Optimization:
Problem Set
Solutions

Annalisa Molino
University of Rome “Tor Vergata”
annalisa.molino@uniroma2.it

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1 Compute the maxima, minima or saddle points of the following functions

a. $f(x, y) = \sqrt{1 + x^2 + y^2}$,

b. $g(x, y) = x^3 + y^3 + xy$.

Solution

a. Since the function square root is strictly increasing on its domain we can study the function $\tilde{f}(x, y) = x^2 + y^2$, where we left out the constant 1. The gradient of $\tilde{f}(x, y)$ is given by

$$\nabla \tilde{f}(x, y) = (2x, 2y).$$

Hence, setting $\nabla \tilde{f}(x, y) = (0, 0)$, we obtain the stationary point $(0, 0)$. In order to find the nature of the stationary point we have to compute the Hessian matrix

$$Hf(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The matrix is positive definite, thus the point $(0, 0)$ is a local minimum for the function $\tilde{f}(x, y)$, so is for $f(x, y)$.
Moreover, $(0, 0)$ is an absolute minimum for $\tilde{f}(x, y)$ because $f(0, 0) = 0$ and $f(x, y) \geq 0$ for every $(x, y) \in \mathbb{R}^2$, so is for $f(x, y)$.

b. The gradient of $g(x, y)$ is given by

$$\nabla g(x, y) = (3x^2 + y, 3y^2 + x).$$

Hence, setting $\nabla g(x, y) = (0, 0)$ we obtain two stationary points $(0, 0)$ and $(-\frac{1}{3}, -\frac{1}{3})$. In order to figure out the nature of the stationary points we have to compute the Hessian matrix

$$Hg(x, y) = \begin{bmatrix} 6x & -1 \\ -1 & 6y \end{bmatrix},$$

which yields

$$Hg(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$Hg(-\tfrac{1}{3}, -\tfrac{1}{3}) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

The matrix is indefinite at $(0, 0)$ and negative definite at $(-\frac{1}{3}, -\frac{1}{3})$. Therefore $(0, 0)$ is a saddle point and $(-\frac{1}{3}, -\frac{1}{3})$ is a local maximum for g.

2 Find the stationary points of the following functions and discuss the behavior of the functions in those points

a. $f(x, y) = x^3 - 3xy^2 + y^4$,

b. $g(x, y) = x^4 + x^2y^2 - 2x^2 + 2y^2 - 8$,

c. $h(x, y) = x^2 + y^3 - xy$.

Solution

a. The gradient of $f(x, y)$ is given by

$$\nabla f(x, y) = (3x^2 - 3y^2, -6xy + 4y^3).$$

Hence, setting $\nabla f(x, y) = (0, 0)$ we obtain the stationary points $(0, 0)$, $(\frac{3}{2}, \frac{3}{2})$ and $(\frac{3}{2}, -\frac{3}{2})$. In order to figure out the nature of the stationary points we have to compute the Hessian matrix

$$Hf(x, y) = \begin{bmatrix} 6x & -6y \\ -6y & -6x + 12y^2 \end{bmatrix},$$

which yields

$$Hf(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$Hf(\tfrac{3}{2}, \tfrac{3}{2}) = \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix},$$

$$Hf(\tfrac{3}{2}, -\tfrac{3}{2}) = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix}.$$

The matrix is positive definite at $(\frac{3}{2}, \frac{3}{2})$ and $(\frac{3}{2}, -\frac{3}{2})$, therefore they are local minima for f. Since $Hf(0, 0) = 0$, by the Hessian criterion nothing can be said about $(0, 0)$. In this case we can study the sign of one of the partial derivative of f, the one that is analytically simpler. Let us study the sign of

$$f_x = 3x^2 - 3y^2.$$

We have that $f_x > 0$ for $x < -y$ and $x > y$, thus $x = -y$ is a curve of maxima and $x = y$ is a curve of minima for y fixed. The function of one variable $\phi_1(y) = f(y, y) = y^3(y - 2)$ has a maximum in 0, while the function of one variable $\phi_2(y) = f(-y, y) = y^3(y + 2)$ has a minimum in 0. It follows that $(0, 0)$ is a saddle point for f. Indeed $f(0, 0) = 0$, $f(y, y) < 0$ and $f(-y, y) > 0$ for all $y \neq 0$ sufficiently small.

