Assume that output is a function of capital K, employment L, labour augmenting (Harrod Neutral) technology A and human capital H. Effective labour is equal to A times L. As before assume A grows exponentially at rate g and L grows exponentially at rate n.

1) $Y = F(K,H,AL)$

F is an aggregate production function with constant returns to scale and decreasing marginal product of capital and labour. That is

2) $F(aK,aH,aAL) = aF(K,H,AL)$ and

$F_{KK}(K,H,AL) < 0$, $F_{LL}(K,H,AL) < 0$ and $F_{HH}(K,H,AL) < 0$
Where $F_K(K,H,AL)$ is the partial derivative of $F(K,L)$ with respect to K and $F_{KK}(K,H,AL)$ is the second partial derivative of F with respect to K.

H is human capital. It is knowledge which is acquired with time and effort (consider the time and effort we are expending now – a little bit of mine (for which I am well paid) and a whole lot of yours (for which you are not paid and which comes with the opportunity cost that you are working but not for pay).

Importantly human capital is not like technology. It is not a public good freely available to everyone. I find the phrase "human capital" ugly, but acquiring knowledge with time and effort is more like Solow's investment in capital than his magic exogenous technological progress.

GDP is total production of goods and services and is equal to the sum of production of knowledge by dedicated teaching and studying and production of the physical good which is good for everything – shmoo (notably in the original context of the shmoo – the comic L'il Abner – there is no detectible accumulation of knowledge and certainly no time devoted to formal education). An extremely important assumption is that the production functions for producing the good and for producing human capital are identical. This implies that the ratio of physical capital used to produce goods to unskilled labor used to produce goods is equal to the ratio of physical capital used to produce human to unskilled labor used to produce human capital. It also implies that the ratio of physical capital to human capital in the two sectors is the same. This is an unrealistic assumption which makes everything simple - we can pretend that an amount of (good+knowledge) is produced then divided up into consumption, physical capital investment, and increased knowledge (human capital investment).

Assume no depreciation of physical or human capital (human capital depreciates when people forget or when people die).

Aside from the introduction of human capital, let's go

back to Solow and just make simple assumptions about choices which fit the data.
Assume investment in physical capital is equal to a constant S_K times GDP so

3) $\dot{K} = S_K Y$

Assume that investment in human capital is equal to a constant S_H times GDP so

4) $\dot{H} = S_H Y$

Constant returns to scale implies

5) $y = F(k,h,1) \equiv f(k,h)$

As always taking the log and the time derivative implies

6) $\dot{k} = S_K f(k,h) - (n+g)k$

and

7) $\dot{h} = S_H f(k,h) - (n+g)h$

This is enough to draw a phase diagram and figure out everything which will happen given initial h and k.
Cobb Douglas example (as in the book) 8) $Y = aK^{\alpha} H^{\beta} (AL)^{(1-\alpha-\beta)}$

where $\alpha+\beta$ is less than or equal to 1.

9) $\frac{\dot{k}}{k} = aS_K k^{\alpha-1} h^{\beta} - (n + g)$

10) $\frac{\dot{h}}{h} = aS_h k^{\alpha} h^{\beta-1} - (n + g)$

This is enough to draw a phase diagram and figure out everything that happens as a function of initial k and h.

There are two steady states. If h and k are zero, there is no production, so h and k remain zero forever. There is another steady state where

11) $aS_h k^{\alpha} h^{\beta-1} = (n + g) = aS_K k^{\alpha-1} h^{\beta}$

so

12) $S_H/h = S_K/k$

The stocks of capital and human capital are proportional to the fractions of total goods and services devoted to capital accumulation and learning.

