LECTURE 1 : THE INFINITE HORIZON REPRESENTATIVE AGENT MODEL

In this lecture I will discuss how a rational utility maximizer would choose how much to consume, draw testable implications and discuss statistical tests of the hypothesis that aggregate consumption behaves as it would if there were a single rational representative agent who lived forever. The lecture summarizes the work reported by Hall in (Hall 198?) on the reading list.

First consider a consumer who lives for two periods. For simplicity I will first assume that he knows that his income will be w_1 in the first period and w_2 in the second. The consumer can save or borrow at the interest rate r. For simplicity there is only one kind of consumption good.
For a two period lifetime the budget constraint

1) $c_1 + c_2/(1+r) <= w_1 + w_2/(1+r)$

where c_i is consumption in period i and w_i is income in period i. Since the consumer wants to consume, 1 holds with equality. The right side of equation 1 is called permanent income.

The consumer chooses c_1 to maximize the intertemporal utility function

2) $\underset{c_1,c_2}{\text{maximize}}\ U = u(c_1) + u(c_2)/(1+d)$ subject to 1

d is the subjective rate of discounting of future happiness. This very simple problem can be solved in many ways. The simplest is to use equation 1 to solve for c_2 as a function of c_1

and plug the result into equation 2.

3) $c_2 = w_1(1+r) + w_2 - c_1(1+r)$

so

4) $U = u(c_1) + [u(w1(1+r) + w_2 - c_1(1+r))]/(1+d)$

Equation 4 gives the first order condition equation 5 which is a discrete time Euler equation

5) $0 = u'(c_1) - u'(c_2)(1+r)/(1+d)$

Equation 5 states that $u'(c_1) = u'(c_2)$, it is conventional to write $u'(c_1) = u'(c_2)/(1+r)$ = lambda, and call lambda a Lagrange multiplier. Lagrange generalized the trivial derivation above, but basically he did the same thing. maximizing 2 subject to the constraint 1 is equivalent to unconstrained maximization of equation 6 with respect to c_1 and c_2

6) $\max_{c_1,c_2} u(c_1) + u(c_2)/(1+d) - \lambda[c_1 + c_2/(1+r) -w_1 - w_2/(1+r)]$

It is fairly easy to generalize the two period case to the infinite horizon case. Now assume that the consumer lives forever, consumes c_i in period i and earns w_i in period i. The consumer chooses c_i in each period to maximize the intertemporal utility function

$$7)\quad U = \sum_{t=1}^{\infty} u(c_t)/(1+d)^t$$

subject to the budget constraint

$$8)\quad \sum_{t=1}^{\infty} (c_t - w_t)/(1+r)^t <= S_1$$

Where S_1 is wealth at the beginning of period 1.

By noting the analogy with equation 6 and appealing to Lagrange you might convince yourself that maximizing 7 subject to 8 is equivalent to unconstrained maximization of equation 9

$$9)\quad \max \sum_{t=1}^{\infty} [\ u(c_t)/(1+d)^t] - \lambda\{\sum_{i=1}^{\infty} [(c_t-w_t)/(1+r)^t\}$$

So long as w_t is known for all t, λ is constant. 9 gives an infinite number of first order conditions, one for
each time period all described by equation 10 for each i

$$10)\quad u'(c_t) = \lambda\ (1+d)^t/(1+r)^t$$

Since λ is unknown equation 10 is not very useful by itself but λ can be eliminated by comparing equation 10 for t and for t+1 giving equation 11 for each t

$$11)\quad u'(c_{t+1}) = u'(c_t)[(1+d)/(1+r)]$$

The important point is that if d and r are constant, the derivative of the utility function is multiplied by the same amount each period. Since I have assumed that w is known with certainty each period I make a clearly false prediction. If I Assume that u'(c) is a decreasing function of c, I predict that consumption either decreases always or increases always. One way to explain up and down fluctuations in consumption in the intertemporal utility maximizing framework is that consumers do not know their future income and adjust their consumption as they learn about it.

To adjust to an uncertain world ì assume that consumers maximize the expected value of the intertemporal utility function and know the probability of having any income in the future, that is by assume rational expectations As consumers learn more about their income stream, they adjust their consumption, so future consumption is not known exactly. Also consumption can decrease and increase as it does.

Assume that at time t consumers don't know w_{t+1}, w_{t+2} etcetera but do know their expected value this means that at each time t the consumer maximizes the expected value of future consumption taking expectations conditional on all information available at time t

$$12)\quad \underset{C_d, s>=t}{\text{MAXIMIZE}}\ E_t\{\sum_{i=t}^{\infty} [u(c_s)/(1+d)^{s-t}]\}$$

subject to the constraint

$$13)\quad \sum_{s=t}^{\infty} (c_s-w_i)/(1+r)^{s-t} \leq S_t$$

Where S_t is financial wealth at time t (which can be negative).