b. Leaving out the constant -8, the gradient of $g(x,y)$ is given by

$$\nabla g(x,y) = (4x^3 + 2xy^2 - 4x, 2x^2y + 4y).$$

Hence, setting $\nabla g(x,y) = (0,0)$ we obtain the stationary points $(0,0)$, $(1,0)$ and $(-1,0)$. In order to figure out the nature of the stationary points we have to compute the Hessian matrix

$$Hg(x,y) = \begin{bmatrix} 12x^2 + 2y^2 - 4 & 4xy \\ 4xy & 2x^2 + 4 \end{bmatrix},$$

which yields

$$Hg(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$Hg(1,0) = \begin{bmatrix} 8 & 0 \\ 0 & 6 \end{bmatrix}$$

and

$$Hg(-1,0) = \begin{bmatrix} 8 & 0 \\ 0 & 6 \end{bmatrix}.$$

The matrix is positive definite at $(1,0)$ and $(-1,0)$, therefore they are local minima for g. Since $Hg(0,0) = 0$, by the Hessian criterion nothing can be said about $(0,0)$. We can proceed as in point a. Let us study the sign of

$$f_y = 2x^2y + 4y = 2y(x^2 + 2).$$

$f_y > 0$ for $y > 0$, thus $y = 0$ is a curve of minima for x fixed. The function of one variable $\phi(x) = f(x,0) = x^4 - 2x^2 - 8$ has a local maximum in 0. It follows that $(0,0)$ is a saddle point for f. Indeed $f(0,0) = -8$, $f(x,0) < -8$, $f(0,y) > -8$ for all $x \neq 0$ and $y \neq 0$ sufficiently small.

c. The gradient of $h(x,y)$ is given by

$$\nabla h(x,y) = (2x - y, 3y^2 - x).$$

Hence, setting $\nabla h(x,y) = (0,0)$ we obtain the stationary points $(0,0)$ and $(\frac{1}{12}, \frac{1}{6})$. In order to figure out the nature of the stationary points we have to compute the Hessian matrix

$$Hh(x,y) = \begin{bmatrix} 2 & -1 \\ -1 & 6y \end{bmatrix},$$

which yields

$$Hh(0,0) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

and

$$Hh(\tfrac{1}{12}, \tfrac{1}{6}) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

The matrix is indefinite at $(0,0)$ and positive definite at $(\frac{1}{12}, \frac{1}{6})$. Therefore $(0,0)$ is a saddle point and $(\frac{1}{12}, \frac{1}{6})$ is a local minimum point.

3 Find the minima, maxima, or saddle points of the following functions

a. $f(x_1, x_2, x_3) = x_1 x_2 x_3$ s.t. $x_1^2 + x_2^2 + x_3^2 - 1 = 0$,

b. $f(x_1, x_2) = x_1^2 + 3x_2$ s.t. $\frac{x_1^2}{4} + \frac{x_2^2}{9} - 1 = 0$.

Solution

a. $f(x_1, x_2, x_3) = x_1 x_2 x_3$ s.t. $x_1^2 + x_2^2 + x_3^2 - 1 = 0$.

Let us note first that the constraint

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 - 1 = 0\} \equiv S((0,0,0);1)$$

is a sphere of $\mathbb{R}^3$ with centre $(0,0,0)$ and radius 1. Hence, it is a compact subset of $\mathbb{R}^3$. Therefore, since the function f is continuos on $\mathbb{R}^3$, by the Weierstrass theorem there exist a maximum and a minimum point for f on $S((0,0,0);1)$. The Lagrangian of f is given by

$$\mathcal{L}(x, \lambda) = x_1 x_2 x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1).$$

Thus, the FOCs are

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} & = x_2 x_3 + 2\lambda x_1 = 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} & = x_1 x_3 + 2\lambda x_2 = 0, \\ \frac{\partial \mathcal{L}}{\partial x_3} & = x_1 x_2 + 2\lambda x_3 = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} & = x_1^2 + x_2^2 + x_3^2 - 1 = 0. \end{cases}$$

It follows that

– if $\lambda = 0$ then

$$\begin{cases} x_2x_3 = 0, \\ x_1x_3 = 0, \\ x_1x_2 = 0, \\ x_1^2 + x_2^2 + x_3^2 - 1 = 0, \end{cases}$$

whose possible solutions are $(\pm 1, 0, 0), (0, \pm 1, 0)(0, 0, \pm 1)$.

