Problem Set 4: Solutions

Theory of Banking - Academic Year 2015-16

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Exercise 1: Collateralizable Debt Contracts

Consider an entrepreneur who can choose between two investment projects, a and b, which need I to be activated. The outcome of every project $i = a, b$ is risky,

$$\tilde{X}_i = \begin{cases} X_i & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i. \end{cases}$$

Assume that both project are valuable, and project a is less risky than b:

$$p_a X_a > p_b X_b > I, \quad 1 > p_a > p_b > 0, \quad X_b > X_a.$$

The entrepreneur has no initial endowment, hence must borrow the amount I, but owns some $W > 0$ which can be used as collateral for the loan. Borrower and lender are risk neutral, and limited liability holds, in that the entrepreneur's utility cannot be negative. Let the bank offer a collateralizable debt contract (I, R, C), in which I is fixed, R represents the gross interest rate of the debt contract, C the amount of collateral which the bank can seize in case of failure.

1) *Compute the expected payoff of the entrepreneur and the expected profit of the bank.*

$$V(i, R, C) = p_i(X_i - R) + (1 - p_i)(-C) = p_i(X_i - R + C) - C$$

$$\Pi(i, R, C) = p_i R + (1 - p_i)C - I = p_i(R - C) + C - I$$

2) *Define the bank optimal contract and the entrepreneur's project choice under symmetric information (the bank observes which project is actually implemented).*
The optimal contract from the bank's point of view is to ask for $R = X_a$, $C = 0$ and to require that project a is carried out. In this case $V(X_a, a) = 0$, and so every entrepreneur gets the same utility in case she receive a loan or not .

3) *Write down the profit maximisation problem of that bank in case of asymmetric information. Explain the role of every constraint.*

$$\begin{aligned} &\max_{i,R,C}\ p_i(R-C)+C-I \\ \text{s.t.}\ \ & p_i(X_i-R+C)-C\geq 0 \\ & C\leq W \\ & p_i(X_i-R)-(1-p_i)C\geq p_j(X_j-R)-(1-p_j)C \end{aligned}$$

4) *Examine the Incentive Compatibility constraint, and find the interest rate threshold as a function of the collateral. Explain what is the effect of having introduced collateral on the incentives of the entrepreneur.*

$$p_a(X_a-R)-(1-p_a)C\geq p_b(X_b-R)-(1-p_b)C$$

$$R\leq \frac{p_aX_a-p_bX_b}{p_a-p_b}+C \qquad \Rightarrow \qquad R\leq \hat{R}+C$$

Notice that in a contract without collateral that constraint would be $R\leq \hat{R}$. The use of collateral induces the entrepreneur to choose project a. In contrast to interest payments, collateral requirements have positive incentive effects. They effectively punish the borrower when his project fails, thus creating a motive to lower the probability of bankruptcy by choosing project a.

5) *Define the relevant range of collateral, which are admissible for the entrepreneur, given his initial collateralizable wealth and the limited liability constraint.*
The lender must choose whether he wants to implement project a or project b. If he decides to implement project b , he can appropriate the entire surplus of the enterprise, by setting $R^*=X_b$ and $C^*=0$, for an expected payoff p_bX_b-I.
If he wants to implement project a, it is most profitable for him to set $R^*=R+C^*$, the maximum compatible with the incentive compatibility condition, and to set $C^*=\min\{W,p_a(X_a-\hat{R})\}$, the maximum compatible with both the constraints $C^*\leq W$ and $V_a(R+C^*,C^*)\geq 0$.

6) *Explain what are the possible (optimal) contractual offers of the bank, as a function of the results obtained in* 4) *and* 5). *Is there any case in which credit rationing may occur?*

- If $p_a\hat{R}+\min\{W,p_a(X_a-\hat{R})\}<p_bX_b$, the lender prefers to implement project b and to appropriate the entire surplus of the project by setting $R^*=X_b$, $C^*=0$. As in the case without collateral, this case does not involve credit rationing because the loan applicants are indifferent about whether they receive loans or not.

