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Calculus

Problem Set 3

$\hookrightarrow$ **Topics**

Tangent planes, differentiability, gradient, level curves, directional derivatives. Eigenvalues and eigenvectors.

Exercise 1

Find the tangent plane of the following function:

- $f(x, y) = x^3 - y^3$ in $P(0, 1, -1)$.
- $f(x, y) = x^y + y^x$ in $P(1, 1, 2)$.

Solution

$f(x, y)$ is differentiable in $(x_0, y_0) \Leftrightarrow \exists$ a tangent plane in $(x_0, y_0, f(x_0, y_0)$ to $f(x, y)$ tangent plane equation:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Consider the function $f(x, y) = x^3 - y^3$ in $P(0, 1, -1)$.

$z = f(0, 1)+f_x(0, 1)(x-0)+f_y(0, 1)(y-1) = -1+3(0)^2x-3(1)^2(y-1) = -1-3y+1 = 2-3y$

Consider now the function $f(x, y) = x^y + y^x$ in $P(1, 1, 2)$.

$z = f(1, 1)+f_x(1, 1)(x-1)+f_y(1, 1)(y-1) = 2+1(x-1)+1(y-1) = 2+x-1-y-1 = x+y$

Exercise 2

Consider the following function

$$f(x, y) = \begin{cases} x & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases} .$$

Compute the directional derivatives in the origin along an arbitrary direction v. Show that the equality

$$\langle \nabla f(0, 0), v \rangle = \frac{\partial f}{\partial v}(0, 0) \tag{1}$$

does not hold in general.

What can you conclude about the differentiability of the function f?

Solution

For any direction $v \neq (1,0), (-1,0)$ the partial derivative is

$$\frac{\partial f}{\partial v}(0,0) = 0$$

In particular

$$\frac{\partial f}{\partial y}(0,0) = 0$$

On the other hand

$$\frac{\partial f}{\partial x}(0,0) = 1$$

so that

$$\nabla f(0,0) = (1,0)$$

Therefore, for any direction $v = (v_1, v_2) \neq (1,0), (-1,0), (0,1), (0,-1)$ one has

$$\langle \nabla f(0,0), v \rangle = \langle (1,0), (v_1, v_2) \rangle = v_1 \neq 0 = \frac{\partial f}{\partial v}(0,0)$$

The function cannot be differentiable otherwise equation 1 would be true for any direction v.

Exercise 3

Check the continuity and the differentiability at $(0,0)$ of $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \sqrt{x^2 + y^2}.$$

Continuity

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2}$$

In polar coordinates:

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} &= \lim_{\rho\to 0^+} \sqrt{\rho^2(\cos^2\theta + \sin^2\theta)} \\ &= \lim_{\rho\to 0^+} |\rho| \\ &= 0 \end{aligned}$$

Therefore, the function f is continuos in $(0,0)$

Differentiability

Consider the function along the x axis, $y = 0$
$f(x,0) = \sqrt{x^2} = |x|$ the absolute value function is continuous in $(0,0)$ but not differentiable.

It is the same if you consider the function along the y axis, $x = 0$
$f(0,y) = \sqrt{y^2} = |y|$
Therefore, the function is continuous but not differentiable in the point.

Exercise 4

Check that the
gradient is orthogonal to the correspondent level curve when $f(x,y)$ is:

$$f(x,y) = y - x$$

$L_c = \{(x,y) \in \mathbb{R}^{\nvDash} \mid y - x = c\}$ This is the set of straight line with slope 1
Gradient
$\triangledown f(x,y) = (-1,1), \forall (x,y)$ Therefore, the gradient is orthogonal to the correspondent level curve.

Exercise 5

Check that the
gradient is orthogonal to the level curve $L_c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 1\}$ at the points $P(0,1)$ and $P'(0,-1)$ when $f(x,y)$ is:

$$f(x,y) = y^2 - x^2$$

$L_c = \{(x,y) \in \mathbb{R}^{\nvDash} \mid y - x = c\}$
consider c=1, the hiperbola you get intersect the y axis at P and P'

The gradient is $\triangledown f(x,y) = (-2x, 2y)$

At P it is $\triangledown f(x,y) = (0,2)$

At P' it is $\triangledown f(x,y) = (0,-2)$

Exercise 6

Consider the function $f(x,y) = x^2 + 2xy + 2y^2$ in the neighborhood of the point $(2,1)$. Determine a direction v in which the directional derivative $\frac{\partial f}{\partial v}$ at the point $(2,1)$ is null.

Solution

$$\nabla f(x,y) = (2(x+y), 2(x+2y)) \qquad \Rightarrow \qquad \nabla f(2,1) = (6,8)$$

Therefore for a direction $v = (v_1, v_2)$ one has

$$0 = \frac{\partial f}{\partial v}(2,1) = \langle \nabla f(2,1), v\rangle = \langle (6,8), (v_1,v_2)\rangle = 6v_1 + 8v_2$$

so that $v_2 = -\frac{3}{4}v_1$ and therefore

$$(v_1, v_2) = \left(\pm\frac{4}{5}, \mp\frac{3}{5}\right)$$

Exercise 7

Compute the eigenvalues of the following matrices:

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}; B = \begin{pmatrix} 3 & 6 \\ 9 & 18 \end{pmatrix}; C = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}.$$

Solution

We are looking for a λ such that $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0.$$

$$(5-\lambda)(2-\lambda) - 4 = 0; \lambda^2 - 7\lambda + 6 = 0;$$

Therefore, $\lambda_1 = 6$ and $\lambda_2 = 1$.

Notice that this matrix is singular, we expect that one λ is equal to zero

$$\det(B - \lambda I) = \begin{vmatrix} 3-\lambda & 6 \\ 9 & 18-\lambda \end{vmatrix} = 0.$$

$$(3-\lambda)(18-\lambda)-54=0; \lambda^2-21\lambda=0;$$

Therefore, $\lambda_1=21$ and $\lambda_2=0$.

$$\det(C-\lambda I)=\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 5-\lambda & 0 \\ 2 & 0 & 7-\lambda \end{vmatrix}=0.$$

$(6-\lambda)(5-\lambda)(7-\lambda)-2(5-\lambda)2-(-2)(7-\lambda)(-2)=0; \lambda^3+18\lambda^2-99\lambda+162=0$

$(\lambda-3)(\lambda-6)(-\lambda+9)=0$

Therefore, $\lambda_1=9$, $\lambda_2=6$ and $\lambda_3=3$.