

## THE UTILITY MAXIMIZATION PROBLEM

16/11/202

$$\begin{aligned} \text{Max } & u(x) \\ \text{s.t. } & p \cdot x \leq w \\ & x \geq 0 \end{aligned}$$

with  $L$  commodities

$$\begin{aligned} \text{Max } & u(x_1, x_2, \dots, x_L) \\ \text{s.t. } & p_1 x_1 + p_2 x_2 + \dots + p_L x_L \leq w \\ & x_\ell \geq 0 \quad \forall \ell = 1, \dots, L \end{aligned}$$

Take  $L = 2$

## A SUMMARY ON CONSTRAINED OPTIMIZATION

$$\text{Max } u(x_1, x_2)$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 \leq \omega$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Examine interior solutions

Set-up the Lagrangian function

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda [w - p_1 x_1 - p_2 x_2]$$

then find the stationary point of  $\mathcal{L}(\cdot)$   
by solving

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 0 \quad x_1 > 0 \quad (1)$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 0 \quad x_2 > 0 \quad (2)$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = 0 \quad \lambda \geq 0 \quad (3)$$

these FOC are necessary for an optimum.

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \quad \Leftrightarrow \quad \frac{\partial u(x_1, x_2)}{\partial x_1} = \lambda p_1$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 \quad \Leftrightarrow \quad \frac{\partial u(x_1, x_2)}{\partial x_2} = \lambda p_2$$

↳ in matrix notation

$$\nabla u(x_1, x_2) = \lambda p \quad x > 0$$

with

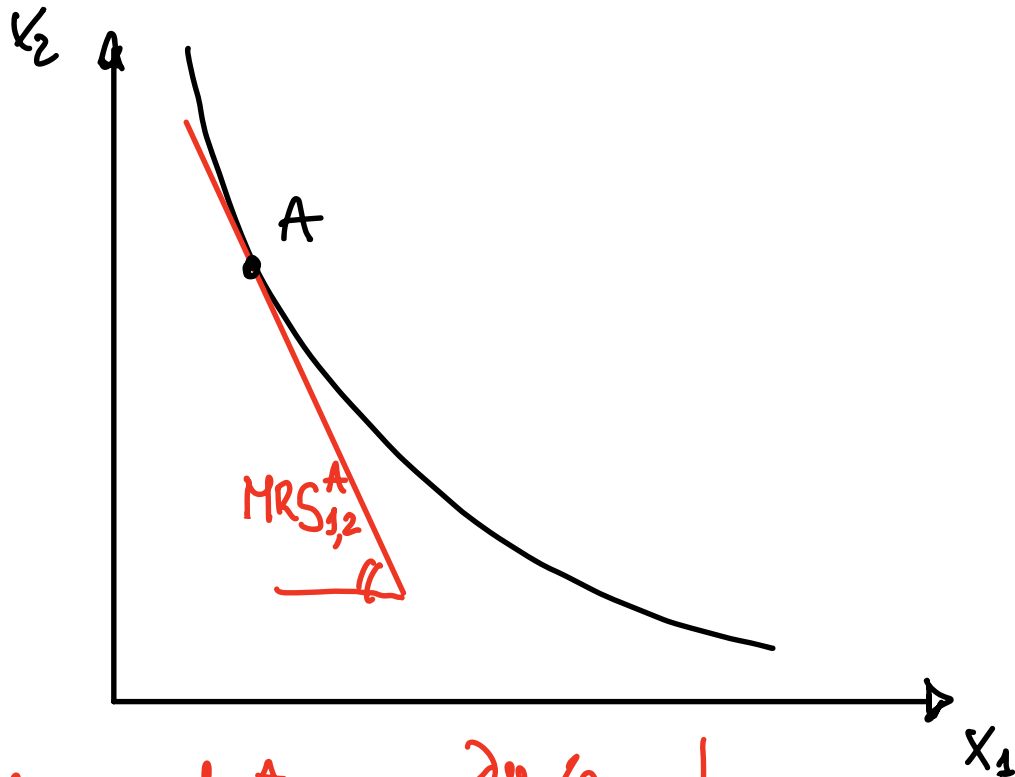
$$\nabla u(x_1, x_2)_{(1 \times L)} = \left[ \frac{\partial u(x_1, x_2)}{\partial x_1} \quad \frac{\partial u(x_1, x_2)}{\partial x_2} \right]$$

$$p = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$$

$(1 \times L)$

the gradient vector of  $u(x_1, x_2)$  and the price vector  $p$  are proportional