The analysis above can be repeated with expected values and gives equation 14

If c_t is optimal and the plan which gives c_s s>t as a function of new information is optimal, then there is no other c'_s and new plan c'_s i > t which improves expected utility. In particular there is no change in which $c'_s = c_s$ for all s > t+1 which increases expected utility. Given the budget constraint this implies that there is no δ such that expected utility is increased if $c'_t = c_t + \delta$ and $c'_{t+1} = c_{t+1} - (1+r)\delta$. Note c_{t+1} is not known at t it depends on things the consumer learns after t, but the rational consumer plans c_{t+1} as a function of this new information and can imagine changing the plan by consuming $(1+r)\delta$ less in every case. When considering modified consumption plans of this type it is clear that the FOC is that the derivative of expected utility with respect to δ is zero at δ = 0 or

14) $E_t[(1+r)u'(c_{t+1})/u'(c_t)] = 1+d$

Assuming constant r is silly. If if r_{t+1} is interest paid in period t+1 then the budget constraint becomes

13.1) $$\sum_{s=t}^{\infty} (c_s-w_i) \prod_{s=t}^{\infty} 1/(1+ r_{s+1}) \leq S_t$$

14 becomes 14.1

14) $E_t[(1+ r_{t+1})u'(c_{t+1})/u'(c_t)] = 1+d$

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most economists eager to test equations 14 and 14.1s have assumed constant elasticity of substitution utility functions of the form

$$u(c) = [c^{(1-\theta)}]/(1-\theta)$$

which implies that $u'(c) = c^{-\theta}$ and turns equation 14.1 into the highly useful equation 15

15) $E_t(1+ r_{t+1})(c_{t+1}/c_t)^{-\theta} = (1+d)$

Which making a bold approximation for logarithms implies

16) $E_t(\log(c_{t+1})) - \log(c_t) = (- d)/\theta$

The second lecture consumption will have a lot to do with the fact that this approximation isn't exact.

16 can be written in the usable form

17) $\log(c_{t+1}) = \log(c_t) + (E_t(r_{t+1}) - d)/\theta + \varepsilon_{t+1}$

where ε_{t+1} is a disturbance term which is according to the rational expectations hypothesis uncorrelated with any lagged information.

Equation 17 is testable. If r is a known constant, it would imply that if log consumption is regressed on lagged log consumption and on lagged variables, the coefficient on log consumption will be one and the other coefficients will be zero. If r is stochastic, it means lagged variables are useful in forecasting the change in log consumption only to the extent they are useful in estimating the expected real interest rate. This means that if the change in log consumption is regressed on r_{t+1} and any lagged variable (say c_{t-1} or GDP_{t-s}) and if 2SLS is used with lagged variables used to instrument r_{t+1} and the lagged variable, then the coefficient on r_{t+1} should be $1/\theta$ and the coefficient on the lagged variable should be zero.

Hall's contribution was to point out that it is not necessary to specify a consumption function in order to derive equation 17.

Put briefly most of the tests have rejected the restrictions implied by equation 17. lagged information often helps predict changes in consumption. This implies that one of the many assumptions made in deriving equation 17 must be false.

Many assumptions have to be made in order to derive equation 17.

1) it is assumed that consumers are expected intertemporal utility maximisers.
2) it is assumed that they have rational expectations.
3) it is assumed that consumers are free to borrow and to borrow at the same interest rate they earn when they save. That is it is assumed that they are not liquidity

constrained. These assumptions are critical and rejection of the restrictions imposed by equation 17 might imply that any (or all) of them are false.

4) A form of the utility function is assumed. It is easy to check that results do not depend on the particular form of the one period utility function u by using different assumptions.

5) More importantly it was assumed that the intertemporal utility function is time separable, that is that it is the discounted sum of functions of consumption at each time. This implies that consumption now does not affect the marginal utility of consumption in the future. This is clearly.

6) It is assumed that utility is separable in consumption and everything else, a sum of pleasure from consuming and happiness given other things. If the pleasure from consumption and the pain from work do not have additively separable effects on welfare, then equation 17 will not hold.

7) Equation 17 was derived for a single consumer. It is usually tested with aggregate data. An additional assumption is made -- that aggregate consumption behaves as if there were a single representative consumer. Nonetheless the model has also been tested and rejected with data on individual consumption.

In this lecture I began assuming that the real interest rate r is known and constant. This assumption was relaxed by testing whether lagged information helps predict changes in consumption only because it helps predict real interest rates. In other words, the real interest rate is instrumented using information available to the consumer when the consumption saving decision was made. The hypothesis is rejected by the data.

This leaves the following possible explanations for the rejection of the predictions of the life-cycle permanent income hypothesis -- that consumers are not utility maximisers, that they do not have rational expectations, that the utility function is not additively separable and that they are not free to borrow.

Appendix 1: The best part again. This is another effort to tell the story needed to get to equation 14.

