Problem Set 3: Solution

Theory of Banking - Academic Year 2016-17

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Exercise 1

Consider an economy á la Diamond and Dybvig (1983), that lasts for three periods ($t = 0, 1, 2$) and with a unique good. The economy is populated by a continuum of ex-ante identical agents (of mass 1) and each agent is endowed, at time $t = 0$, with one unit of the good. There are two types of agents: *patient* and *impatient* agents. Indeed each agent will know, at $t = 1$, whether he has to consume at $t = 1$ (*impatient agent*) or at $t = 2$ (*patient agent*). A non random proportion π_1 of the agents will be impatient and the complementary proportion, $\pi_2 = 1 - \pi_1$, will be patient. The agent's type is not observable.
The agents' utility function is:

$$\begin{cases} \sqrt{C_1} & \text{for impatient agents} \\ \rho\sqrt{C_2} & \text{for patient agents} \end{cases}$$

,s.t. Moreover, in the economy, any agent has access to both a storage technology and a long-run technology:

- the *storage technology* yields a zero net interest per unit of good stored;
- otherwise any agent can invest, at $t = 0$, in a *long-run technology.* This technology yields $R > 1$ units of the good, at $t = 2$, per unit invested, whereas if the investment is prematurely liquidated at $t = 1$ yields only $L < 1$ unit of the good per unit invested.

(a) *Write down the constrained maximization problem of the consumer who lives in autarky.*
Under autarky the consumer solves the following constrained optimization problem for the expected utility function $V(C_1, C_2) = \pi_1\sqrt{C_1} + (1 - \pi_1)\rho\sqrt{C_2}$:

$$\max_{\{C_1,C_2\}} \quad \pi_1\sqrt{C_1} + (1 - \pi_1)\rho\sqrt{C_2} \quad s.t.$$

$$\text{(i)} \qquad x + y \leq 1$$

$$\text{(ii)} \quad C_2^p(y, x) = Rx + y$$

$$\text{(iii)} \quad C_1^i(y, x) = y + Lx$$

where x and y are the shares of the initial endowment invested in the Long Term technology and the Storage technology.

(b) *Characterize the consumption allocation that guarantees the highest utility to the consumer in autarky.*

First note that the constraint (i) is binding at the optimum; therefore we can rewrite (ii) and (iii) in terms of x only:

$$\text{(ii)} \qquad C_2^p(y{=}1{-}x,\, x) = Rx + (1-x)$$

$$\text{(iii)} \qquad C_1^i(y{=}1{-}x,\, x) = (1-x) + Lx$$

Considering that $0 \le x \le 1$, compute the consumption allocation in the extreme points $x{=}0$

$$\text{(ii)} \qquad C_2^p(y{=}1,\, x{=}0) = 1$$

$$\text{(iii)} \qquad C_1^i(y{=}1,\, x{=}0) = 1$$

and $x{=}1$

$$\text{(ii)} \qquad C_2^p(y{=}0,\, x{=}1) = R$$

$$\text{(iii)} \qquad C_1^i(y{=}0,\, x{=}1) = L\,.$$

Using these two feasible allocations, we can recover the budget constraint using the equation of a stright line passing through two points :

$$\frac{C_1 - 1}{L-1} = \frac{C_2 - 1}{R-1}$$

Rearrange the equation to express C_2 as a function of C_1, thus finding the equation of the budget constraint:

$$C_2 = -\frac{R-1}{1-L}\,C_1 + \frac{R+L}{1-L}\,.$$

Note that the slope depends both on R and L. To characterize the optimal allocation three conditions have to be met:

1. It has to be feasible, i.e., it must lie below or on the straight line representing the budget constraint.
2. Its indifference curve must be tangent to the straight line representing the budget constraint. Since the slope of the indifference curve is

   $$-MRS = -\frac{\partial V/\partial C_1}{\partial V/\partial C_2} = -\frac{\pi}{1{-}\pi}\,\frac{\sqrt{C_2}}{\rho\sqrt{C_1}} \quad \text{and the slope of the budget constraint is} \quad \frac{R-1}{1-L}\,,$$

   the tangency condition is $MRS = (R-1)/(1-L)$
3. C_1 must be between L and 1 ($0 \le L \le C_1 \le 1$).This condition comes from the specific characteristics of the investment contracts.

(c) *Compute the (first best) efficient allocation and explain how it depends on ρR.*

To determine the optimal allocation we can consider the problem of a benevolent central planner who wants to maximize the expected utility of the representative consumer, allocating the endowment among the available technologies. If the central planner can observe the consumers' types then the solution of the planner's problem is the efficient one.