## CHARACTERIZATION OF AN INTERIOR SOLUTION (by NECESSARY CONDITIONS)



$$MRS_{1,2}^A = \frac{dx_2^A}{dx_1^A} = - \frac{\partial u / \partial x_1}{\partial u / \partial x_2} \Big|_A$$

measures the variation in  $x_2$  induced by a differential change in  $x_1$  along an indifference curve.

Why at the optimum  $MRS_{1,2} = \frac{P_1}{P_2}$  ?

Assume instead:  $MRS_{12} > \frac{P_1}{P_2}$

$$\Leftrightarrow \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} > \frac{P_1}{P_2}$$

Can any such  
bundle be  
optimal?

the consumer can increase the consumption of good 1 by  $dx_1$  and reduce the consumption of good 2 by  $-\frac{P_1}{P_2} dx_1$  (moving

along the budget line) and increase her utility by

$$\frac{\partial u}{\partial x_1} \cdot dx_1 + \frac{\partial u}{\partial x_2} dx_2 =$$

$$\frac{\partial u}{\partial x_1} dx_1 - \frac{\partial u}{\partial x_2} \cdot \left( \frac{P_1}{P_2} \right) dx_1 > 0$$

Example

$$\text{let } u(x_1, x_2) = x_1^\alpha x_2^\beta$$

$$\begin{aligned} &\text{with } \alpha + \beta = 1 \\ &\alpha \in (0, 1) \\ &\beta \in (0, 1) \end{aligned}$$

UMP

$$\text{Max } x_1^\alpha x_2^{1-\alpha}$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 \leq w$$

$$[x_1 \geq 0, x_2 \geq 0]$$

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^\alpha x_2^{1-\alpha} + \lambda [w - p_1 x_1 - p_2 x_2]$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial x_2} = (1-\alpha) x_1^\alpha x_2^{-\alpha} - \lambda p_2 = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0 \quad \lambda \geq 0$$

$$w = p_1 x_1 + p_2 x_2 \quad (3)$$

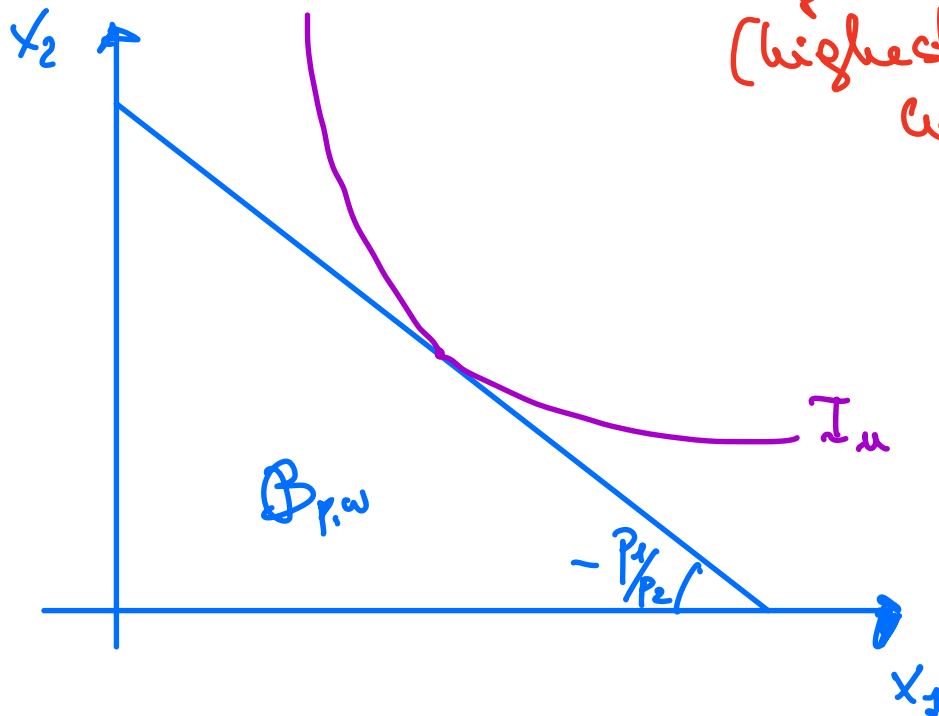
Focus on (1)-(2)

