Exam Statistics 9^{th} September 2015

Maura Mezzetti

1 Exercise 1

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as a Uniform distribution $U(0, \theta)$

1. Find sufficient statistics for θ
2. Find MLE (maximum likelihood estimator) and the MOM (method of moments estimator) of θ
3. Is the MLE unbiased? If not, find an unbiased estimator based on the MLE.
4. Compare the MLE and the MOM of θ with respect to the MSE criterion
5. Verify that X_{max}/θ is a pivotal quantity for θ.
6. Show that $[Xmax; \alpha^{-1/n} X_{max}]$ is a $100(1-\alpha)\%$ confidence interval for θ

2 Solution Exercise 1

1.
$$f(x|\theta) = \begin{cases} \theta^{-1}, & 0 \leq x \leq \theta \\ 0, & otherwise \end{cases}$$

Thus the joint probability distribution function of $X_1, \ldots, X_n$ is

$$f(\underline{x}|\theta) = \begin{cases} \theta^{-n}, & 0 \leq x_i \leq \theta, \, for \; i = 1, \ldots, n \\ 0, & otherwise \end{cases}$$

Since for each i, $x_i \leq \theta$, we can write $\max_i x_i \leq \theta$, and let define $T(x) = max_i \, x_i$:

$$f(\underline{x}|\theta) = \theta^{-n} \prod_i I_{[0,\theta]}(x_i)$$

$$f(\underline{x}|\theta) = \theta^{-n} I_{[max(x_i),\infty]}(\theta)$$

and

$$g(t|\theta) = \begin{cases} \theta^{-n}, & t \leq \theta \\ 0, & otherwise \end{cases}$$

It can be easily verified that $f(x|\theta) = g(T(x)|\theta)h(x)$ for all x and all θ. So the statistics $T(\mathbf{X}) = max_i X_i$ is a sufficient statistics for θ

2.

$$L(\theta|\underline{x}) = \theta^{-n} \prod_i I_{[0,\theta]}(x_i) = \theta^{-n} I_{[max(x_i),\infty]}(\theta)$$

Likelihood function is a decreasing function of θ, therefore the likelihood is maximized when θ is equal to the minimum values $\hat{\theta}_{MLE} = max(x_i)$

$$\begin{aligned} E(X) &= \int x f(x|\theta) dx = \int_0^\theta \frac{x}{\theta} dx = \\ &= |_0^\theta \frac{x^2}{2\theta} = \frac{\theta}{2} \\ \frac{\theta}{2} &= \bar{x} \\ \hat{\theta}_{MOM} &= 2\bar{x} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int x^2 f(x|\theta) dx = \int_0^\theta \frac{x^2}{\theta} dx = \\ &= |_0^\theta \frac{x^3}{3\theta} = \frac{\theta^2}{3} \\ Var(X) &= E(X^2) - E(X)^2 = \frac{\theta^2}{3} - \frac{\theta^2}{4} \\ Var(X) &= \frac{\theta^2}{12} \end{aligned}$$

$$\begin{aligned} Var(\bar{X}) &= \frac{\theta^2}{12n} \\ Var(\hat{\theta}_{MOM}) &= \frac{\theta^2}{3n} \\ MSE(\hat{\theta}_{MOM}) &= \frac{\theta^2}{3n} \end{aligned}$$

3. The maximum likelihood estimator is biased. For $0 \geq x \geq \theta$, the distribution of $Y = \max_{1 \leq i \leq n} X_i$ is

$$F(y) = P \max_{1 \leq i \leq n} X_i \leq y = P\{X_i \leq y\}^n = \left(\frac{y}{\theta}\right)^n$$