The central planner problem can be stated as follows:

$$\max_{\{C_1,C_2\}} \quad \pi_1\sqrt{C_1} + \pi_2\rho\sqrt{C_2} \quad s.t.$$

$$x + y \leq 1$$

$$\pi_1 C_1^i(y, x) \leq y$$

$$\pi_2 C_2^p(y, x) \leq Rx + (y - \pi_1 C_1^i(y, x))$$

Since the three constraints are binding we can rewrite the optimization as:

$$\max_{y} \quad \pi_1 \sqrt{\frac{y}{\pi_1}} + \pi_2 \rho \sqrt{\frac{R(1-y)}{\pi_2}} \tag{1}$$
$$\text{s.t} \quad 0 \leq y \leq 1$$

Thus, consider the First Order Condition and set it equal to zero:

$$\frac{1}{2} \frac{y}{\pi_1}^{-1/2} - \rho R \frac{1}{2} \left(\frac{R(1-y)}{\pi_2} \right)^{-1/2} = 0$$

$$\left(\frac{y}{\pi_1} \right)^{-1/2} = \rho R \left(R \frac{1-y}{\pi_2} \right)^{-1/2}$$

Notice that the FOC is the condition that supports the optimal consumption allocation across time 1 and time 2 with a new parameter (ρ):

$$u'(C_1^\star) = u'(C_2^\star) \rho R$$

It is left to show that $y^\star$ is associated to a maximum. We can consider the second derivative of the expected utility to check that $y^\star$ is indeed a maximum.

$$\frac{\partial \left[\pi_1 \sqrt{C_1} + \pi_2 \rho \sqrt{C_2} \right]}{\partial y} = -\frac{1}{2} \left(\frac{y}{\pi_1} \right)^{-\frac{3}{2}} \left(\frac{1}{\pi_1} \right) - \rho R \left(-\frac{1}{2} \right) \left(\frac{R(1-y)}{\pi_2} \right)^{-\frac{3}{2}} \left(-\frac{R}{\pi_2} \right) \leq 0$$

For any $y \in [0, 1]$ the second derivative of the expected utility is strictly negative, therefore the expected utility is strictly concave and $y^\star$ is associated to a global maximum of the central planner's problem.
From the FOC we get:

$$y^\star = \frac{\pi_1}{\rho^2 R \pi_2 + \pi_1} \quad \text{and} \quad x^\star = \frac{\rho^2 R \pi_2}{\rho^2 R \pi_2 + \pi_1}$$

Once $y^\star$ and $x^\star$ have been determined, then the optimal consumption stream is:

$$C_1^\star = \frac{y^\star}{\pi_1} = \frac{1}{\rho^2 R \pi_2 + \pi_1} \quad \text{and} \quad C_2^\star = \frac{R(1-y^\star)}{\pi_2} = \frac{\rho^2 R^2}{\rho^2 R \pi_2 + \pi_1}$$

Moreover since $u(C) = \sqrt{C}$ is strictly increasing and strictly concave, from the FOC we can state that:

$$\begin{cases} C_2^\star > C_1^\star & \text{if and only if} \quad \rho R > 1 \\ C_2^\star = C_1^\star & \text{if and only if} \quad \rho R = 1 \\ C_2^\star < C_1^\star & \text{if and only if} \quad \rho R < 1 \end{cases}$$

(d) *Assume $\rho = 1$, can the optimal allocation be achieved in a market economy as the one described in class? (The asset traded in the economy is a zero coupon bond, that yields one unit of the $t = 2$ consumption good.)*
In a market economy we know that:

$$C_1 = y + pRx \quad \text{and} \quad C_2 = Rx + \frac{y}{p}$$

where p is the price of the bond.
Moreover, to have an interior solution ($C_1^\star > 0$ and $C_2^\star > 0$), the market p should be equal to $1/R$.
Hence the market allocation is:

$$C_1^M = 1 \quad \text{and} \quad C_2^M = R$$

When $\rho = 1$, the optimal allocation has to satisfy the following condition (FOC rewritten for $\rho = 1$):

$$u'(C_1) - Ru'(C_2) = 0 \Leftrightarrow \frac{1}{\sqrt{C_1}} - \frac{R}{\sqrt{C_2}} = 0$$

Substituting (C_1^M, C_2^M) in the previous equation we get:

$$\frac{1}{\sqrt{C_1^M}} - R\frac{1}{\sqrt{C_2^M}} = 1 - \sqrt{R} = 0$$

which is not true in general.

(e) *Is such a market allocation optimal? Explain.*
Since $R \neq 1$, the *market economy allocation* cannot be an optimal allocation.