$$(1) \quad \alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1$$

$$\alpha \frac{x_2}{x_1} x_1^{\alpha} x_2^{-\alpha} = \lambda p_1$$

$$(2) \quad (1-\alpha) x_1^{\alpha} x_2^{-\alpha} = \lambda p_2$$

At the optimal bundle, the budget line is tangent to the (highest) indifference curve



## ECONOMIC INTERPRETATION OF THE LAGRANGE MULTIPLIER

$\lambda$  measures the mag. effect of changes in  $w$  on the UMP

$\lambda$  is the mag. utility of wealth at the optimum

take  $x(p, w)$  differentiable and interior solution.

$u(x(p, w))$  is the utility at the optimum

What's the effect of a change in  $w$  on  $u(x(p, w))$ ?

$$\begin{aligned}\frac{\partial u(x(p, w))}{\partial w} &= \frac{\partial u(x(\cdot))}{\partial x_1} \cdot \frac{\partial x_1(p, w)}{\partial w} + \\ &+ \frac{\partial u(x(\cdot))}{\partial x_2} \cdot \frac{\partial x_2(p, w)}{\partial w} + \\ &+ \dots + \frac{\partial u(x(\cdot))}{\partial x_L} \cdot \frac{\partial x_L(p, w)}{\partial w}\end{aligned}$$

recall  $D_w x(p, w)$  is  $(L \times 1)$  vector

Hence the change in utility can be rewritten as

$$\nabla u(x(p, w)) \cdot D_w x(p, w) \quad (C^*)$$

At the optimum

$$\nabla u(x(p, w)) = \lambda p$$

Hence  $(C^*)$  can be rewritten as

$$\lambda p \cdot D_w x(p, w) \quad (C^*)$$

At the optimum

$$p \cdot x(p, w) = w$$

hence

$$p \cdot D_w x(p, w) = 1$$

$$\boxed{\frac{\partial u(x(p, w))}{\partial w} = \lambda} \quad (C^*)$$

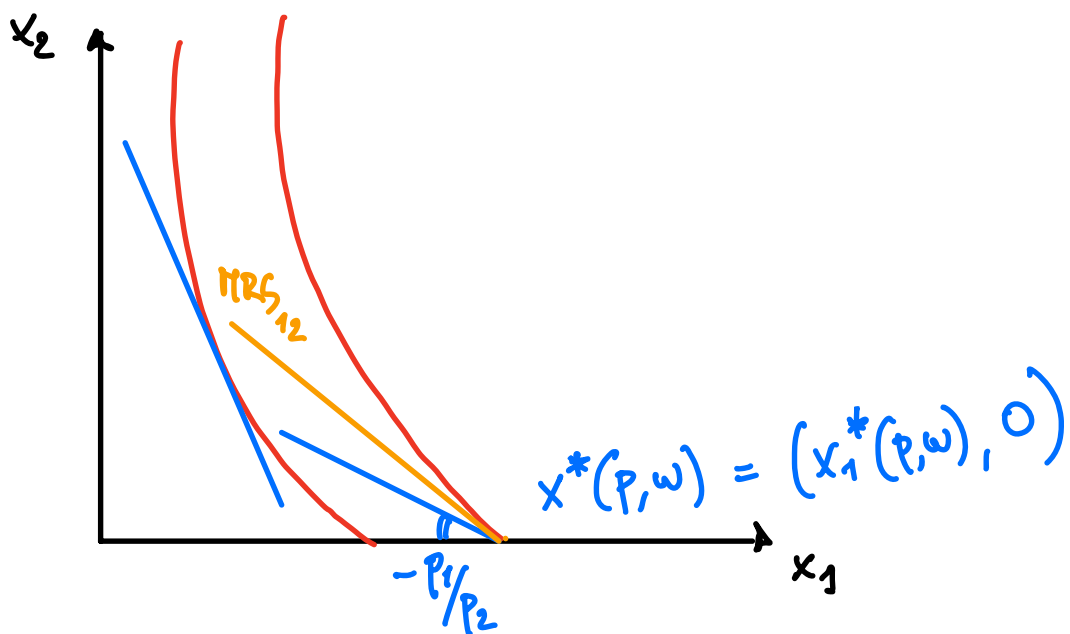
More generally, since optimal bundles may not exclude that  $x_l^*(p, w) = 0$  for some commodity  $l$ , the conditions for an optimum have to be generalized to :

