Solutions to Problem Set 1

Theory of Banking - Academic Year 2016-17

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Exercise 1. *An individual consumer has an income stream (Y_0, Y_1) and can borrow and lend at the interest rate i. For each of the following data points, determine whether the consumption stream (C_0, C_1) satisfies the intertemporal budget constraint.*

	(C_0, C_1)	(Y_0, Y_1)	$(1+i)$
case (a)	(18, 11)	(15, 15)	1.1
case (b)	(10, 25)	(15, 15)	1.8

Table 1: Exercise 1 - Consumption and income streams.

Draw a graph to illustrate your answer in each case.

Exercise 1: Solution.

The intertemporal budget constraint,

$$C_0 - Y_0 \leq \frac{1}{(1+i)}(Y_1 - C_1),$$

represents the set of all allocations which can be achieved by borrowing and lending in the economy. Let us compute the highest level of consumption that a consumer can achieve in each period (i.e. the consumer chooses zero consumption in the other period). C_0^{int} represents the highest level of consumption achievable at $t = 0$ while C_1^{int} represents the highest level of consumption achievable at $t = 1$:

$$C_0^{int} = Y_0 + \frac{1}{1+i}Y_1 \qquad \text{and} \qquad C_1^{int} = C_0^{int}(1+i) = (1+i)Y_0 + Y_1$$

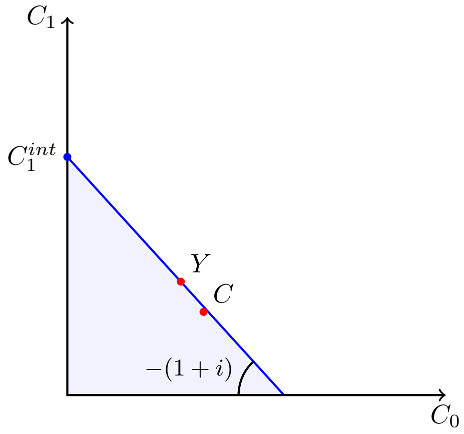
Hence C_1 satisfies the inter-temporal budget constraint if:

$$(C_1 - Y_1) \leq (1+i)(Y_0 - C_0) \quad \Leftrightarrow \quad C_1 \leq Y_1 + (1+i)Y_0 - (1+i)C_0 \quad \Leftrightarrow \quad C_1 \leq C_1^{int} - (1+i)C_0$$

Case (a): (C_0, C_1)=$(18, 11)$; (Y_0, Y_1)=$(15, 15)$ and $(1 + i)$=1.1.
Let's compute C_1^{int}:

$$C_1^{int} = 1.1 \cdot 15 + 15 = 31.5$$

Figure 1: Exercise 1 - case (b)



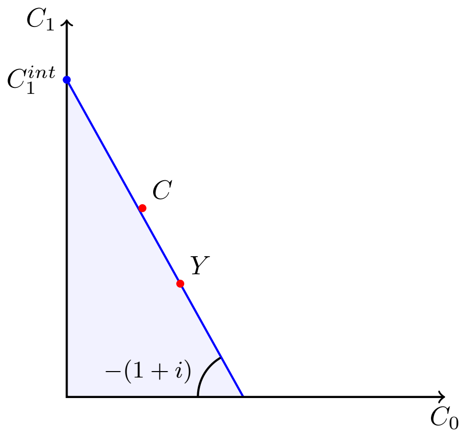
Now we check if the consumption stream can satisfy the inter-temporal budget constraint:

$$11 \leq 31.5 - 1.1 \cdot 18 \Leftrightarrow 11 \leq 11.7$$

The above inequality is satisfied, therefore in case (a) the consumption stream is a feasible allocation.

Case (b): (C_0, C_1)=$(10, 25)$; (Y_0, Y_1)=$(15, 15)$ and $(1+i)$=1.8.

Figure 2: Exercise 1 - case (d)



Let's compute C_1^{int}:

$$C_1^{int} = 1.8 \cdot 15 + 15 = 42$$

Now we check if the consumption stream can satisfy the inter-temporal budget constraint:

$$25 \leq 42 - 1.8 \cdot 10 \Leftrightarrow 25 \leq 24$$

The above inequality is not satisfied, therefore in case (b) the consumption stream is not a feasible allocation.

Exercise 2. *Mr. Jones has an income stream* $(Y_0, Y_1) = (100,\ 50)$ *and can borrow and lend at the interest rate* $i = 0.11$. *His preferences are represented by the additively separable utility function:*

$$U(C_0, C_1) = \frac{C_0^{1-\eta}}{1-\eta} + 0.9\frac{C_1^{1-\eta}}{1-\eta},$$

where the risk-aversion is given by the parameter $\eta = 2$.