Recall we had gotten to the most interesting part, maximizing utility over an infinite horizon. I had just described the budget constraints equations 14 and 15 then I jumped to the first order condition and result (also called 14 sorry again). This can be derived in a manner strictly analogous to the finite horizon case.
consider an alleged solution c^*_1, c^*_2, c^{*3} &c This consumption path implies wealth at each period S^*_1, S^*_2, S^{*3} &c. If this is optimal it is impossible to find an improvement and in particular it is impossible to find an improvement with the additional restriction that S_t is equal to S^*_t for every t not equal to i+1. This means that the consumer chooses c_i to maximise 13

$$13)\ U= \sum_{j=i}^{\infty} u(c_j)/d^{j-i}$$

given the constraint that $S_{i+2} = S^*_{i+2}$ et cetera

This gives

new 1) $c_{i+1} = (1+r)(S^*_i + w_i - c_i) + w_{i+1} - S^*_{i+2}/(1+r)$
and $c_j = c^*_j$ for j greater than i+2.

this gives

new 2) $dU/dc_i = u'(c_i) - u'(c_{i+1})(1+r)/(1+d)$

If the alleged solution is really an optimum this must equal zero as asserted. The problem of finding optimal consumption for each period can be quite tedious. The problem of checking that the stated first order condition holds is as you have seen trivial. The only trick (and this is very common) is the trick of arguing that if their is no improvement which satisfies the original budget constraint, then there must be no improvement that satisfies it and additional restrictions.

In this additional bit I have solved the problem under uncertainty. The exact same technique for looking for an improvement works under uncertainty. The only difference is that If I specify S^{*i+2} I must imagine specifying it as a function of new information such as w_{i+1}.

I could also use the restriction

new 3) $c_{i+1} = c^*_{i+1} - (1+r)(c_i - c^{*i})$

this leaves wealth and consumption the same for all periods i+2 and after so the first order condition new 2 holds under certainty and the first order condition 14

14) ${}_tE[u'(c_{i+1}/c_i) = (1+d)/(1+r)$

holds even if wages are uncertain.

Appendix II An implicit assumption which I used above

Recall equation 13 read

$$13)\quad \sum_{s=t}^{\infty} (c_s-w_s)/(1+r)^{s-t} \le S_t$$

Where S_t is financial wealth at time t and can be negative.

Between equation 13 and equation 14 there should be an argument as follows: If c_t is optimal and the plan which gives c_s s>t as a function of new information is optimal then there is no other c'_t and new plan c'_s s > t which improves expected utility. In particular there is no change in which $c'_s = c_s$ for all s > t+1 which increases expected utility. Given the budget constraint this is equivalent to saying there is no δ such that expected utility is increased if $c'_t = c_t + \delta$ and $c'_{t+1} = c_{t+1} - (1+r)\delta$. Note c_{t+1} is not known at t it depends on things the consumer learns after t, but the rational consumer plans c_{t+1} as a function of this new information and can imagine changing the plan by consuming (1+r)δ less in every case. When considering modified consumption plans of this type it is clear that the FOC is that the derivative of expected utility with respect to δ is zero at δ = 0 or

$$14)\ E_i[u'(c_{t+1})/u'(c_t)] = (1+d)/(1+r)$$

Now the main point. I assume that the budget constraint must be satisfied with certainty. Disappointingly low w must be balanced by low c. This may not always be possible if the required c is negative. Then the consumer goes bankrupt. Creditors would not loan at the safe rate r to a consumer who might go bankrupt. For creditors to be willing to loan any amount the consumer wants to borrow at the same interest rate consumers receive on savings it is necessary that consumers do not want to risk bankruptcy -- that they choose to borrow only so much that there consumption is certainly strictly positive in each period. Consumers will choose to do this if the consequences of zero or extremely low consumption are sufficiently horrible, that is, if the slope of the u(c) goes to infinity sufficiently quickly as c goes to 0.
Returning to the argument behind equation 14, for the FOC to hold it is necessary that the consumer not choose to be at a corner. I argue that the derivative of expected utility with respect to δ must be zero at δ = 0. Otherwise it would be possible to increase expected utility for δ slightly positive or slightly negative. For this argument to be valid, slightly positive and slightly negative δ must be feasible. In other words c_t must be positive making it possible to reduce c_t by a small amount and c_{t+1} must be a random variable bounded away from zero (certainly greater than or equal to some positive amount) making it certain that c_{t+1} can be reduced by (1+r)δ for some positive δ. If this is not always true with certainty for every t, equation 14 is not valid.

The assumption that consumers are free to borrow any amount at the same real interest rate, and the assumption that lenders have rational expectations together require and imply that consumers will never choose to risk bankruptcy which should imply equation 14. If consumers are willing to risk bankruptcy (as we certainly are) creditors will charge different interest rates depending on the risk of bankruptcy or refuse to lend at all (as they certainly do). The possibility that consumers might choose to risk bankruptcy not only implies that we sometimes violate our budget constraint, but also implies that rational creditors are not willing to lend us any amount that we wish to borrow at the same interest rate. This might explain why equation 14 does not hold in practice and is as noted the most popular proposed explanation.