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} \leq 0 \quad x_1 \geq 0 \quad x_1 \left[ \frac{\partial \mathcal{L}}{\partial x_1} \right] = 0 \quad (1')$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} \leq 0 \quad x_2 \geq 0 \quad x_2 \left[ \frac{\partial \mathcal{L}}{\partial x_2} \right] = 0 \quad (2')$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} \geq 0 \quad \lambda \geq 0 \quad \lambda \left[ \frac{\partial \mathcal{L}}{\partial \lambda} \right] = 0 \quad (3')$$

# A CORNER SOLUTION



$$x_2^*(p, w) = 0$$

the tangency condition here cannot hold!

$$MRS_{1,2} \neq \frac{p_1}{p_2}$$

$$\text{FOC} \quad \frac{\partial u_1(x_1, x_2)}{\partial x_1} = \lambda p_1 \quad \text{for } x_1 > 0$$

$$\frac{\partial u_2(x_1, x_2)}{\partial x_2} \leq \lambda p_2 \quad \text{for } x_2 = 0$$

Hence, the consumer would like to

reduce the consumption of  $x_2$  as much as she can (to increase  $\partial u / \partial x_2$ ) ... the lower bound on consumption is  $x_2 = 0$ .

Hence, at the optimum for a consumer with those preferences

$$MRS_{12} > \frac{p_1}{p_2}$$

In case we admit corner solutions, the non-negativity constraints  $x_1 \geq 0$ ,  $x_2 \geq 0$  should be explicitly included in the Lagrangian function each with its own multiplier

It's useful to write them

$$\begin{aligned} -x_1 &\leq 0 \\ -x_2 &\leq 0 \end{aligned}$$

so that the Lagrangian function becomes

$$\begin{aligned} L(x_1, x_2, \lambda, \mu_1, \mu_2) = & u(x) + \lambda [W - p_1 x_1 - p_2 x_2] - \\ & + \mu_1 x_1 + \mu_2 x_2 \end{aligned}$$

and the FOC will need to be adjusted

the FOCs are necessary and sufficient for an optimal bundle if the utility function is strictly quasi-concave ( $\approx$  are strictly convex)

$\Rightarrow$  the bundle obtained as a result of the system of FOCs is the unique maximizer of the UMP.

Recall that

$$\text{Min } p_1 x_1 + p_2 x_2$$

$$\text{s.t. } u(x_1, x_2) \geq \bar{u}$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$



$$\text{Max } -(p_1 x_1 + p_2 x_2)$$

$$\text{s.t. } -u(x_1, x_2) \leq -\bar{u}$$

$$-x_1 \leq 0$$

$$-x_2 \leq 0$$

Hence you can solve the EMP using the same approach developed for UMP.

Of course, the multipliers of the two problems do not have the same interpretations.

The system of FOC relative to each commodity  $x_i$  in the two problems at the beginning of the page yields to the same optimal bundle.