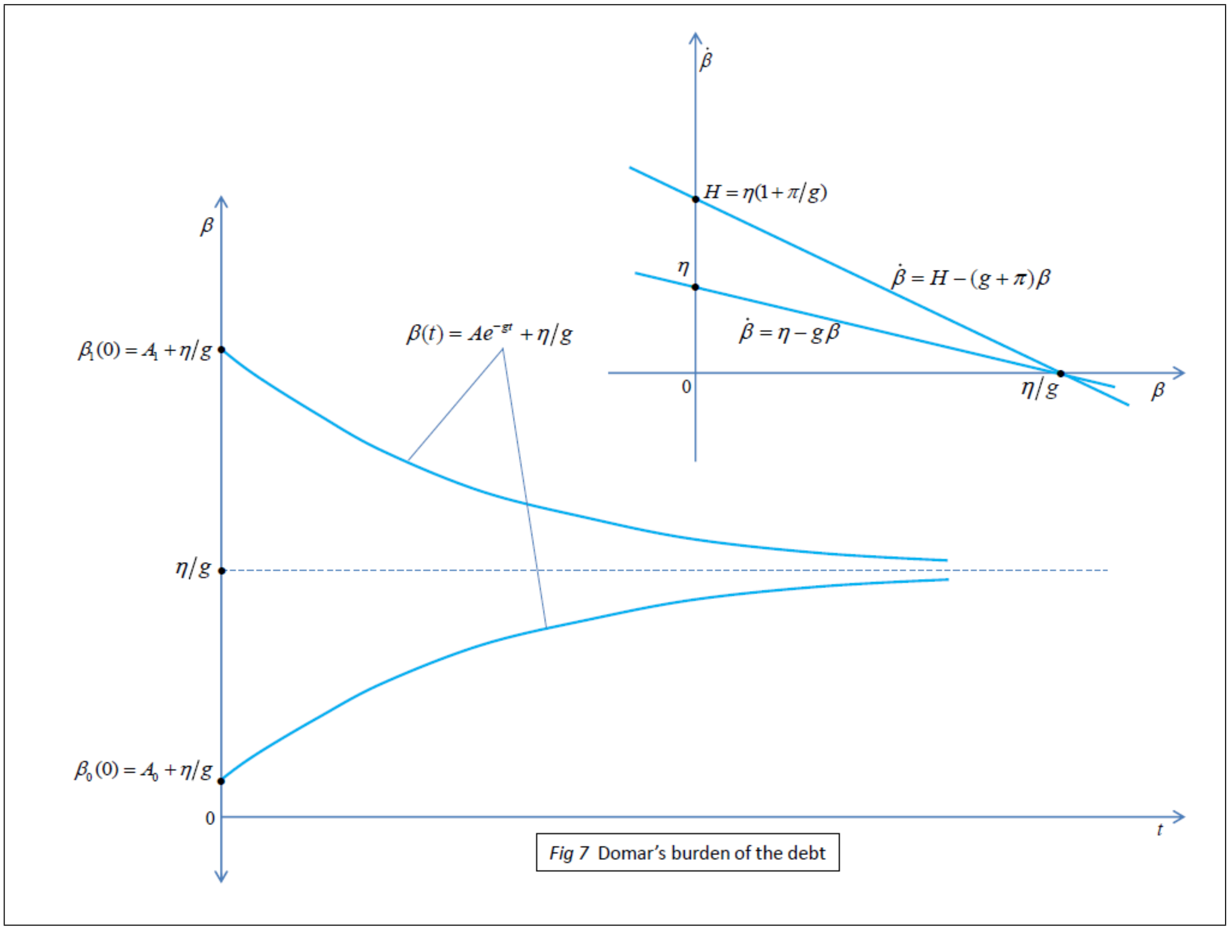

Fig 7 Domar's burden of the debt

9 - Combining the dynamics of the *Debt/GDP* and *Capital/Output* ratios

First we define *real disposable income adjusted for inflation*

$PQ - T + rB$

$T = \tau(PQ + (r + \pi)B)$

$T/P = \tau(Q + (r + \pi)b)$

$(PQ - T + rB)/P = Q - T/P + rb$

$= Q - \tau(Q + (r + \pi)b) + rb$

$= (1 - \tau)Q + ((1 - \tau)r - \tau\pi))b$

$= (1 - \tau)Q + (1 - \tau)rb = (1 - \tau)(Q + rb)$

By assumption C is equal to $(1 - s)$ times the *net* real adjusted disposable income

