Problem Set 2

Theory of Banking - Academic Year 2016-17

Maria Bachelet
maria.jua.bachelet@gmai.com

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Exercise 1

Consider an agency relationship in which the principal contracts the agent, whose effort determines the outcomes. Assume that uncertainty is represented by three states of nature $\{\epsilon_1, \epsilon_2, \epsilon_3\}$. The agent can choose between two effort levels. The results are shown in Table 1.

		states of nature		
		ϵ_1	ϵ_2	ϵ_3
efforts	e=6	60000	60000	30000
	e=4	30000	60000	30000

Table 1: Effort Levels and States of Nature

The principal and the agent both believe that the probability of each state is 1/3. The objective functions of the principal and the agent are, respectively:

$$B(x, w) = x - w$$

$$U(w, e) = \sqrt{w} - C(e)$$

where x is the monetary outcome of the relationship, w is the monetary wage that the agent receives and $C(e) = e^2$ is the cost of effort. Assume that the agent only accepts the contract if he obtains an expected utility level of at least 114 (his reservation utility level).

(a) What is the attitude of each player towards risk?

(b) What is the wage that a principal proposes to an agent that exerts effort $e = 6$? Determine the agent's and the principal's utility at this contract.

(c) What is the wage that a principal proposes to an agent that exerts effort $e = 4$? Determine the agent's and the principal's utility at this contract.

(d) What would the optimal level of effort and the optimal wage be in a situation in which the principal can contract on agent's effort and on wage?

Solution

(a) The agent is risk averse (i.e. $\partial^2 U(w,e)/\partial w^2 = -1/4 \cdot (w)^{-3/2}$) and the principal is risk neutral (i.e. $\partial^2 B(x,w)/\partial w^2 = 0$).

(b) When the principal can condition the wage on the level of effort, the optimal contract implies that the principal bears all the risk and the agent's participation constraint is binding. Thus, the optimal wage is found by imposing: $U(w,e) = \sqrt{w} - C(e) = 114$. The expected profits are: $\mathbb{E}(\Pi(e)) = \pi_1(e)x_1 + \pi_2(e)x_2 - w(e)$.

When $e = 6$ we have:

- optimal wage: $w^{1/2} - 6^2 = 114 \Rightarrow w_{(e=6)} = 22500$
- expected profits: $\frac{2}{3}(60000) + \frac{1}{3}(30000) - 22500 = 27500$
- utility: $22500^{1/2} - 6^2 = 114$

(c) When $e = 4$ we have:

- optimal wage: $w^{1/2} - 4^2 = 114 \Rightarrow w_{(e=4)} = 16900$
- expected profits: $\frac{2}{3}(30000) + \frac{1}{3}(60000) - 16900 = 23100$
- utility: $16900^{1/2} - 4^2 = 114$

(d) In a situation of symmetric information, the principal will set $w = 22500$ in order to induce effort $e = 6$ and obtain higher expected profits. This result relies on the principal risk neutrality (the amount of money for which the individual is indifferent between the gamble and the certain amount, namely the certainty equivalent, equals the expected profits). Indeed, a strictly risk averse agent accepts the gamble only if the expected utility of the consumption is sufficiently higher than the expected consumption.

Exercise 2

Consider an insurance economy, with two states of nature ($s \in \{s_1, s_2\}$, with s_1 = no accident and s_2 = accident) and complete and perfect information. Let the consumer be risk-averse and the insurer be risk-neutral. The consumer is endowed with $W > 0$ units of the consumption good, but with a certain probability he will have an accident and suffer a loss $0 < L < W$.
The agent can be of two types:

i) *good type* (type G), the probability of an accident is π_G;

ii) *bad type* (type B), the probability of an accident is π_B.

Assume that $\pi_G < \pi_B$. Consider the Cartesian space defined over the contingent resources available in case of no accident (x_1) -x axis- and in case of accident (x_2) -y axis-.lLet the agents' utility function be $\sqrt{x}$.

1) Write down the expected utility function for each type of consumer.

2) Characterize the indifference curves of each consumer' type in the Cartesian space (x_1, x_2). Are they increasing? Are they convex?

3) Compare the marginal rate of substitution (in absolute value) along the indifference curves for the different types, what can you conclude?

Consider now a perfectly competitive insurance market. Insurance is provided to the consumer by means of contracts which specify a premium, α, that the consumer pays when signing the contract, and a total reimbursement β that the consumer receives in case of accident.

4) Write down the expected profits of the insurance company for each type of consumer.

5) Write down the contingent consumptions (x_1, x_2) for the consumer in case he decides to buy insurance.

6) Characterize the insurer's iso-profit lines in the Cartesian space $(x_1,\ x_2)$. Are they increasing? Are they convex?

7) Represent the iso-profit function that corresponds to insurer's zero expected profits (ZPL) in the Cartesian space $(x_1,\ x_2)$, for every type of consumer. Show that these iso-profit functions pass through the endowment point $(W,\ W-L)$ and have a slope equal to $-(1-\pi_i)/\pi_i$ with $i=G,B$.