- If $\begin{cases} W<p_a(X_a-\hat{R}) \\ p_a\hat{R}+W>p_bX_b \end{cases}$,
the bank finds it most profitable to set $R^*=\hat{R}+W$ and $C^*=W$, thus implementing project a. In this case, the enterpreneur's payoff is $V_a(\hat{R}+W,W)=p_a(X_a-\hat{R})-W$, which is strictly positive. The insufficiency of the bank's funds leads to equilibrium credit rationing because the loan applicants who are rejected envy those who are accepted and would gladly offer to pay more than the interest $\hat{R}+W$ that the bank is asking.

- If $\begin{cases} W \geq p_a(X_a - \hat{R}) \\ p_a[\hat{R} + (X_a - \hat{R})] > p_b X_b \end{cases}$,
the lender again wants to implement project a , this time however by setting $R^* = p_a X + (1-p)\hat{R}$ and $C^* = p_a(X_a - \hat{R})$. The enterpreneur's expected payoff then is zero, which means that the individual rationality constraint $V_a(R^*, C^*) \geq 0$ is binding. Even though some loan applicants are rejected, the equilibrium does not involve rationing because those loan applicants who are rejected do not care and are unwilling to offer more than the lender is asking.

Exercise 2: Credit rationing

Consider an economy populated by 100 entrepreneurs, each entrepreneur can undertake two type of investment projects, but has no initial resources. To start the project, an entrepreneur can only rely on external financing, through which he should raise $I = 1$.
The first project (project a) yields, if successful, a return $R^a = 2$, and in case of failure 0, the probability of success is $p^a = 0.9$. The second project (project b) yields, if successful, a return $R^b = 2.5$, and in case of failure 0 and the probability of success is $p^b = 0.45$.
Each lender can raise funds at a net interest rate equal to r_d and can only offer a standard debt contract that prescribes a loan equal to 1 and a repayment $R = (1 + r)$ with $r \in \mathbb{R}_+$. Moreover, no collateral can be required ($C = 0$). The banking sector is competitive (meaning that banks have to break even).
Entrepreneurs and lenders are risk neutral. The entrepreneur is protected by limited liability. The supply of deposits is given by $D = A + 20r_d$.
Moreover assume that we can apply the Law of Large Numbers to approximate the share of successful projects by p^a, in case firms find optimal to undertake project a, and p^b if the optimal investment choice is b.

1. *Which of the two projects has the highest net present value?*

$$NPV^a = p^a R^a - 1 = 1.8 - 1 = 0.8$$

$$NPV^b = p^b R^b - 1 = 1.125 - 1 = 0.125$$

The project that guarantees the highest NPV is project a.

2. *Assume that the project choice is contractible. Banks are going to implement the efficient project and $A = 91.2$.*

2a) *Write down the expected profit function of the banks per unit invested.*

$$V(a, r) = p^a(1 + r) - (1 + r_d) = 0.9(1 + r) - (1 + r_d)$$

2b) *Determine the equilibrium deposit rate (r_d) that clears the deposit market, i.e. such that supply equals demand.*
The deposit rate that clears the market is given by:

$$Supply_d = Demand_d \Leftrightarrow 91.2 + 20r_d^\star = 100 \Leftrightarrow r_d^\star = \frac{8.8}{20} = 0.44$$

2c) *Determine the interest rate (r) charged by banks in equilibrium, in such a competitive loan market (recall that banks have to break even).*

Since banks should break even:

$$p^a(1+r^\star) - (1+r_d^\star) = 0 \Leftrightarrow (1+r^\star) = \frac{1.44}{0.9} \Leftrightarrow r^\star = \frac{1.44}{0.9} - 1 = 0.6$$

Notice that $r^\star > r_d^\star$.

