Time, Uncertainty and Liquidity

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Theory of Banking, 2017

Time, uncertainty and liquidity

Let us start with some basic notions that will be useful throughout the course.

- Inter-temporal budget constraint
- Contingent commodities and risk sharing
- Attitude towards risk
- Insurance and risk pooling

Inter-temporal budget constraint/1

- Consider a two-period economy $t = 0, 1$.

- There exists one unique homogeneous consumption good.

- Suppose the consumer has an income stream (Y_0, Y_1) at time 0 and 1, respectively.

- Let the utility be a function of consumption in the two dates: $U(C_0, C_1)$

- Let (C_0, C_1) be the date-0 and date-1 consumption flows.

To set up the consumer's problem we need to define the set of feasible allocations.

- This set is given by the *inter-temporal budget constraint*.

Inter-temporal budget constraint/2

• An allocation belongs to the feasible set for the consumer if the income stream (Y_0, Y_1) can be transformed into a consumption stream (C_0, C_1) by *borrowing and lending*.

• Suppose the consumer can borrow and lend at a constant, safe interest rate $i > 0$ per period.

Let a bank be the institution offering such a contract.

• Borrowing: the consumer borrows 1 unit of the good at $t = 0$ and promises to give back $(1 + i)$ units at $t = 1$.

• Lending: the consumer lends $\frac{1}{1+i}$ unit of the consumption good at $t = 0$ in exchange for a promise to receive back 1 unit at $t = 1$.

Inter-temporal budget constraint/3

Let B define borrowing, i.e. $B > 0$ if the consumer is a net borrower at $t = 0$ ($B < 0$, otherwise).

$$B = C_0 - Y_0$$

Suppose the consumer decides to consume $C_0 > Y_0$ at $t = 0$, he is a net borrower. To be able to repay the debt with interest i,

$$(1 + i)B \leq Y_1 - C_1$$

i.e. consumption C_1 must be lower than income Y_1 by an amount equal to principal+interest.

In other words,

$$C_0 - Y_0 \leq \frac{1}{1 + i}(Y_1 - C_1)$$

Inter-temporal budget constraint/4

Similarly, let S define savings, i.e. $S > 0$ if the consumer is a net lender at $t = 0$ ($S < 0$, otherwise).

$$S = Y_0 - C_0$$

Suppose the consumer decides to consume $C_0 \leq Y_0$, he could save $S > 0$ and deposit it at a bank. In $t = 1$ he will receive $(1 + i)S$ which he can use to finance consumption:

$$C_1 - Y_1 \leq (1 + i)S$$

Using the definition of S, we can rewrite:

$$C_0 - Y_0 \leq \frac{1}{1 + i}(Y_1 - C_1)$$

Inter-temporal budget constraint/5

This is the inter-temporal budget constraint:

$$C_0 - Y_0 \leq \frac{1}{1+i}\left(Y_1 - C_1\right)$$

it represents the set of all allocations which can be achieved by borrowing and lending, hence are feasible.

It is also true that all consumption pairs (C_0, C_1) that satisfy the inter-temporal budget constraint can be reached by some form of borrowing and lending.

Hence, an income stream (Y_0, Y_1) can be transformed into a consumption stream (C_0, C_1) through borrowing and lending at a fixed interest rate i if and only if it satisfies the inter-temporal budget constraint.

Inter-temporal budget constraint/6

Consider again the inter-temporal budget constraint:

$$C_0 + \frac{1}{1+i}C_1 \leq Y_0 + \frac{1}{1+i}Y_1$$

Intercepts:

• if the consumer decides to forgive consumption at $t = 1$, then at $t = 0$ he can consume $C_0 = Y_0 + \frac{1}{1+i}Y_1$

• if the consumer decides to forgive consumption at $t = 0$, then at $t = 1$ he can consume $C_1 = (1 + i)Y_0 + Y_1$,

Inter-temporal budget constraint/7

Consider again the inter-temporal budget constraint:

$$C_0 + \frac{1}{1+i}C_1 \leq Y_0 + \frac{1}{1+i}Y_1$$

• Suppose now that consumption in $t = 0$ is increased by ΔC_0. What is the effect on future consumption?

Future consumption will be reduced. By how much?

• The inter-temporal budget constraint implies that $\Delta C_1 = -(1+i)\Delta C_0$, because for every unit of consumption chosen at $t = 0$ the consumer is renouncing to $(1+i)$ units in $t = 1$

• The slope of the inter-temporal budget constraint is -(1+i).

Wealth and present values

Consider again the inter-temporal budget constraint:

$$C_0 + \frac{1}{1+i} C_1 \leq Y_0 + \frac{1}{1+i} Y_1$$

• We can interpret the RHS of the equation as the present value of income stream:

$$PV(Y_0, Y_1) \equiv Y_0 + \frac{1}{1+i} Y_1$$

sometimes also referred to as **wealth, W**.
and the present value of the consumption stream

$$PV(C_0, C_1) \equiv C_0 + \frac{1}{1+i} C_1$$

• The inter-temporal budget constraint requires that the present value of consumption must be at most equal to consumer's wealth.

• Another interpretation of the inter-temporal budget constraint can be obtained by considering C_0 and C_1 as two distinct commodities with respective prices p_0 and p_1.

$$p_0 C_0 + p_1 C_1 \leq p_0 Y_0 + p_1 Y_1$$

• Arbitrage considerations imply that, if C_0 is the numeraire good i.e. $p_0 = 1$, $p_1 = \frac{1}{1+i}$ otherwise consumers could have arbitrage opportunities by borrowing and lending and could make arbitrarily large profits.

