Univariate Time Series.

Questions:

1. Provide the definitions of stationarity and invertibility for a linear stochastic process. Given the ARIMA(p, d, q) process

$$(1 - \phi L)(1 + 0.5L)y_t = m + (1 + \theta L)\varepsilon_t \tag{1}$$

where $\varepsilon_t \sim WN$, establish the conditions on the parameters ϕ and θ under which y_t is an invertible ARIMA$(1, 1, 1)$ process.

2. Under the conditions derived in (1), compute the unconditional expected value and the auto-covariance function of Δy_t.

3. Derive formulae for the optimal h-step ahead forecast of Δy_t.

Solutions:

1. Regarding the definitions of stationarity and invertibility, see respectively pp. 6-7 and 28-29 of the slides. Notice that process (1) has the autoregressive polynomial

$$\phi(L) = (1 - \phi L)(1 + 0.5L) \tag{2}$$

and the moving-average polynomial

$$\theta(L) = (1 + \theta L) \tag{3}$$

In view of pp. 28-29 of the slides and of Equation (3), we easily conclude that process (1) is invertible if $|\theta| < 1$. In view of Equation (2), we immediately understand that the polynomial $\phi(L)$ has a root equal to -2, which lies outside the unit circle, and another root equal ϕ^{-1}. Hence, in order to have that y_t is I(1), ϕ must be equal to 1. The resulting ARIMA$(1, 1, 1)$ process follows the equation

$$(1 + 0.5L)\Delta y_t = m + (1 + \theta L)\varepsilon_t \tag{4}$$

where $|\theta| < 1$.

2. In view of p. 33 of the slides and of Equation (4), the moments of Δy_t are obtained as follows

$$\begin{aligned}
\mathrm{E}(\Delta y_t) &= m - 0.5\mathrm{E}(\Delta y_t) = m - 0.5\mu = m/(1 + 0.5) \\
\gamma(0) &= \mathrm{E}[\Delta y_t(\Delta y_t - \mu)] = \mathrm{E}[(m - 0.5\Delta y_{t-1})(\Delta y_t - \mu)] + \mathrm{E}[\varepsilon_t(\Delta y_t - \mu)] \\
&+ \mathrm{E}\{\theta\varepsilon_{t-1}[-0.5(\Delta y_t - \mu) + \varepsilon_t + \theta\varepsilon_{t-1})\} \\
&= -0.5\gamma(1) + \sigma^2(1 + \theta - 0.5 + \theta^2) \\
\gamma(1) &= \mathrm{E}[\Delta y_t(\Delta y_{t-1} - \mu)] = \mathrm{E}[(m - 0.5\Delta y_{t-1})(\Delta y_{t-1} - \mu)] + \mathrm{E}[\varepsilon_t(\Delta y_{t-1} - \mu)] \\
&+ \mathrm{E}\{\theta\varepsilon_{t-1}[-0.5(\Delta y_{t-2} - \mu) + \varepsilon_{t-1} + \theta\varepsilon_{t-2})\} \\
&= -0.5\gamma(0) + \theta\sigma^2 \\
\gamma(k) &= \mathrm{E}[\Delta y_t(\Delta y_{t-k} - \mu)] = -0.5\gamma(k - 1), \quad k > 1
\end{aligned}$$

3. In view of p. 36 of the slides and of Equation (4), we get

$$
\begin{aligned}
\Delta y_t(1) &= m - 0.5\Delta y_t + \theta\varepsilon_t \\
\Delta y_t(h) &= m - 0.5\Delta y_t(h-1) = m[1 - 0.5 + \cdots + (-0.5)^{h-2}] + (-0.5)^{h-1}\Delta y_t(1) \\
&= m[1 - 0.5 + \cdots + (-0.5)^{h-1}] + (-0.5)^{h}\Delta y_t + (-0.5)^{h-1}\theta\varepsilon_t, \quad h > 1
\end{aligned}
$$