– If $\lambda \neq 0$ then subtracting the 2^{nd} equation from the 1^{st}

$$x_2x_3 + 2\lambda x_1 - x_1x_3 - 2\lambda x_2 = 0.$$

Therefore

$$\underbrace{(x_1 - x_2)}_{A}\underbrace{(2\lambda - x_3)}_{B} = 0.$$

∗ CASE 1 $\mathbf{A = 0}$ **and** $\mathbf{B \neq 0}$.
Subtracting the 3^{nd} equation from the 2^{st} we obtain as before

$$\underbrace{(x_2 - x_3)}_{C}\underbrace{(2\lambda - x_1)}_{D} = 0.$$

· CASE 1.a $\mathbf{A = 0}$ **and** $\mathbf{C = 0}$.
We have $x_1 = x_2 = x_3$. Substituting into the fourth equation (the constraint) we obtain

$$x_1 = x_2 = x_3 = \frac{1}{\sqrt{3}} \text{ or } x_1 = x_2 = x_3 = -\frac{1}{\sqrt{3}}.$$

· CASE 1.b $\mathbf{A = 0}$ **and** $\mathbf{D = 0}$.
We have $x_1 = x_2 = 2\lambda$. Substituting into the third equation we obtain $4\lambda^2 + 2\lambda x_3 = 0$. Therefore $2\lambda(2\lambda + x_3) = 0$. We do not consider the solution $\lambda = 0$ for our initial assumption $\lambda \neq 0$. The only part that could be equal to zero is $2\lambda + x_3 = 0$, i.e. $x_3 = -2\lambda$. Substituting into the constraint $x_1 = x_2 = x_3 = 2\lambda$ we obtain $\lambda = \pm\frac{1}{\sqrt{12}}$. Substituting this value of λ in order to obtain the values of x, we obtain the following solutions

$$x_1 = \frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}, x_3 = -\frac{1}{\sqrt{3}}$$

or

$$x_1 = -\frac{1}{\sqrt{3}}, x_2 = -\frac{1}{\sqrt{3}}, x_3 = \frac{1}{\sqrt{3}}.$$

* CASE 2 $\mathbf{A \neq 0}$ **and** $\mathbf{B = 0}$.
In this case we have $x_3 = 2\lambda$. Substitute this result into the second equation we obtain $2\lambda(x_1 + x_2) = 0$. The possible solutions are $\lambda = 0$ (which we do not consider because we have impose $\lambda \neq 0$) or $x_1 = -x_2$.
Now substituting $x_1 = -x_2$ into the third equation, we obtain $-x_1^2 + 4\lambda^2 = 0$, i.e. $x_1 = \pm 2\lambda$. We have the possible combinations $x_1 = 2\lambda, x_2 = -2\lambda, x_3 = 2\lambda$ or $x_1 = -2\lambda, x_2 = 2\lambda, x_3 = 2\lambda$. As before, substituting this result into the constraint, we obtain $\lambda = \pm\frac{1}{\sqrt{12}}$.
This gives us the four possible solutions:
if $\lambda = +\frac{1}{\sqrt{12}}$

$$x_1 = \frac{1}{\sqrt{3}}, x_2 = -\frac{1}{\sqrt{3}}, x_3 = \frac{1}{\sqrt{3}}$$

or

$$x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}, x_3 = \frac{1}{\sqrt{3}},$$

if $\lambda = -\frac{1}{\sqrt{12}}$

$$x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}, x_3 = -\frac{1}{\sqrt{3}}$$

or

$$x_1 = \frac{1}{\sqrt{3}}, x_2 = -\frac{1}{\sqrt{3}}, x_3 = -\frac{1}{\sqrt{3}}.$$

All the possible solutions are

$$(\pm 1, 0, 0); (0, \pm 1, 0); (0, 0, \pm 1);$$

$$(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}); (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}); (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}); (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}});$$

$$(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}); (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}); (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}); (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}).$$

Computing the values of the function f in all the stationary points we have

$$f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) =$$

$$= f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{1}{3\sqrt{3}} \rightarrow \text{global maxima}$$

$$f(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = f(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = f(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) =$$

$$= f(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = -\frac{1}{3\sqrt{3}} \to \text{global minima}$$
$$f(\pm 1, 0, 0) = f(0, \pm 1, 0) = f(0, 0, \pm 1) = 0.$$

b. $f(x_1, x_2) = x_1^2 + 3x_2$ s.t. $\frac{x_1^2}{4} + \frac{x_2^2}{9} - 1 = 0$.
Let us note first that the constraint

$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \frac{x_1^2}{4} + \frac{x_2^2}{9} - 1 = 0\right\} \equiv E((0,0,0); 2, 3)$$

is an ellipsis of $\mathbb{R}^3$ with centre $(0,0,0)$ and semi-axes 2 and 3. Hence, $E((0,0,0); 2, 3)$ is a compact subset of $\mathbb{R}^3$. Therefore, since the function f is continuos on $\mathbb{R}^3$, by the Weierstrass theorem there exist a maximum and a minimum point for f on $E((0,0,0); 2, 3)$. The Lagrangian of f is given by

$$\mathcal{L}(\lambda, x) = x_1^2 + 3x_2 + \lambda \left(\frac{x_1^2}{4} + \frac{x_2^2}{9} - 1 \right).$$