$C = (1 - s)[(1 - \tau)Q + ((1 - \tau)r - \tau\pi)b]$

$G = \gamma Q$

$I = \dot{K}$

The (closed economy) national accounting equation in real terms $C+G+I = Q$ becomes

$(1 - s)[(1 - \tau)Q + ((1 - \tau)r - \tau\pi)b] + \gamma Q + \dot{K} = Q$

This yields

$$\begin{aligned} (1 - \gamma)Q - (1 - s)(1 - \tau)Q - (1 - s)((1 - \tau)r - \tau\pi)b &= \dot{K} \\ [s(1 - \tau) - (\gamma - \tau)]\, Q - (1 - s)((1 - \tau)r - \tau\pi)b &= \dot{K} \end{aligned}$$

Dividing by Q and cancelling for simplicity (see above) the inflation-dependent term $\tau\pi$ yields

$$[s(1 - \tau) - (\gamma - \tau)] - (1 - s)(1 - \tau)r\beta = \frac{\dot{K}}{Q}$$

Now, introducing a new notation for the *capital/output* ratio $\frac{K}{Q} = v$, we see that

$$\begin{aligned} \dot{\left(\frac{K}{Q}\right)} &= \dot{v} = \frac{\dot{K}Q - K\dot{Q}}{Q^2} = \frac{\dot{K}}{Q} - \frac{K}{Q}\frac{\dot{Q}}{Q} = \frac{\dot{K}}{Q} - gv \\ &\rightarrow \frac{\dot{K}}{Q} = \dot{v} + gv \end{aligned}$$

This yields the equation of the dynamics of the *capital/output* ratio v as a function of v itself and β

$$\begin{aligned} \dot{v} + gv &= [s(1 - \tau) - (\gamma - \tau)] - (1 - s)(1 - \tau)r\beta \\ \dot{v} &= [s(1 - \tau) - (\gamma - \tau)] - (1 - s)(1 - \tau)r\beta - gv = \dot{v}(v, \beta) \end{aligned}$$

which in turn yields the condition for the constant (steady state) *capital/output* ratio v_{ss}

$$\dot{v} = 0 \rightarrow \beta = \frac{s(1 - \tau) - (\gamma - \tau)}{(1 - s)(1 - \tau)r} - \frac{g}{(1 - s)(1 - \tau)r}v \rightarrow \beta(v)$$

Since we want to investigate the relationship between the dynamics of the *capital/output* ratio and the dynamics of the *debt/GDP* ratio we add here the already explained dynamic eq of the latter, which, for comparability with the former, we represent not only as a function of β but also as a constant function of v

$$\dot{\beta} = (\gamma - \tau) - (g - (1 - \tau)r)\,\beta \rightarrow \dot{\beta}(v, \beta)$$

with its associated condition for tha constant (steady state) *debt/GDP* ratio β_{ss}

$$\dot{\beta} = 0 \rightarrow \beta = \frac{(\gamma - \tau)}{(g - (1 - \tau)r)}$$

The *combined dynamics* of the *Debt/GDP* and *Capital/Output* ratios defined by these eqs is represented graphically in **Figure 8**.