• Borrowing and lending at a constant interest rate is equivalent to having a market in which present and future consumption can be exchanged at constant prices (p_0, p_1)

Individual's optimal consumption choice/1

Given the inter-temporal budget constraint, the individual's maximization problem is

$$\begin{aligned} & max \qquad U(C_0, C_1) \\ s.t. \qquad & C_0 + \frac{1}{1+i} C_1 \leq W \equiv \quad Y_0 + \tfrac{1}{1+i} Y_1 \end{aligned}$$

Individual's optimal consumption choice/1

The optimality condition is given by:

$$MRS = \frac{\frac{\partial U}{\partial C_0}(C_0^*, C_1^*)}{\frac{\partial U}{\partial C_1}(C_0^*, C_1^*)} = (1+i)$$

which represents the tangency point between the inter-temporal budget constraint and the highest achievable indifference curve.

Individual's optimal consumption choice/2

- The optimality condition can be rewritten as:

$$\frac{\partial U}{\partial C_0}(C_0^*, C_1^*) = (1+i)\frac{\partial U}{\partial C_1}(C_0^*, C_1^*)$$

to emphasize that the marginal utility for consumption at date 0 must be equal to the value of the marginal utility of saving the same unit until date 1, consuming it at time 1 when it will be worth (1+i).

Lagrange method/1

- To solve the individual's maximization problem

$$\begin{aligned} max \qquad & U(C_0, C_1) \\ s.t. \qquad & C_0 + \frac{1}{1+i} C_1 \leq W \end{aligned}$$

we can alternatively use the Lagrange method. Write the Lagrange function:

$$\mathcal{L}(C_0, C_1, \lambda) = U(C_0, C_1) + \lambda \left(W - C_0 - \frac{1}{1+i} C_1 \right)$$

then maximize it w.r.t (C_0, C_1, λ).

Lagrange method/2

- Necessary conditions for $(C_0^*, C_1^*, \lambda^*)$ to be a stationary point for $\mathcal{L}$ are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_0}(C_0^*, C_1^*, \lambda^*) &= \frac{\partial U}{\partial C_0}(C_0^*, C_1^*) - \lambda^* = 0 \\
\frac{\partial \mathcal{L}}{\partial C_1}(C_0^*, C_1^*, \lambda^*) &= \frac{\partial U}{\partial C_1}(C_0^*, C_1^*) - \frac{\lambda^*}{(1+i)} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda}(C_0^*, C_1^*, \lambda^*) &= W - C_0 - \frac{1}{1+i} C_1 = 0
\end{aligned}$$

leading to the same result:

$$\frac{\partial U}{\partial C_0}(C_0^*, C_1^*) = (1+i)\frac{\partial U}{\partial C_1}(C_0^*, C_1^*)$$

Contingent commodities and risk sharing/1

Dealing with uncertainty can be done by defining:

• the set of possible *states of nature*;
A state of nature is a complete description of all uncertain, exogenous factors that can affect the outcome of an individual's decision.

• the set of *contingent commodities*, which are identified by physical characteristics, the date and the state of nature at which they are delivered.

• a set of *preferences* for each individual, defined over bundles of contingent commodities;

• a (well-defined) *probability distribution* over the states of nature.

Von Neumann-Morgenstern approach to individual choice under uncertainty.

Contingent commodities and risk sharing/2

• The individual problem then becomes to choose the bundle of contingent commodities that maximizes his utility, within the set of feasible allocations (budget constraint).

• For the optimal allocation to satisfy efficient allocation of risk, we must assume that there is a set of complete markets, i.e. there is a market for each contingent commodity. An economy with complete markets is often referred to as an *Arrow-Debreu* economy.

• The optimal allocation satisfies the property that it lies at the tangency between the highest achievable indifference curve and the budget constraint.

Attitude towards risk

The Von Neumann-Morgenstern approach to choice under uncertainty implies that individual attitude towards risk translates into preferences structure.

Hence,

• an individual is *risk averse* if her utility is a strictly concave function, i.e. if the marginal utility of consumption is decreasing.

• an individual is *risk lover* if her utility is a strictly convex function, i.e. if the marginal utility of consumption is increasing.

• finally, an individual is *risk neutral* if her utility is a linear function, i.e. if the marginal utility of consumption is constant.

Insurance and risk pooling

• In an economy with two agents $\{a, b\}$ and two states of nature $\{H, L\}$, the set of efficient allocations can be identified by means of the Edgeworth box, and corresponds to the consumption points which belong to the *contract curve*.

• An efficient allocation is given by (C_{aH}, C_{aL}), (C_{bH}, C_{bL}) such that the marginal rates of substitutions for agent a and b are equalized:

$$\frac{U_a'(C_{aH})}{U_a'(C_{aL})} = \frac{U_b'(C_{bH})}{U_b'(C_{bL})}$$

• Risk is shared among agents according to their respective preferences.

• In the particular case in which agent a is risk neutral, the efficiency condition becomes:

$$\frac{U_b'(C_{bH})}{U_b'(C_{bL})} = 1$$

which implies $C_{bH} = C_{bL}$, i.e. consumption is independent of the state of nature.

• Hence, the risk neutral agent bears the entire risk due to uncertainty, providing complete insurance to the risk-averse agent.