8) Show that if the riskiness of the consumer is public information, every consumer's type achieves full insurance and the insurance premium is increasing with the consumer's riskiness.

Solution

1. The expected utility function can be written in terms of x_1, the consumption of the unique consumption good in case s_1 (no accident), and of x_2, the consumption of the good in case s_2(accident), where $x_1 = W$ and $x_2 = W - L$,

$$V_i(x_1, x_2) = (1-\pi_i)\sqrt{x_1} + \pi_i\sqrt{x_2} \quad \text{for} \quad i = G, B.$$

2. $\forall (x_1, x_2) \in \mathbb{R}^2_+$ the marginal rate of substitution is:

$$MRS_i(x_1, x_2) = \frac{\partial V_i(x_1,x_2)/\partial x_1}{\partial V_i(x_1,x_2)/\partial x_2} = \frac{1-\pi_i}{\pi_i}\frac{\sqrt{x_2}}{\sqrt{x_1}} \quad \text{for} \quad i = G, B.$$

Consider the total differential of the consumer's expected utility:

$$dV_i(x_1,x_2) = 0 \Leftrightarrow dV_i(x_1,x_2) = \frac{\partial V_i(x_1,x_2)}{\partial x_1}dx_1 + \frac{\partial V_i(x_1,x_2)}{\partial x_2}dx_2 = 0 \Leftrightarrow$$

$$\Leftrightarrow (1-\pi_i)\frac{1}{\sqrt{x_1}}dx_1 + \pi_i\frac{1}{\sqrt{x_2}}dx_2 = 0 \quad \text{for} \quad i = G, B$$

Then $\forall (x_1, x_2) \in \mathbb{R}^2_+$ the slope of the consumer's indifference curves is:

$$\frac{dx_2}{dx_1} = -MRS_i = -\left(\frac{1-\pi_i}{\pi_i}\right)\frac{\sqrt{x_2}}{\sqrt{x_1}} = -\left(\frac{1-\pi_i}{\pi_i}\right)x_2^{\frac{1}{2}}x_1^{-\frac{1}{2}} \quad \text{for} \quad i = G, B \tag{1}$$

The indifference curve of both type of consumer are strictly decreasing $\forall (x_1, x_2) \in \mathbb{R}^2_+$.
Let's now determine if the indifference curve are concave in the Cartesian space (x_1, x_2):

$$\frac{d^2x_2}{dx_1^2} = -\left(\frac{1-\pi_i}{\pi_i}\right)\left[-\frac{1}{2}x_1^{-\frac{3}{2}}x_2^{\frac{1}{2}} + \frac{1}{2}x_1^{-\frac{1}{2}}x_2^{-\frac{1}{2}}\frac{dx_2}{dx_1}\right] \quad \text{for} \quad i = G, B \tag{2}$$

Substituting Eq.(1) into Eq.(2) we obtain $\frac{d^2x_2}{dx_1^2} > 0$ for any $x_1 > 0$, $x_2 > 0$ and hence the indifference curves of both types of consumer are strictly convex $\forall (x_1, x_2) \in \mathbb{R}^2_+$.

3. For each $(x_1, x_2) \in \mathbb{R}^2_+$, since $\pi_G < \pi_B$, we have:

$$MRS_G(x_1, x_2) = \left(\frac{1-\pi_G}{\pi_G}\right) x_2^{\frac{1}{2}} x_1^{-\frac{1}{2}} > \left(\frac{1-\pi_B}{\pi_B}\right) x_2^{\frac{1}{2}} x_1^{-\frac{1}{2}} = MRS_B(x_1, x_2)$$

Given a consumption bundle $(x_1, x_2) \in \mathbb{R}^2_+$, the indifference curve of the good consumer passing through (x_1, x_2) is steeper than the one of the bad consumer.