(a) *Define the marginal utility of consumption at* $t = 0$ *and* $t = 1$, *respectively.*

(b) *Write down the consumer's intertemporal budget constraint and the first order condition that must be satisfied by the optimal consumption stream.*

(c) *Use the first order condition and the consumer's inter-temporal budget constraint to find the consumption stream* $(C_0^\star, C_1^\star)$ *that maximizes utility.*

(d) *How much will the consumer save at* $t = 0$? *How much will his savings be worth at* $t = 1$?

(e) *Check that he can afford the optimal consumption* $C_1^\star$ *in* $t = 1$.

Exercise 2: Solution.

(a) The marginal utility of a good is the gain in consumer satisfaction from an increase in consumption of one unit of that good. Formally,

$$\frac{\partial U(C_0, C_1)}{\partial C_0} = \frac{1}{C_0^2} \qquad \frac{\partial U(C_0, C_1)}{\partial C_1} = 0.9\frac{1}{C_1^2}$$

(b) The intertemporal budget constraint is:

$$C_0 - Y_0 \leq \frac{1}{1+i}(Y_1 - C_1) \Rightarrow C_0 - 100 \leq \frac{1}{1.11}(50 - C_1)$$

We can rewrite it as:

$$C_0 + \frac{1}{i+i}C_1 \leq Y_0 + \frac{1}{i+i}Y_1 \Rightarrow C_0 + \frac{1}{1.11}C_1 \leq 145.04$$

. To define his optimal consumption stream, the consumer solves the following problem:

$$\max_{(C_1,C_2)} \frac{C_0^{1-\eta}}{1-\eta} + 0.9\frac{C_1^{1-\eta}}{1-\eta} \tag{1}$$

$$\text{s.t.} \quad C_0 + \frac{1}{1.11}C_1 \leq 145.04$$

. We can now write down the Lagrange function:

$$\mathcal{L}(C_0, C_1, \lambda) = \frac{C_0^{1-\eta}}{1-\eta} + 0.9\frac{C_1^{1-\eta}}{1-\eta} + \lambda\left(145.04 - C_0 - \frac{1}{1.11}C_1\right)$$

To find the values of the parameter in which the Lagrangean achieves its maximum, search its stationary point by imposing the first derivatives equal to 0 and compute the First Order Conditions,

$$\begin{aligned}
\frac{\partial \mathcal{L}(C_0, C_1, \lambda)}{\partial C_0} = 0 \Rightarrow &\quad C_0^{-\eta} = \frac{1}{C_0^2} = \lambda \\
\frac{\partial \mathcal{L}(C_0, C_1, \lambda)}{\partial C_1} = 0 \Rightarrow &\quad 0.9 * C_1^{-\eta} = 0.9\frac{1}{C_1^2} = \frac{1}{1.11}\lambda \\
\frac{\partial \mathcal{L}(C_0, C_1, \lambda)}{\partial \lambda} = 0 \Rightarrow &\quad C_0 + \frac{1}{1.11}C_1 \leq 145.04
\end{aligned}$$

To check that this stationary point is a maximum, check that the Hessian (the matrix of the second derivatives) is negative semidefinite.

(c) By combining the first two FOCs,

$$\frac{1}{C_0^2} = \frac{0.9 \cdot 1.11}{C_1^2} \qquad \Rightarrow \qquad C_1 = 0.999C_0.$$

Since the budget constraint is binding, we can plug this equation for C_1 into the intertemporal budget constraint:

$$C_0 + \frac{1}{1.11}C_1 = 145.04 \qquad \Rightarrow \qquad C_0 + 0.9C_0 = 145.04$$

Therefore, the optimal consumption stream is:

$$C_0^\star = 76.34 \qquad C_1^\star = 76.26.$$

(d) The consumer at $t = 0$ saves an amount of income equal to:

$$S_0 = Y_0 - C_0 = 100 - 76.34 = 23.66$$

If the amount saved in $t = 0$ is lent, in $t = 1$ the consumer will get $1.11 \cdot S_0 = 26.26$.

(e) The optimal consumption in $t = 1$ can be supported by: Y_1 and $S_0(1 + i)$.

$$C_1^\star = Y_1 + S_0(1 + i) = 50 + 26.26 = 76.26$$

Therefore the consumer can afford the optimal consumption in the second period.

Exercise 3. *Consider a portfolio choice in a two-period economy with one risky and one riskless assets. The safe asset gives back the initial investment (it yields one unit of the good at date 1 for each unit invested at date 0). The risky asset returns $R_H = 1.15$ with probability $\pi = \frac{3}{5}$ and $R_L = 0.8$ with probability $(1 - \pi)$ per unit invested. Consider a consumer who owns initial wealth $W_0 = 100$ with preferences represented by $log(C)$.*

1. *Denote the fraction of wealth invested in the risky asset by θ. Write down the contingent consumptions in the high and low state, respectively C_H and C_L.*

2. *Write down the mathematical relation between consumptions C_H and C_L, and represent it in a cartesian space (C_H, C_L). Compute the slope of the frontier of the set of feasible allocations for the consumer. Explain.*

