Macroeconometrics

Lecture 2

Filtering Part 1

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Filtering theory: Linear filters

- Signal extraction is carried out by performing linear operations on y_t.
- Cycle estimates are often obtained in this way. The constructive principle may differ (band-pass filtering - nonparametric; Wiener -Kolmogorov filter - parametric; penalised least squares - semiparametric).
- We aim at characterising such linear estimators

A *linear time invariant* filter is represented as follows:

$$\mathrm{w}(L) = \sum_{j=1}^{\infty} \mathrm{w}_j L^j$$

$$\mathrm{w}(L) y_t = \cdots + \mathrm{w}_1 y_{t-1} + \mathrm{w}_0 y_t + \mathrm{w}_{-1} y_{t+1} + \cdots$$

The filter is said to be *symmetric* if $\mathrm{w}_j = \mathrm{w}_{-j}$.

$$\mathrm{w}(L) = \mathrm{w}_0 + \sum_{j=1}^{\infty} \mathrm{w}_j (L^j + L^{-j}).$$

The sequence of weights $\{\mathrm{w}_j, j = -h_1, \ldots, h_2\}$ is known as the *impulse response* of the filter (w_j is the partial derivative with respect to y_{t-j}).

- Finite Impulse Response (FIR) filters: h_1, h_2 are finite.
- Infinite Impulse Response (IIR) filters: either h_1 or h_2, or both, are infinite.

Obviously, an IIR filter is not realisable, as it requires infinite observations, but it can be approximated or projected onto the finite available sample.

Examples

Differencing filter : $w(L) = 1 - L$: $w_0 = 1, w_1 = -1$, (asymmetric FIR)

Seasonal Differences : $w(L) = 1 - L^s$: $w_0 = 1, w_{s-1} = -1$ (asymmetric FIR)

Summation filter : $w(L) = 1 + L + \cdots + L^{s-1}$, also denoted $S(L)$ (asymmetric FIR)

Arithmetic Moving Average : $w(L) = \frac{1}{3}L^{-1} + \frac{1}{3} + \frac{1}{3}L$ (symmetric FIR)

Integration filter (asymmetric IIR)

$$\mathrm{w}(L) = 1/(1-L) : \mathrm{w}_j = 1, j \geq 0$$
$$(1-L)^{-1} = 1 + L + L^2 + \cdots$$

Ideal band-pass filter (symmetric IIR)

$$\mathrm{w}(L) = \frac{\omega_2 - \omega_1}{\pi} + \sum_{j=1}^{\infty} \frac{\sin(\omega_2 j) - \sin(\omega_1 j)}{\pi j}(L^j + L^{-j}).$$

EMWA filter (symmetric IIR)

$$\mathrm{w}(L) = \frac{1}{1 + \lambda[(1-L)(1-L^{-1})]}$$

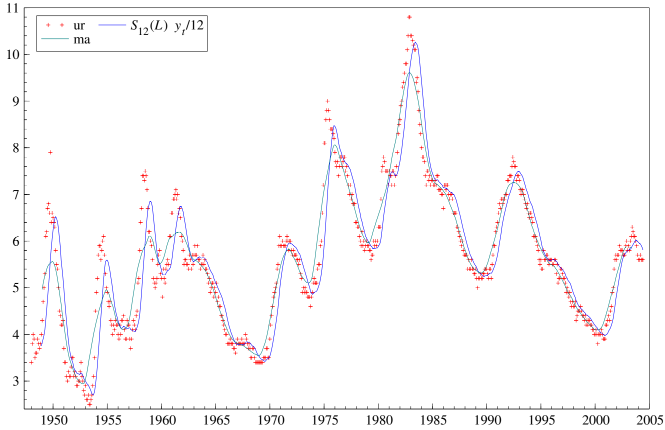
Gain and Phase

The effects of a linear filter applied on a series y_t, give $z_t = \mathrm{w}(L)y_t$, are twofold:

Amplitude effect : the filter alters the amplitude of the fluctuations.

Phase effect : the cyclical components of the original series are displaced in time.

Figure: U.S. monthly unemployment rate. Series y_t and linear filters



The polar representation of the FRF, $\mathrm{w}(e^{-\imath\omega}) = G(\omega)e^{-\imath Ph(\omega)}$, is written in terms of two crucial quantities:

- The **gain**: $G(\omega) = |\mathrm{w}(e^{-\imath\omega})| = \sqrt{\mathrm{w}_{\mathcal{R}}(\omega)^2 + \mathrm{w}_{\mathcal{I}}(\omega)^2}$,
  The **gain** measures the amplitude effect of the filter, so that if at some frequencies the gain is less than one, then those frequency components **will be attenuated** in the filtered series.
- The **phase**: $Ph(\omega) = \arctan(-\mathrm{w}_{\mathcal{I}}(\omega)/\mathrm{w}_{\mathcal{R}}(\omega))$.
  The **phase** measures the displacement, or the phase shift, of the signal.

- If $f_y(\omega)$ denotes the spectrum of y_t, the spectrum of $\mathrm{w}(L)y_t$ is equal to $|\mathrm{w}(e^{-i\omega})|^2 f_y(\omega)$, and therefore the square of the gain function (also known as the *power transfer function*) provides the factor by which the spectrum of the input series is multiplied to obtain that of the filtered series.
- In the important special case when $\mathrm{w}(L)$ is symmetric, the phase displacement is zero, and the gain is simply $G(\omega) = |\mathrm{w}_0 + 2\sum_{j=1}^{m} \mathrm{w}_j \cos(\omega j)|$.

