Today we will discuss economic growth. This was a topic of interest to economic theorists in general in the 60s then for a while it was left to development economists. Since roughly 1986 it has become a hot topic infested with math again. In the 60s economic theorists, in particular Robert Solow, concluded that long run economic growth was largely caused by technological progress. This was assumed to be exogenous and not economists' business. Economic factors such as incentives to save and invest or to invest in human capital (learn and teach) were alleged to affect the level of output but not its (deterministic) trend. For this discovery Solow was awarded the Nobel prize. In his Nobel lecture he said he didn't believe it anymore, since he didn't think one could study even long run trends assuming e.g. full employment. Anyway, I will assume full employment and lots of other crazy things. On the other hand I promise not to put any of this on the exam.

Assume that output is a function of capital K, employment L, and human capital or labour augmenting (Harrod Neutral) technology A. AL is effective labour equal to A times L.

1) $Y = F(K,AL)$

F is an aggregate production function with constant returns to scale and decreasing marginal product of capital and labour. That is

2) $F(aK,aAL) = aF(K,AL)$ and

$F_{KK}(K,AL) < 0$ and $F_{LL}(K,AL) < 0$.
Where $F_K(K,AL)$ is the partial derivative of $F(K,L)$ with respect to K and $F_{KK}(K,AL)$ is the second partial derivative of F with respect to K.

These are in fact heroic assumptions since we have assumed that capital and labour can be measured by a single number. In particular we assume that people with more human capital are better at everything than people with less, so that we can measure labour adjusted for human capital L in "efficiency units". We assume that each persons wage is proportional to his or her productivity in all jobs which depends only on his or her quantity of human capital (which can be described by one number) so the returns to schooling are fixed. We also assume that capital goods can be converted to and from consumption goods at will. That is, that one can eat factories. We, further, assume that capital depreciates at rate δ so

3) $K_t^{dot} = -\delta K_t + Investment_t = -\delta K_t + Y_t - Consumption_t$.

Where superscript dot indicates the time derivative. Finally we assume perfect competition so labour and capital are paid their marginal products. Thus $W_t = F_L(K_t,A_tL_t)$ and $r_t = F_K(K_t,L_t)$. Perfect competition and constant returns to scale imply that

4) $W_tL_t = Y_t - K_tr_t$

I will be very casual about consumption savings choices. In fact will be very casual about all individual choices and generally consider the social planners problem assuming an omniscient omnipotent benevolent social planner (e.g. God). This means that I don't have to worry much about A. I assume that $L^{dot}_t = nL_t$ where n is the rate of growth of employment and $A^{dot}_t = gA_t$ where g is the rate of labour augmenting technological progress.

Now define $k_t = K_t/(A_tL_t)$, $y_t = Y_t/(A_tL_t)$, $c_t = C_t/(A_tL_t)$ and $w_t = W_t/A_t$ Given the assumption of constant returns to scale $F(K_t,L_t) = L_tF(k_t,1)$. Define $f(k) = F(K_t,L_t)/L_t = F(k_t,1)$. This means that $y_t = f(k_t)$. Note that

5) $r_t = F_K(K_t,A_tL_t) = A_tL_tF_K(k_t,1) = A_tL_t df(K_t/L_t)/dK_t = f'(k_t)$

and

6) $w_t = f(k_t) - k_tf'(k_t)$.

with all variables divided by AL equation 3 becomes

7) $k_t^{dot} = f(k_t) - c_t - (\delta + n + g)k_t$

This describes the options open to society. To analyze them it is useful to consider the choices open to a social planner who can dictate consumption. To simplify still more compare steady states assuming that the capital labour ratio and the consumption labour ratios are fixed.

8) $c = f(k) - (\delta + n + g)k$.

Differentiating gives the first order condition for the optimal steady state capital labour ratio - the golden rule

9) $0 = f'(k^*) - (\delta + n + g)$

here the point is that there is a capital labour ratio so high that it would be crazy to maintain a higher one. Capital is always nice, but the cost of replacing depreciated capital and making new factories for new workers and finally of making the new tools that more highly skilled workers need can be greater than the marginal product of the extra capital. While savings and investment can bring k up to k^* it can't increase C/(AL) indefinitely. Eventually output and consumption per capita Y/L and C/L grow at rate g and the growth is entirely due to increased A (technology) not increased k. If A is exogenous that is all there is to it, long run growth is exogenous and can't be affected by even the social planners choice of k_t.

One simple illustration of the temporary effects of saving and investment on the rate of growth is obtained by assuming that a fraction s of Y is saved. Then equation 10 describes the growth of k

10) $k^{dot} = sf(k) - (\delta + n + g)k$

IF f(k)/k goes to zero as k goes to infinity, this implies that k reaches a steady state value. Once k has reached this value Y grows at rate n + g. k can't grow forever so an increased s can't increase the long run asymptotic growth rate of Y.

Assume instead that there are a capitalist class which owns the capital and a working class which receives wages. Assume that capitalists save all of their income and workers save none of their income. Then $k^{\cdot}$ is described by equation 11

11) $k^{dot} = kf'(k) - (\delta + n + g)k$

if f'(k) is decreasing in k (if the production function is concave) then the economy will reach the steady state in which

9) $f'(k) = \delta + n + g$

why that's the golden rule steady state which maximizes consumption per unit of (effective) labour. Also note that all the workers are the only people that consume. This means that in the long run the capitalist system will achieve the maximum consumption per worker consistent with the exogenous progress of technology provided that capitalists save all of their income and workers save none and remain propertyless (proletarian). The existence of this steady state (without even exogenous technological progress) was noted by Marx who neglected to mention that it achieved the maximum possible level of consumption for workers (Capital chapter 25 part I, parts 2-4 present a model of embodied technological progress with irrational firms wildly at variance with more recent models and the experience of workers following publication of the book, but nonetheless possibly an accurate description of England in 1867 or underdeveloped countries today).

The choice of optimal steady states is closely related to the choice of optimal savings for a social planner who maximizes the present discounted value of a function of c say that given by

$$12)\quad \max V = \int_t^{\infty} e^{-(s-t)\rho'} U(C_s/(A_s N_s)\ ds$$

This is a simple problem in optimal control or dynamic programing and is, in fact, the example used by Intriligator (not on reading list). It gives a steady state capital labour ratio described by

$$13)\quad f'(k) = \delta + n + g + \rho'$$

which implies that if the planner is impatient she obtains a lower steady state consumption level in exchange for more consumption now. This is also what would result if many individual consumers decided how much to consume provided A is exogenous. This might be interpreted as implying that the results about optimal steady states described above have implications for what actually happens.