3. *Write down the agent maximization problem. Identify the indifference curve and compute the MRS (Marginal Rate of Substitution)*

4. *Find the optimal portfolio choice θ^*.*

Exercise 3: Solution

1.

$$C_H = R_H\theta W_0 + (1 - \theta)W_0 = 115\theta + (1 - \theta)100 = 100 + 15\theta,$$

$$C_L = R_L\theta W_0 + (1 - \theta)W_0 = 80\theta + (1 - \theta)100 = 100 - 20\theta.$$

2. Rewrite the equation of C_S in terms of W_0,

$$C_S = [R_S\theta + (1-\theta)]W_0 \quad \Rightarrow \quad W_0 = \frac{C_S}{1+\theta(R_S-1)}.$$

Therefore,

$$\frac{C_L}{1+\theta(R_L-1)} = \frac{C_H}{1+\theta(R_H-1)} \quad \Rightarrow \quad C_L = \left[\frac{1+\theta(R_L-1)}{1+\theta(R_H-1)}\right] C_H.$$

For $\theta = 1,\ C_L = \frac{R_L}{R_H} C_H = \frac{0.8}{1.15} C_H = 0.696 C_H$.
To compute the slope of the feasible set, compare the portfolio in which all income is invested in the safe asset with the portfolio in which income is all invested in the risky asset. The change in C_H is $\Delta C_H = W_0 - W_0 R_H$ and the change in C_L is $\Delta C_L = W_0 - W_0 R_L$. Thus, the slope is

$$\frac{\Delta C_L}{\Delta C_H} = \frac{W_0 - W_0 R_L}{W_0 - W_0 R_H} = \frac{1-R_L}{1-R_H} = \frac{0.2}{-0.15} = -1.33$$

In general, when $R_L < 1 < R_H$ the slope is negative.

3. The agent maximizes the expected utility of her future consumption. Her decision problem is

$$\begin{aligned} &\max_{\theta} V(C_H, C_L) = \frac{3}{5}\log(C_H) + \frac{2}{5}\log(C_L) \\ &\text{s.t. } C_H = 100 + 15\theta \\ &\qquad C_L = 100 - 20\theta. \end{aligned}$$

To derive the slope of the indifference curve, let us compute the MRS

$$MRS = \frac{dC_L}{dC_H} = -\frac{\pi U'(C_H)}{(1-\pi)U'(C_L)} = \frac{\pi C_L}{(1-\pi)C_H}$$

The slope of the indifference curve is equal to $-$MRS

$$-MRS = -\frac{\frac{3}{5}(100-20\theta)}{\frac{2}{5}(100+15\theta)}.$$

4. For an interior solution $0 < \theta^* < 1$ of the optimal portfolio choice, the slope of the indifference curve must be equal to the slope of the feasible set (tangency condition),

$$\begin{aligned} -\frac{\pi U'(C_H)}{(1-\pi)U'(C_L)} = \frac{1-R_L}{1-R_H} \quad \Rightarrow \quad & -\frac{\frac{3}{5}(100-20\theta)}{\frac{2}{5}(100+15\theta)} = -\frac{0.2}{0.15} \\ & 3(0.15)(100-20\theta) = 2(0.2)(100+15\theta) \\ & 0.45(100-20\theta) = 0.4(100+15\theta) \\ & 45 - 9\theta = 40 + 6\theta \qquad \Rightarrow \quad \theta^* = \frac{5}{15} = \frac{1}{3}. \end{aligned}$$

This is, of course, the solution of the maximization problem in part (3).
Alternatively, the solution may be found by substituting the constraints in the utility function. In this way the expected utility $V(C_H, C_L)$ is a function of θ only,

$$\max_{\theta\in(0,1)} V(\theta) = \frac{3}{5}\log(100+15\theta) + \frac{2}{5}\log(100-20\theta).$$

Compute the first derivative of the expected utility function with respect to θ:

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{3}{5}\frac{15}{(100+15\theta)} - \frac{2}{5}\frac{20}{(100-20\theta)}.$$

To find the value of θ in which the expected utility function achieves its maximum, search its stationary point by imposing the first derivative equal to 0 and compute the First Order Condition,

$$\begin{aligned} \frac{\partial V(\theta)}{\partial \theta} = 0 \quad \Rightarrow \quad & \frac{3}{5}\frac{15}{(100+15\theta)} = \frac{2}{5}\frac{20}{(100-20\theta)} \\ & 9(100-20\theta) = 8(100+15\theta) \\ & 900-180\theta = 800+120\theta \qquad\qquad \Rightarrow \ \theta^* = \frac{1}{3} \end{aligned}$$

Since the objective function only depends on θ, to check that this stationary point is a maximum we just need to check that the second derivative with respect to θ is negative.

$$\frac{\partial V(\theta)}{\partial^2 \theta} = -\frac{9 \cdot 15}{(100+15\theta)^2} - \frac{20 \cdot 8}{(100-20\theta)^2} < 0.$$