The FOCs are

$$\begin{cases} \frac{\partial L}{\partial x_1} & = 2x_1 + \frac{1}{2}\lambda x_1 = 0, \\ \frac{\partial L}{\partial x_2} & = 3 + \frac{2}{9}\lambda x_2 = 0, \\ \frac{\partial L}{\partial \lambda} & = \frac{x_1^2}{4} + \frac{x_2^2}{9} - 1. \end{cases}$$

It follows that

– if $x_1 \neq 0$, then $\lambda = -4$. Substituting $\lambda = -4$ into the second condition we have $x_2 = \frac{3^3}{2^3}$. Thus, the last constraint gives

$$\frac{x_1^2}{2^2} + \frac{3^6}{2^6 3^2} - 1 = 0$$

that is

$$x_1^2 = \frac{2^6 - 3^4}{2^4} = -\frac{17}{16}$$

which yield a non real root.

– if $x_1 = 0$, then from the constraint we obtain

$$x_2 = \pm 3.$$

Since $f(0, -3) = -9$ and $f(0, 3) = 9$, the point $(0, -3)$ is a global minimum and the point $(0, 3)$ is a global maximum for f on $E((0,0,0); 2, 3)$.

4 Find the maxima points of the following functions

a. $f(x,y) = xy$ s.t. $x^2 + y^2 = 1$,

b. $f(x,y) = x^\alpha y^{1-\alpha}$ s.t. $w = p_x x + p_y y$.

Solution

a. $f(x,y) = xy$ s.t. $x^2 + y^2 = 1$.

Since (x,y) satisfy the relationship $x^2 + y^2 - 1 = 0$, on this curve the function f can be rewritten as

$$f(x,y) = xy = xy + \frac{1}{2}(x^2 + y^2 - 1) = \frac{1}{2}(x^2 + y^2 + 2xy) - \frac{1}{2} = \frac{1}{2}(x+y)^2 - \frac{1}{2}.$$

In this way it is easy to check that the minimum is attained on the line $x = -y$, that intersects the curve in the points $(\sqrt{2}/2, -\sqrt{2}/2)$ and $(-\sqrt{2}/2, \sqrt{2}/2)$, and therefore these are the minima for f. Similarly,

$$f(x,y) = xy = xy - \frac{1}{2}(x^2 + y^2 - 1) = -\frac{1}{2}(x^2 + y^2 - 1) = -\frac{1}{2}(x-y)^2 + \frac{1}{2}.$$

In this way it is easy to check that he maximum is attained on the line $x = y$, that intersects the curve in the points $(\sqrt{2}/2, \sqrt{2}/2)$ and $(-\sqrt{2}/2, -\sqrt{2}/2)$, and therefore these are the maxima for f.

b. $f(x,y) = x^\alpha y^{1-\alpha}$ s.t. $w = p_x x + p_y y$.

$$\mathcal{L}(x,y,\lambda) = x^\alpha y^{1-\alpha} + \lambda(p_x x + p_y y - w),$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = \alpha x^{\alpha-1} y^{1-\alpha} + \lambda p_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = (1-\alpha) x^\alpha y^{-\alpha} + \lambda p_y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = p_x x + p_y y - w = 0, \end{cases}$$

$$\begin{cases} \alpha x^{\alpha-1} y^{1-\alpha} = -\lambda p_x \\ (1-\alpha) x^\alpha y^{-\alpha} = -\lambda p_y \\ w = p_x x + p_y y. \end{cases}$$

It follows that

$$\frac{\alpha x^{\alpha-1} y^{1-\alpha}}{(1-\alpha) x^\alpha y^{-\alpha}} = \frac{p_x}{p_y},$$

that is,

$$\frac{\alpha}{1-\alpha} \frac{y}{x} = \frac{p_x}{p_y}.$$

Substitute $y = x\frac{p_x}{p_y}\frac{1-\alpha}{\alpha}$ into the constraint to get

$$w = \frac{xp_x}{\alpha}.$$

Therefore

$$x(p_x, w) = \frac{w\alpha}{p_x}$$

$$y(p_y, w) = \frac{w(1-\alpha)}{p_y}$$

is a maximum point for f.