2d) *Compute the firms' expected profits, Π_F^e.*

The firms' expected profits are:

$$\pi(a, r^\star) = p^a(R^a - (1+r^\star)) = 0.9(2-1.6) = 0.36$$

2e) *Compute the depositors' and the bank's expected profits, Π_D^e and Π_B^e. Finally, compute the social welfare $SW = \Pi_F^e + \Pi_D^e + \Pi_B^e$.*

We could roughly refer to the social welfare (SW) as the sum of: the firms' expected profits, depositors' expected profits and banks' expected profits.

$$SW = 100 \cdot 0.8 = 80 = 100 \cdot 0.36 + 100 \cdot 0.44$$

3. *Assume now that the project choice is not contractible, then:*

3a) *Write down the expected profits of the firm as a function of the project's choice.*

Since the project is not contractible, firms' expected profits are:

$$\pi(c,r) = \begin{cases} p^a[R^a - (1+r)] & \text{if} \quad p^a[R^a-(1+r)] \geq p^b[R^b - (1+r)] \\ p^b[R^b - (1+r)] & \text{if} \quad p^b[R^b-(1+r)] > p^a[R^a - (1+r)] \end{cases}$$

3b) *Define the firms' optimal choice over projects as function of the loan rate, r.*

Since:

$$p^a[R^a - (1+r)] \geq p^b[R^b - (1+r) \quad \Rightarrow \quad r \leq \frac{p_aR_a - p_bR_b}{p_a - p_b} - 1 \quad \Rightarrow \quad r \leq \frac{1}{2}$$

To characterize the optimal investment choice we can refer to the correspondence $c(r)$:

$$c(r) = \begin{cases} a & \text{if} \quad r \leq 1/2 \\ b & \text{if} \quad r > 1/2 \end{cases}$$

3c) *What is the expected gross return of the bank per unit invested, as function of r? Characterize this relationship in the Cartesian space, in which r is on the horizontal axis.*

The expected gross return of the bank is given by:

$$GR(r) = \begin{cases} p^a(1+r) & \text{if} \quad r \leq 1/2 \\ p^b(1+r) & \text{if} \quad r > 1/2 \end{cases}$$

Moreover the firms' individual rationality constraint is violated when:

$$p^b[2.5 - (1+r)] < 0$$

Then we can refer to value of $r \in [0, 1.5]$:

The banks' gross expected returns are non monotonic in r and reach a maximum at $r = 0.5$, indeed $GR(0.5) = 1.35$ whereas $GR(1.5) = 1.125$.

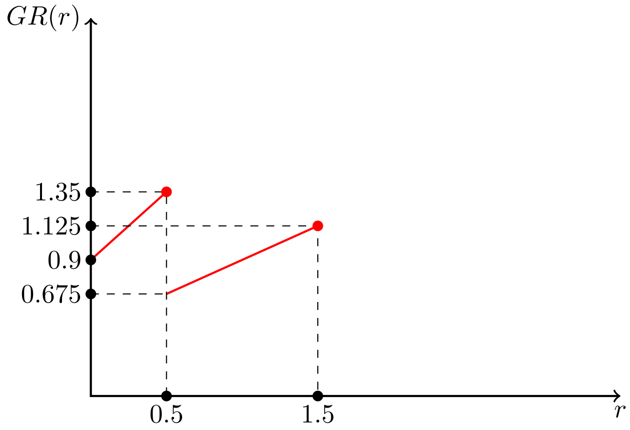

Figure 1: Banks' gross expected returns as function of the loan rate.

4. *Keep assuming that the project choice is not contractible. Characterize the equilibrium in the deposit market.*

   4a) *Characterize the market-clearing deposit rate as a function of A.*
   As usual, if the total demand of deposits is 100, the market clearing condition is given by:

   $$100 = A + 20r_d \Leftrightarrow r_d^{clear} = \frac{100 - A}{20}$$

   4b) *Which value of A leads to credit rationing?*
   Any $A \in \mathbb{R}$ such that the deposit clearing rate (r_d^{clear}) is higher than $\hat{r}_d$ leads to credit rationing, where $\hat{r}_d$ has to satisfy:

   $$p^a \cdot 1.5 - (1 + \hat{r}_d) = 0 \Leftrightarrow 1.35 - 1 = \hat{r}_d \Leftrightarrow \hat{r}_d = 0.35$$

   Then:

   $$r_d^{clear} > \hat{r}_d \Leftrightarrow \frac{100 - A}{20} > 0.35 \Leftrightarrow A < 93$$

   Hence we have credit rationing if the deposit rate required to clear the deposit market is so high, that the corresponding loan rate exceeds it's critical value ($r = 0.5$).