Plugging in $h = (S_H/S_K)k$ into 11 gives

13) $(n + g) = aS_K k^{\alpha+\beta-1} (S_H/S_K)^{\beta}$

so steady state $k = ((aS_K^{1-\beta} S_h^{\beta})/(n+g))^{1/(1-\alpha-\beta)}$

multiply by S_H/S_K to see that

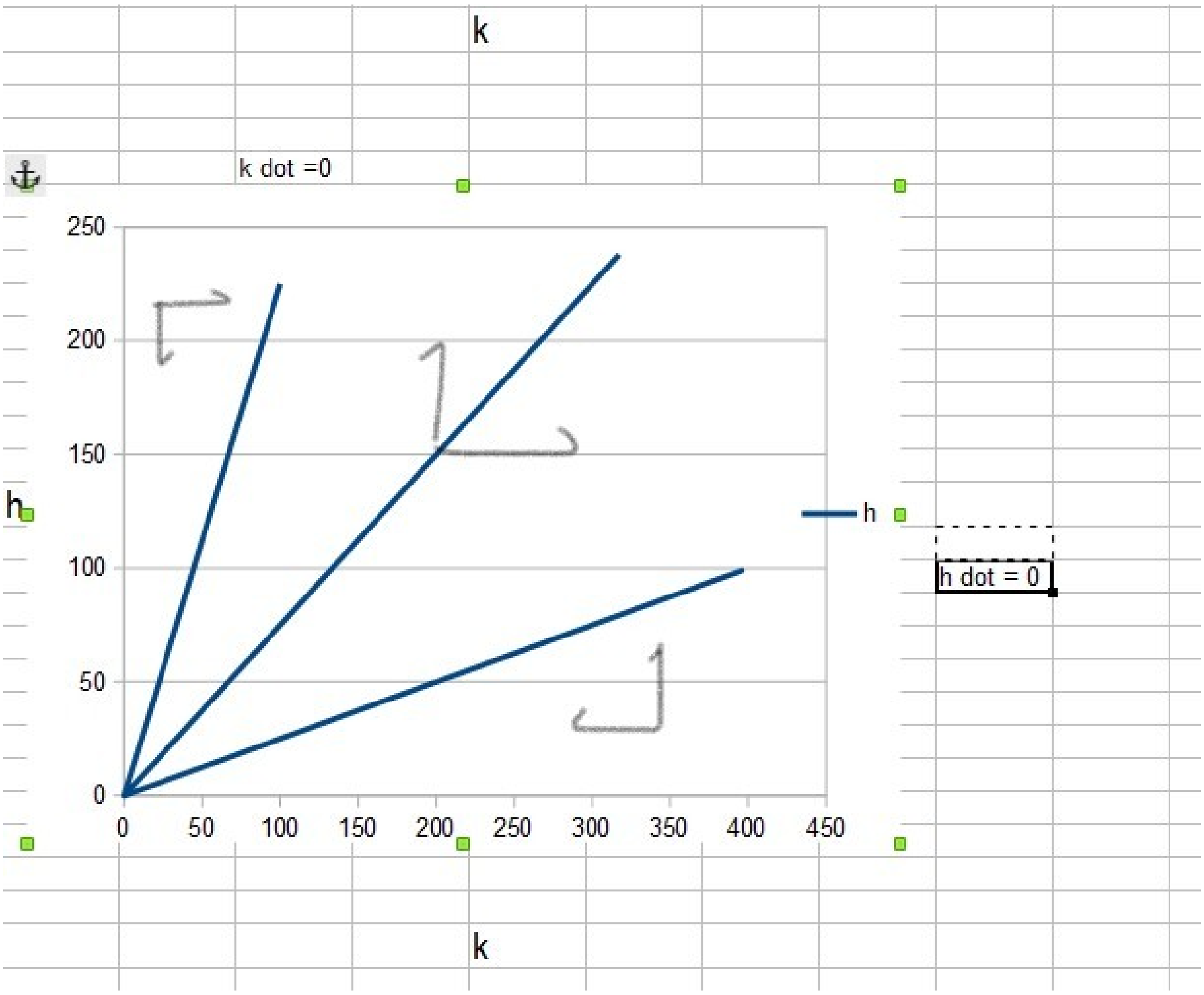
steady state $h = ((aS_K^{\alpha} S_H^{1-\alpha})/(n+g))^{1/(1-\alpha-\beta)}$

One interesting special case occurs if $\alpha+\beta = 1$. In that case depending on a,n,g, S_H, and S_K there is either unbounded growth of h and k or they shrink to zero.

If $aS_K k^{\alpha+\beta-1}(S_H/S_K)^{\beta} > (n + g)$ there is unbounded growth of h and k.

If $aS_K k^{\alpha+\beta-1}(S_H/S_K)^{\beta} < (n + g)$ then h and k shrink to zero.

This means that the model can be an exogenous growth model where the rate of growth converges to balanced growth at rate n+g or an endogenous growth model where the rate of growth depends on the choices S_H and S_K .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200
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h
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h
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