Thus, the density

$$f_{(n)}(y) = \frac{ny^{n-1}}{\theta^n}$$

The mean

$$E_Y = \frac{n}{n+1}\theta$$
$$E_{\hat{\theta}_{MLE}} = \frac{n}{n+1}\theta$$

An unbiased estimator for θ could be

$$\hat{\hat{\theta}} = \frac{n+1}{n}\hat{\theta}_{MLE} = \frac{n+1}{n} max X_i$$

4. Given the density of $Y = max x_i$

$$f_{(n)}(y) = \frac{ny^{n-1}}{\theta^n}$$

The variance can be computed as:

$$\begin{aligned}
E(Y^2) &= \int_0^\theta x^2 f_{(n)}(y)dy = \int_0^\theta \frac{ny^{n+1}}{\theta^n}dy = \frac{n\theta^{n+2}}{(n+2)(\theta^n)} \\
E(Y^2) &= \frac{n}{n+2}\theta^2 \\
Var(Y) &= E(Y^2) - E(Y)^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\right)^2\theta^2 = \frac{n}{(n+2)(n+1)^2}\theta^2 \\
Bias(\hat{\theta}_{MLE}) &= E(\hat{\theta}_{MLE}) - \theta = \frac{n}{n+1}\theta - \theta = \frac{-1}{n+1}\theta \\
MSE(\hat{\theta}_{MLE}) &= \frac{n}{(n+2)(n+1)^2}\theta^2 + \frac{1}{(n+1)^2}\theta^2 = \frac{2}{(n+2)(n+1)}\theta^2
\end{aligned}$$

$$\begin{aligned}
MSE_{MLE}(\hat{\theta}) &\leq MSE_{MOM}(\hat{\theta}) \\
\frac{2}{(n+2)(n+1)}\theta^2 &\leq \frac{\theta^2}{3n}
\end{aligned}$$

5. For $0 \geq x \geq \theta$, the distribution of $Y = \max_{1 \leq i \leq n} X_i$

$$f_{(n)}(y) = \frac{ny^{n-1}}{\theta^n}$$

.

Let $z = \frac{y}{\theta}$, thus $dy = \theta dz$

$$f_Z(z) = nz^{n-1} \qquad for 0 \leq z \leq 1$$

X_{max}/θ is a pivotal quantity for θ.

6. Cumulative distribution function for z is

$$\begin{aligned}
F_Z(z) &= z^n \qquad for 0 \leq z \leq 1, \\
F_Z(\alpha^{\frac{1}{n}}) &= \alpha \\
1 - F_Z(\alpha^{\frac{1}{n}}) &= 1 - \alpha \\
P(\alpha^{\frac{1}{n}} \leq z \leq 1) &= 1 - \alpha \\
P\left(\alpha^{\frac{1}{n}} \leq \frac{X_{max}}{\theta} \leq 1\right) &= 1 - \alpha \\
P\left(X_{max} \leq \theta \leq \alpha^{-\frac{1}{n}} X_{max}\right) &= 1 - \alpha
\end{aligned}$$

3 Exercise 2

Let T_1 and T_2 be two independent (for example, obtained from different samples) unbiased estimators of a parameter θ with variances σ_1^2 and σ_2^2 respectively. Find the UMVUE for θ among all linear combinations of T_1 and T_2 . What is its variance?

4 Solution Exercise 2

$$\begin{aligned}
T &= a_1 T_1 + a_2 T_2 \\
E(T) &= a_1 E(T_1) + a_2 E(T_2) \\
E(T) &= (a_1 + a_2)\theta
\end{aligned}$$

To be unbiased: $a_1 + a_2 = 1$, $a_2 = 1 - a_1$

$$\begin{aligned}
Var(T) &= a^2 Var(T_1) + (1-a)^2 Var(T_2) \\
E(T) &= a^2 \sigma_1^2 + (1-a)^2 \sigma_2^2
\end{aligned}$$

$$\begin{aligned}
\frac{dVar(T)}{da} &= 2a\sigma_1^2 - 2(1-a)\sigma_2^2 \\
2a\sigma_1^2 + 2(1-a)\sigma_2^2 &= 0 \\
a\star &= \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}
\end{aligned}$$

The UMVUE estimator for θ among all linear combinations of T_1 and T_2 is

$$T = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} T_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} T_2$$

5 Exercise 3

Choose one of the following questions:

1. Provide correct statement for Cramér-Rao inequality

2. Provide correct statement for Likelihood Principle and Describe method to reach Maximum Likelihood Estimation