(f) *Is there any value of ρ such that the optimal allocation can be achieved in a market economy? Explain.*
To answer the question we can substitute (C_1^M, C_2^M) into the FOC:

$$u'(C_1^M) - \rho Ru'(C_2^M) = 0 \Leftrightarrow 1 - \rho\sqrt{R} = 0 \Rightarrow \rho = \frac{1}{\sqrt{R}}$$

Exercise 2

In the same framework of the exercise above, assume now $\rho R > 1$.

a) *Can the optimal allocation be implemented by a financial intermediary by means of a deposit contract? Explain how the optimal contract will work. Characterize the bank's balance sheet.*
If $\rho R > 1$ then $C_2^\star > C_1^\star$ and the bank can offer the following deposit contract to the agents: if the consumer turns out to be impatient, she can withdraw an amount $C_1^\star$ from the bank account at time $t = 1$, otherwise if she turns out to be patient the bank promises to pay an amount $C_2^\star$ in $t = 2$. Such contract is incentive compatible because $C_2^\star > C_1^\star$. In order to fulfill such promises the bank collects deposits in $t = 0$ and invest $y^\star$ and $x^\star$ in the storage technology and the long term investment, thus the bank's balance sheet is:

Assets	Liabilities
Storage Technology ($y^\star$)	Deposits (1)
Long Term Investment ($x^\star$)	

If a bank run occurs, every consumer withdraws money from the bank at $t = 1$. The bank would be able to fulfil its initial promise if and only if the value of the short term asset and the liquidation value of the long term investment are big enough, therefore:

$$y^\star + Lx^\star \geq C_1^\star$$

substituting the optimal allocations and consumption choices:

$$\pi_1 + L\rho^2 R\pi_2 \geq 1$$

$$\pi_1 + L\rho^2 R(1 - \pi_1) \geq 1$$

$$\rho^2 LR \geq 1.$$

b) *Now consider an economy in which at $t = 0$ the financial intermediary is financed entirely by consumers with all their endowments and each consumer receives a share of the bank. At $t = 0$, the financial intermediary announces the dividend per share that will be distributed at $t = 1$. At $t = 1$ the financial intermediary distributes a dividend to every shareholder, $d > 0$ (after each consumer has known his type and before he has to consume). At $t = 2$ the financial intermediary is liquidated and all its cash-flow is distributed to shareholders on a pro-rata basis. At $t = 1$, after dividends have been distributed and before agents have to consume, a (secondary) market for share and dividends opens in which consumers can buy or sell their assets at a price p.*
Can the optimal allocation be achieved in such an equity economy for the intermediary? Explain the result.

This "equity-economy" has been introduced by Jacklin (1983). Each agent receives a share of the bank at time $t = 0$, dividends d in $t = 1$. At time $t = 1$, once the agent discovers her type, she can either buy new shares of the bank on the market (these are claims on consumption in $t = 2$) or sell her shares and consume both the proceeds and the dividends. The bank performs the usual allocation of financial resources between storage and long term investment. In this case the amount of liquid resources that the bank needs at time $t = 1$ corresponds to the amount of dividends the bank has to pay, the remainder is invested LT yielding R in $t = 2$. Therefore:

$$C_1 = p + d$$

$$C_2 = \left(1 + \frac{d}{p}\right) R(1 - d)$$

To close the economic model, we impose the market clearing condition for the market of shares. There are π_1 impatient agents, each willing to sell shares, and there are π_2 patient agents that are willing to use their dividends to buy claims on future consumption. Therefore, the market clearing condition is:

$$\pi_1 = \pi_2 \frac{d}{p},$$

that yields to the equilibrium price

$$p = \frac{\pi_2}{\pi_1} d.$$

We rewrite consumption as:

$$C_1 = d + \frac{\pi_2}{\pi_1} d = \frac{d}{\pi_1}$$

$$C_2 = R(1-d) + R(1-d)\frac{\pi_2}{\pi_1} = \frac{R(1-d)}{\pi_2}.$$

Finally the bank solves the following maximization problem:

$$\max_d \pi_1 \sqrt{\frac{d}{\pi_1}} + \pi_2 \rho \sqrt{\frac{R(1-d)}{\pi_2}} \tag{2}$$

$$s.t. \quad 0 \leq d \leq 1$$

Notice that (2) is analogous to (1) where d replaces y. Therefore the two yields to the same results. You can verify that:

$$d^\star = y^\star$$

$$p^\star = \frac{\pi_2}{\rho^2 R \pi_2 + \pi_1},$$

and the optimal consumption levels $C_1^\star$, $C_2^\star$ are those found in point (c) of Exercise 1.

3) *Assume $\rho R < 1$, then*:

3.1) *Determine the optimal deposit contract for a financial intermediary.*

If $\rho R < 1$ then $C_1^\star > C_2^\star$ and there is no incentive compatible deposit contract. Indeed, if $C_1^\star > C_2^\star$, for a patient agent is optimal to withdraw money in $t = 1$ and carry consumption to $t = 2$ using the storage technology, since this yields an higher consumption than waiting to obtain $C_2^\star$. Incentive compatibility is violated.