5 Compute the maxima , minima or saddle points of the following function

$$f(x, y) = x^2 + 3y$$

$$\text{s.t } \frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Solution

The constraint is an ellipsis, thus is a compact set. Since the function f is continuous by the Weierstrass theorem it has a maximum and a minimum. Parametrize the ellipsis as

$$\begin{cases} x = 2\cos\theta \\ y = 3\sin\theta \end{cases}$$

with $\theta \in [0, 2\pi)$. It follows that

$$f(\theta) = 4\cos^2\theta + 9\sin\theta = 4(1 - \sin^2\theta) + 9\sin\theta = -4\sin^2\theta + 9\sin\theta + 4,$$

$$f'(\theta) = -8\sin\theta\cos\theta + 9\cos\theta = \cos\theta(9 - 8\sin\theta).$$

Impose $f'(\theta) = 0$. We get

- $\cos\theta = 0$, therefore $\theta = \frac{\pi}{2}$, $\theta = \frac{3}{2}\pi$,
- $9 - 8\sin\theta = 0$, that is $\sin\theta = \frac{9}{8} > 1$, therefore no solutions.

Furthermore, since $9 - 8\sin\theta > 0$ forall θ, we have

$$f'(\theta) > 0 \qquad \text{iff} \qquad \cos\theta > 0.$$

Therefore $\theta = \frac{\pi}{2}$ is a global maximum, and $\theta = \frac{3}{2}\pi$ is a global minimum point for f. The global maximum correspond to the point $(0, 3)$, and the global minimum to the point $(0, -3)$.

6 Find the (local) maxima and minima of the function

$$f(x, y) = xy - y^2 + 3$$

subject to the constraint

$$g(x, y) = x + y^2 - 1 = 0$$

using:

1. a parametric representation of the constraint;
2. Lagrange multipliers.

Are they global?

Solution

First of all notice that, since the constraint set is not compact, we cannot use the Weierstrass theorem to claim that the function f assumes global maximum and minimum.

1. A parametrization of the parabola is

$$\begin{cases} x = 1 - t^2 \\ y = t \end{cases}$$

for $-\infty < t < +\infty$. It follows that

$$f(t) = (1 - t^2)t - t^2 + 3 = -t^3 - t^2 + t + 3$$

is a function of one variable on the domain $(-\infty, +\infty)$. Therefore solving

$$f'(t) = -3t^2 - 2t + 1 = 0$$

we get the stationary points $t_1 = -1$ and $t_2 = \frac{1}{3}$. Since $f'(t) > 0$ for $-1 < t < \frac{1}{3}$, $t_1 = -1$ is a relative minimum and $t_2 = \frac{1}{3}$ is a relative maximum. Notice that this maximum and minimum are not global, indeed

$$\lim_{t \to +\infty} f(t) = -\infty$$

and

$$\lim_{t \to -\infty} f(t) = +\infty.$$

In correspondence we have that $(0, -1)$ is a relative minimum and $\left(\frac{8}{9}, \frac{1}{3}\right)$ is a relative maximum for f.

2. The Lagrangian for the problem is

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y) = xy - y^2 + 3 + \lambda(x + y^2 - 1).$$

Solving the first order conditions

$$\begin{cases} y + \lambda = 0 \\ x - 2y + 2\lambda y = 0 \\ x + y^2 - 1 = 0 \end{cases}$$

we get the two critical points $(0, -1, 1)$ and $\left(\frac{8}{9}, \frac{1}{3}, -\frac{1}{3}\right)$. In order to find the nature of the stationary points we have to study the bordered Hessian matrix

$$Hf(x, y, \lambda) = \begin{bmatrix} 0 & 1 & 2y \\ 1 & 0 & 1 \\ 2y & 1 & -2 + 2\lambda \end{bmatrix}$$

in such points.

$$Hf(0, -1, 1) = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}.$$

Since $\det(Hf(0, -1, 1)) < 0$, $(0, -1)$ is a relative minimum for f.

$$Hf\left(\frac{8}{9}, \frac{1}{3}, -\frac{1}{3}\right) = \begin{bmatrix} 0 & 1 & \frac{2}{3} \\ 1 & 0 & 1 \\ \frac{2}{3} & 1 & -\frac{8}{3} \end{bmatrix}.$$

Since $\det(Hf\left(\frac{8}{9}, \frac{1}{3}, -\frac{1}{3}\right)) > 0$, $(\frac{8}{9}, \frac{1}{3})$ is a relative maximum for f. These maximum and minimum are not global, because for example we have

$$\lim_{x \to +\infty} f(x, y) = +\infty$$

and

$$\lim_{x \to -\infty} f(x, y) = -\infty.$$