5. *Let $A = 91.2$. Again, derive the market-clearing deposit rate. Which will be in this case the market-clearing loan rate? Which loan rate will a bank charge? What is the aggregate volume of loans?*
   If the demand for deposits is 100 then the clearing deposit rate will be defined through:

   $$r_d^{clear} = \frac{100 - 91.2}{20} = 0.44 > 0.35$$

   Since $0.9 \cdot 1.5 - 1.44 < 0$, the associated loan rate at which banks break even is:

   $$p^b(1 + r^{BE}) = 1.44 \Leftrightarrow r^{BE} = \frac{1.44}{0.45} - 1 = 2.2$$

   But this cannot be the loan rate charged by banks: since $2.2 > 1.5$, nobody would apply for the loan. The banks will set as loan rate the rate which maximizes the total gross expected return, i.e. $r = 0.5$ (figure 1). At that loan rate, the deposit rate at which the banks break even is 0.35 (see 4b):

   $$0.9 \cdot (1 + r) - 1 = r_d^{BE}(r) \Leftrightarrow r_d^{BE}(0.5) = 0.35$$

The associated credit volume is obtained through the clearing condition in the deposit market:

$$\text{Credit Volume} = Supply_d = 91.2 + 20 \cdot 0.35 = 98.2$$

We observe that $98.2 < 100$, i.e., the supply of funds is not enough to cover the demand. This leads to credit rationing. We talk about credit rationing since:

- not all entrepreneurs are able to access the loan
- the utility of borrowers is higher than the utility of non-borrowers (as we shall verify in the subsequent point 6).

We just have to verify that credit rationing is the only possibile equilibrium for this economy.

In this case if a bank tries to charge a loan rate higher than 0.5 raising funds in the deposit market at $r_d^{BE}(0.5)$ then the firm will undertake project b and the bank will get strictly negative expected profits. Moreover at the deposit rate ($r_d^{BE}(0.5)$) a bank that tries to charge a loan rate $r < 0.5$ raising funds in the deposit market will get strictly negative expected profits.
Consider now any loan rate $r' \in (0.5, 1.5]$ (such loan rate will induce the entrepreneur to undertake project b) and the associated deposit rate at which banks break even, $r_d^{BE}(r')$. Such loan rate cannot be an equilibrium loan rate. Indeed notice that $r_d^{BE}(r')$ is increasing in r', and that for $r_d^{BE}(1.5) = 0.125$ the supply of deposit is lass than 100:

$$S_d(0.125) = 91.2 + 20(0.125) = 93.7$$

Since there is always credit rationing for $r' \in (0.5, 1.5]$, then a bank can raise funds in the deposit market at $r_d^{BE}(r')$ and charge for instance a loan rate $r = 0.5$, doing so the firm will undertake project a and the bank is able to make strictly positive profits. So any $r' \in (0.5, 1.5]$ cannot be an equilibrium loan rate. Consider now any loan rate $r'' \in (0, 0.5)$, also in this case there is credit rationing (above we have checked that $S_d(r^{BE}(0.5)) < 100$), then a bank can always raise funds in the deposit rate at $r^{BE}(r'')$ and charging a loan rate $r = 0.5$ is able to make strictly positive profits. So any $r'' \in (0, 0.5)$ cannot be an equilibrium loan rate.

6. *Calculate the expected profits of those firms who receive a loan and compare them to the results from 2d).*

$$\pi(a, 1.5) = 0.9(2 - 1.5) = 0.45$$

The expected profits for firms who gets a credit loan is higher than the case in which the investment choice is contractible, but not all firms receive a loan.

7. *Calculate the social welfare SW, illustrate its composition and compare with the result obtained in 2e).* The social welfare is given by:

$$98.2 \cdot 0.8 = 78.56$$

The social welfare is made by depositors' profits and firms' profits:

$$0.35 \cdot 98.2 + 0.45 \cdot 98.2 = 0.80 \cdot 98.2 = 78.56$$

Depositors' profits are lower and not all depositors can trade. Moreover, total SW is lower with respect to the case of symmetric information.