Example: the differencing filter

- Take this simple filter $\mathrm{w}(L) = 1 - L$.
- The frequency response function of the filter is

$$\mathrm{w}(e^{-\imath\omega}) = 1 - e^{-\imath\omega} = 1 - \cos\omega + \imath\sin\omega$$

- The gain is

$$G(\omega) = \sqrt{(1-\cos\omega)^2 + \sin^2\omega} = \sqrt{2(1-\cos\omega)}$$

and the squared gain is

$$|\mathrm{w}(e^{-\imath\omega})|^2 = 2(1-\cos\omega).$$

Nonparametric approach to cycle measurement: Band-Pass filters

- A *low-pass* filter is a filter that passes low frequency fluctuations and reduces the amplitude of fluctuations with frequencies higher than a cutoff frequency ω_c (see e.g. Percival and Walden, 1993).
- The frequency response function of an ideal low-pass filter takes the following form for $\omega \in [0, \pi]$:

$$\mathrm{w}_{lp}(\omega) = \begin{cases} 1 & \text{if } \omega \leq \omega_c \\ 0 & \text{if } \omega_c < \omega \leq \pi \end{cases}$$

  The notion of a high-pass filter is complementary, its frequency response function being $\mathrm{w}_{hp}(\omega) = 1 - \mathrm{w}_{lp}(\omega)$.

The band-pass filter

- The notion of a band–pass filter is relevant to business cycle measurement: the traditional definition, ascribed to Burns and Mitchell (1946), considers all the fluctuations with a specified range of periodicities, namely those ranging from one and a half to eight years. Thus, if s is the number of observations in a year, the fluctuations with periodicity between $1.5s$ and $8s$ are included.
- Baxter and King (1999, BK henceforth) argue that the ideal filter for cycle measurement is a band-pass filter.

- Now, given the two business cycle frequencies, $\omega_{c1} = 2\pi/(8s)$ and $\omega_{c2} = 2\pi/(1.5s)$, the band-pass filter is

$$\mathrm{w}_{bp}(L) = \frac{\omega_{c2} - \omega_{c1}}{\pi} + \sum_{j=1}^{\infty} \frac{\sin(\omega_{c2} j) - \sin(\omega_{c1} j)}{\pi j}(L^j + L^{-j}). \tag{1}$$

Notice that $w_{bp}(L)$ is the contrast between the two low–pass filters with cutoff frequencies ω_{c2} and ω_{c1}.

- The ideal band-pass filter exists and is unique, but as it entails an infinite number of leads and lags, an approximation is required in practical applications.
- BK show that the K-terms approximation to the ideal filter (1), that is optimal in the sense of minimising the integrated mean square approximation error, is obtained from (1) by truncating the lag distribution at a finite integer K. They propose using a three years window, i.e. $K = 3s$.
- They also constrain the weights to sum up to zero, so that the resulting approximation is a detrending filter: denoting the truncated filter $\mathrm{w}_{bp,K}(L) = \mathrm{w}_0 + \sum_1^K \mathrm{w}_j(L^j + L^{-j})$, the weights of the adjusted filter will be $\mathrm{w}_j - \mathrm{w}_{bp,K}(1)/(2K+1)$.
- BK do not entertain the problem of estimating the cycle at the extremes of the available sample; as a result the estimates for the first and last three years are unavailable. Christiano and Fitzgerald (2003) provide the optimal finite-sample approximations for the band pass filter, including the real time filter, using a model based approach.

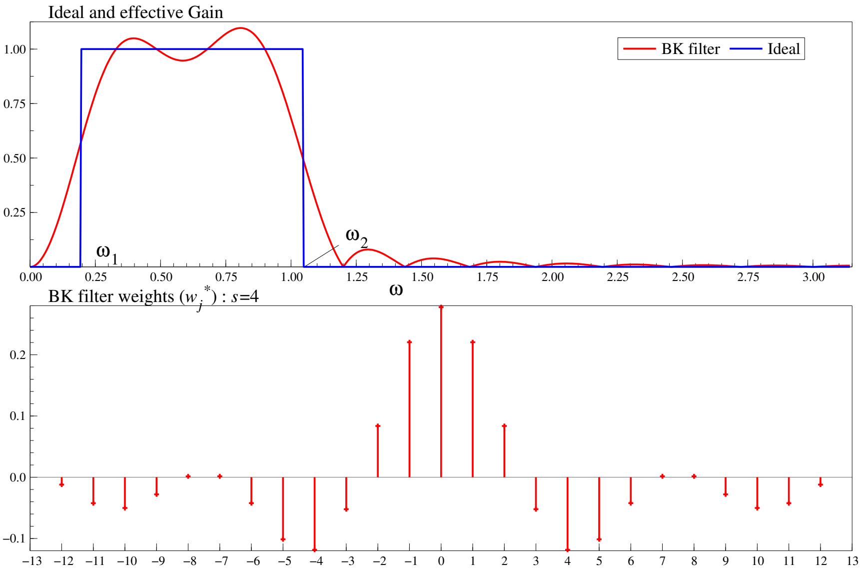


Figure: Baxter and King quarterly cycle filter