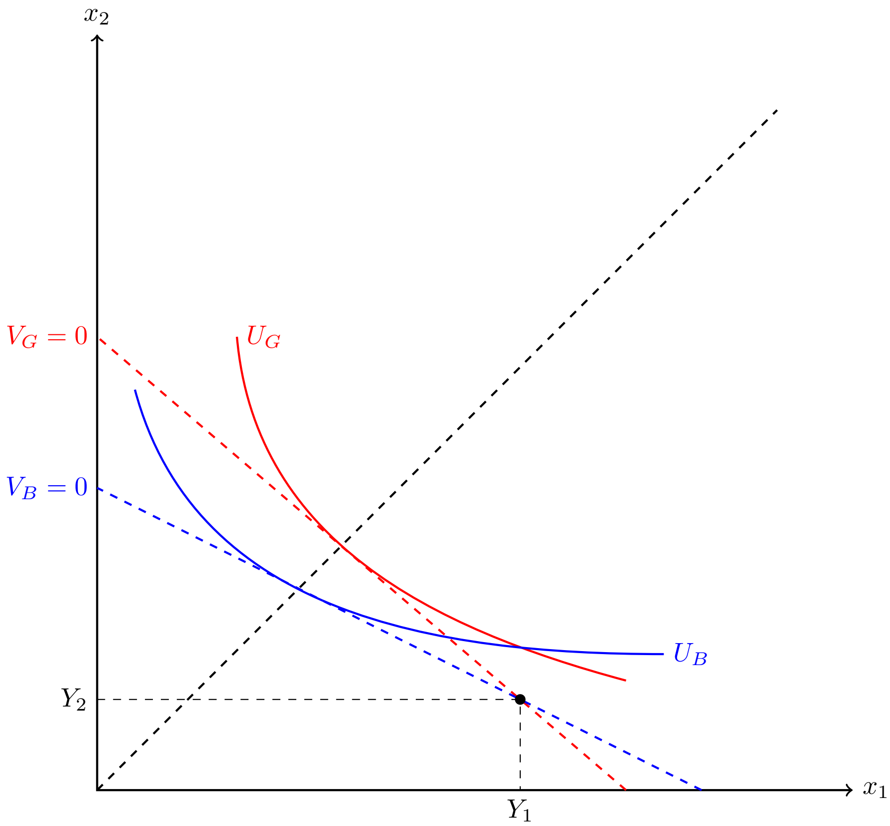

Figure 1: Indifference curves and isoprofit lines.

4. Given the structure of the insurance contract, the insurer's expected profits when facing a consumer of type i are:

$$\Pi_i(\alpha, \beta) = (1-\pi_i)\alpha - \pi_i(\beta - \alpha) \quad \text{with} \quad i = G, B.$$

5. $$x_1^I = W - \alpha \quad \text{and} \quad x_2^I = W - \alpha - L - \beta.$$

6. The insurer's expected profits when facing a consumer of type i are:

$$\Pi_i(x_1, x_2) = (1-\pi_i)\alpha - \pi_i(\alpha - \beta) \quad \text{with} \quad i = G, B.$$

By rearranging the equations for the contingent consumption of the consumer, $\alpha = x_1 - W$ and $\alpha - \beta = x_2 + (W - L)$,, the insurer's expected profits can be written in term of (x_1, x_2),

$$\Pi_i(x_1, x_2) = (1-\pi_i)(W - x_1) - \pi_i[x_2 - (W - L)] \quad \text{with} \quad i = G, B.$$

Suppose that the insurer's expected profits are equal to $c \in \mathbb{R}_+$:

$$(1-\pi_i)(W-x_1) - \pi_i[x_2 - (W-L)] = c$$

$$x_2 = \frac{1}{\pi_i}(W-c) - \left(\frac{1-\pi_i}{\pi_i}\right) x_1 - L \quad \text{for} \quad i \in \{G, B\}\,.$$

The insurer's iso-profit line are linear and decreasing in the Cartesian space $(x_1,\, x_2)$.

7. When $c = 0$, we can check if $(x_1, x_2) = (W, W-L)$ satisfies the insurer's zero iso-profit line for $i \in \{G, B\}$:

$$\Pi_i(x_1, x_2) = (1-\pi_i)(W-x_1) - \pi_i[x_2 - (W-L)].$$

The slope of the consumer's iso-profit line can be easily obtained considering the equations that describe the two zero expected profits conditions (c=0):

$$x_2^0 = \frac{1}{\pi_i} W - \left(\frac{1-\pi_i}{\pi_i}\right) x_1 - L \quad \text{for} \quad i \in \{G, B\}\,,$$

$$\frac{\partial x_2^0(x_1)}{\partial x_1} = -\left(\frac{1-\pi_i}{\pi_i}\right).$$

8. When there is symmetric information (in this case it implies that the principal can observe the consumer's type), the insurer can condition the premium on the consumer's level of riskiness. When choosing α and β in order to maximize his profits, the insurer just takes into consideration the individual rationality constraint (IR) which says that the expected utility of the consumer that signs the insurance contract must be higher - or at least equal - to his "reservation utility" (i.e. the expected utility he would get without the insurance). The insurer problem is:

$$\max_{\alpha,\beta}(1-\pi_i)\alpha + \pi_i(\alpha-\beta)\text{subject to}(1-\pi_i)(w-\alpha) + \pi_i(w-\alpha-L-\beta) >= (1-\pi_i)\alpha + \pi_i(\alpha-\beta)$$

with F.O.C.,

$$-p_i(1-\pi_i)U'(W-p_i\beta) + \pi_i(1-p_i)U'(W-p\beta-L+\beta) = 0.$$

Suppose now that the price p of one unit of insurance is actuarially fair in the sense of it being equal to the expected cost of insurance, $p_i = \pi_i$. Therefore,

$$U'(W-p\beta) = U'(W-p\beta-L+\beta) \quad \Rightarrow \quad \beta^* = L$$

Thus, if insurance is actuarially fair, the decision maker insures completely. The individual's final wealth is then $W - \pi_i L$, regardless of the occurence of the loss.
Since the information on the consumer type is public, the insurer can perfectly discrinate between agents and there are two prices in the market $p_G = \pi_G$ and $p_B = \pi_B$. Moreover, $\alpha_i = p_i\beta^*$. Consequently, $p_G < p_B$ implies $\alpha_G < \alpha_B$.