The intercepts with the v, β axes of this linear phase diagram are

$$\begin{aligned} \beta &= 0 \rightarrow v = \frac{s(1-\tau) - (\gamma - \tau)}{g} \\ v &= 0 \rightarrow \beta = \frac{s(1-\tau) - (\gamma - \tau)}{(1-s)(1-\tau)r} \end{aligned}$$

The intercepts with the v, β axes of this linear phase diagram are

$$\begin{aligned} \beta &= 0 \rightarrow v = \frac{s(1-\tau) - (\gamma - \tau)}{g} \\ v &= 0 \rightarrow \beta = \frac{s(1-\tau) - (\gamma - \tau)}{(1-s)(1-\tau)r} \end{aligned}$$

References and readings

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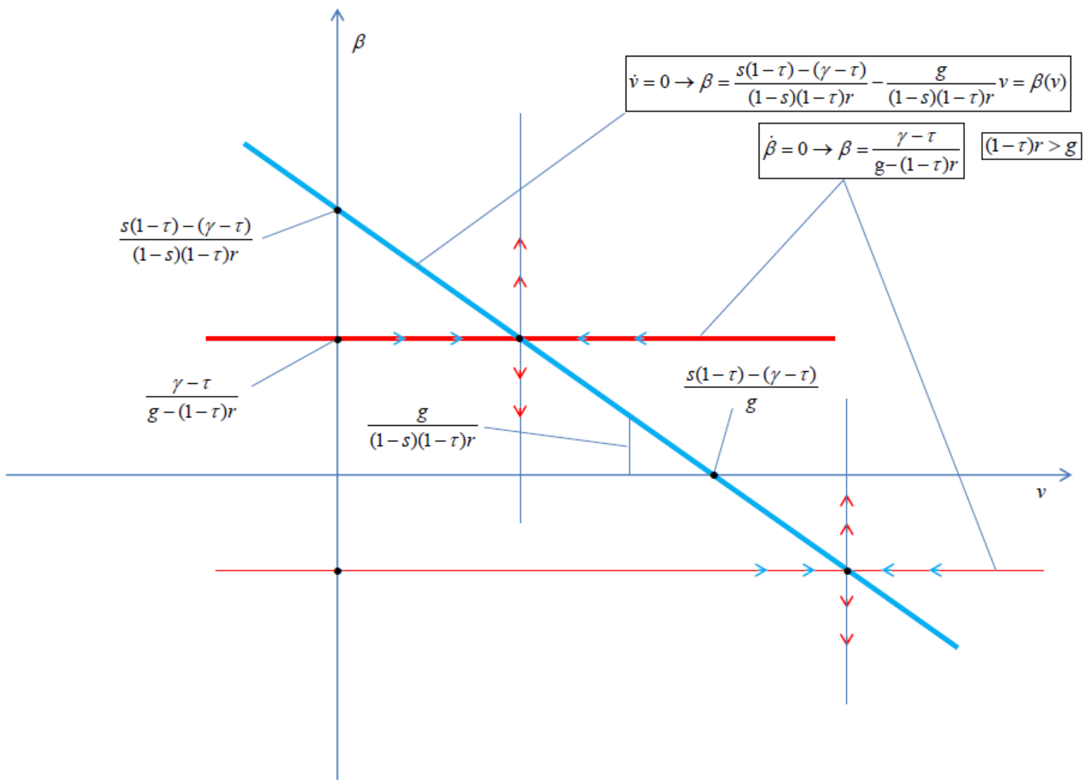

Fig 8 The combined dynamics of the Debt/GDP and Capital/Output ratios

Sections 1-5 of this chapter are a summary of the analytical architecture of what is known in the literature as the *neoclassical one-sector model of economic growth*, or *Solow model*. They bring together into an organic framework parts from Solow 2000, Chiang 2005, Jones & Vollrath 2013. The final Sections 6-7 apply that framework to the specific subject of the chapter. They are our own elaboration from Solow 2000 Chapter 4: 'A model with two assets'.

Jones & Vollrath 2013 is a recommended comprehensive introduction to the modern theory of long-run economic dynamics.

[8/5/15]