Behavioral risk adjustments

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Introduction (1/2)

Behavioral risk in finance

Behavioral finance is a recent field in economics which studies the impact of psychology on the behavior of practitioners and financial markets from a general point of view

In a more specific contest, **behavioral risk** affects the valuation of all financial instruments with **embedded options**, such as prepayment or extension options. A decision has to be taken, allowing one counterpart to terminate the contract or modify contractual conditions

Behavioral risk arises whenever option holders do not act only on the strength of financial convenience, but follow an uncertain and sub-optimal exercise strategy if seen from the point of view of option seller

However, quite surprisingly, there is no unique definition:

- *Behavioral risk as **prepayment risk***
- *Behavioral risk as **residual risk***

Introduction (2/2)

Instruments with embedded options

Assets or liabilities with embedded prepayment/extension options subject to behavioral risk

Assets

- *Mortgages, residential mortgages*
- *Mortgage-Backed Securities*
- *Callable bonds*
- *Corporate loans, retail loans*
- *Loan commitments*

Liabilities

- *Short-sight/non-maturity deposits*
- *Puttable bonds*
- *Postal bonds (issued e.g. by CDP)*
- *Life insurance policies, annuities*

Agenda

1. Early-Exercise Options: Classical approach
2. Early-Exercise Options: Real world
3. Behavioral Risk: Modeling approach
4. Behavioral Risk: Mathematical framework

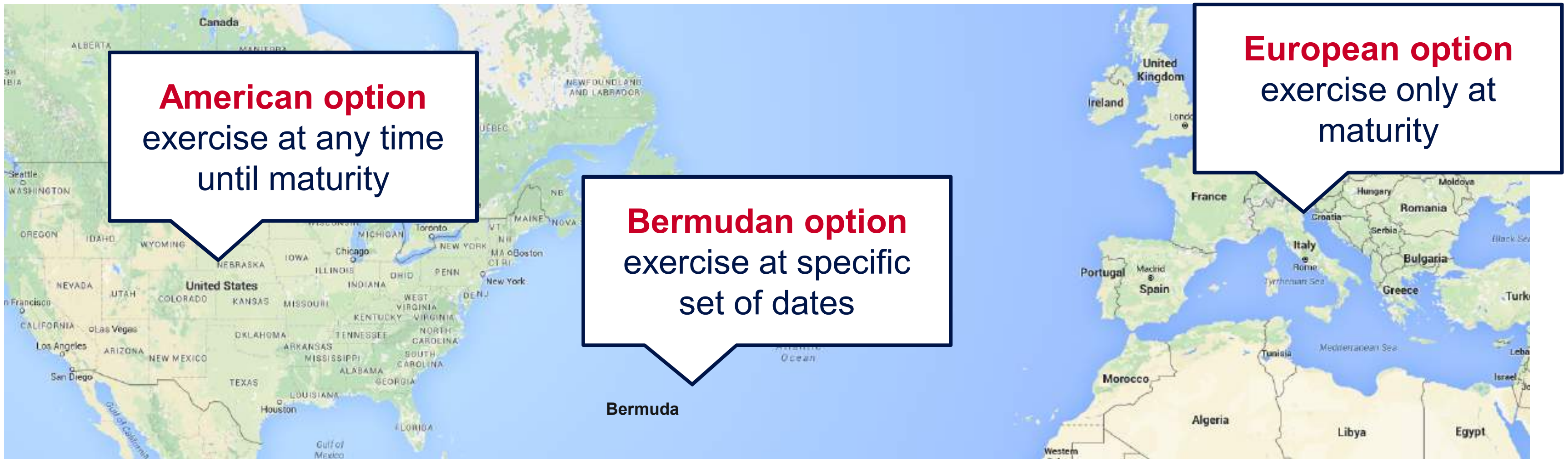
Early-Exercise Options

Classical approach

Early-Exercise Options

Main definitions

- In financial markets a large number of products allows one counterpart (or both) to terminate (or extend) the contract at certain conditions, specified in advance at the inception



- An option holder can benefit from the option owned, while the option seller faces a potential loss, which must be reflected in its price. Ceteribus paribus:

$$V(\text{American}) \geq V(\text{Bermudan}) \geq V(\text{European})$$

The classical pricing approach (1/3)

Main topics

- How to price a financial contract?

Real-World Probability → *Risk-Neutral Probability*

Assumptions

- **Full rationality of investors**
- Absence of arbitrage opportunities
- Market completeness
- Short-selling allowed

- We assign different probabilities to events and define a **market price of risk**. The price is obtained as the cost of a **replicating strategy** and is given by the strategy **maximizing the return** for the investor. In practice, it is computed by means of **backward maximization**

$$\begin{cases} V_T = E_T & h = T \\ V_h = \max\{\, E_h, \mathbb{E}_h\,[V_{h+1}]\,\} & h < T \end{cases}$$

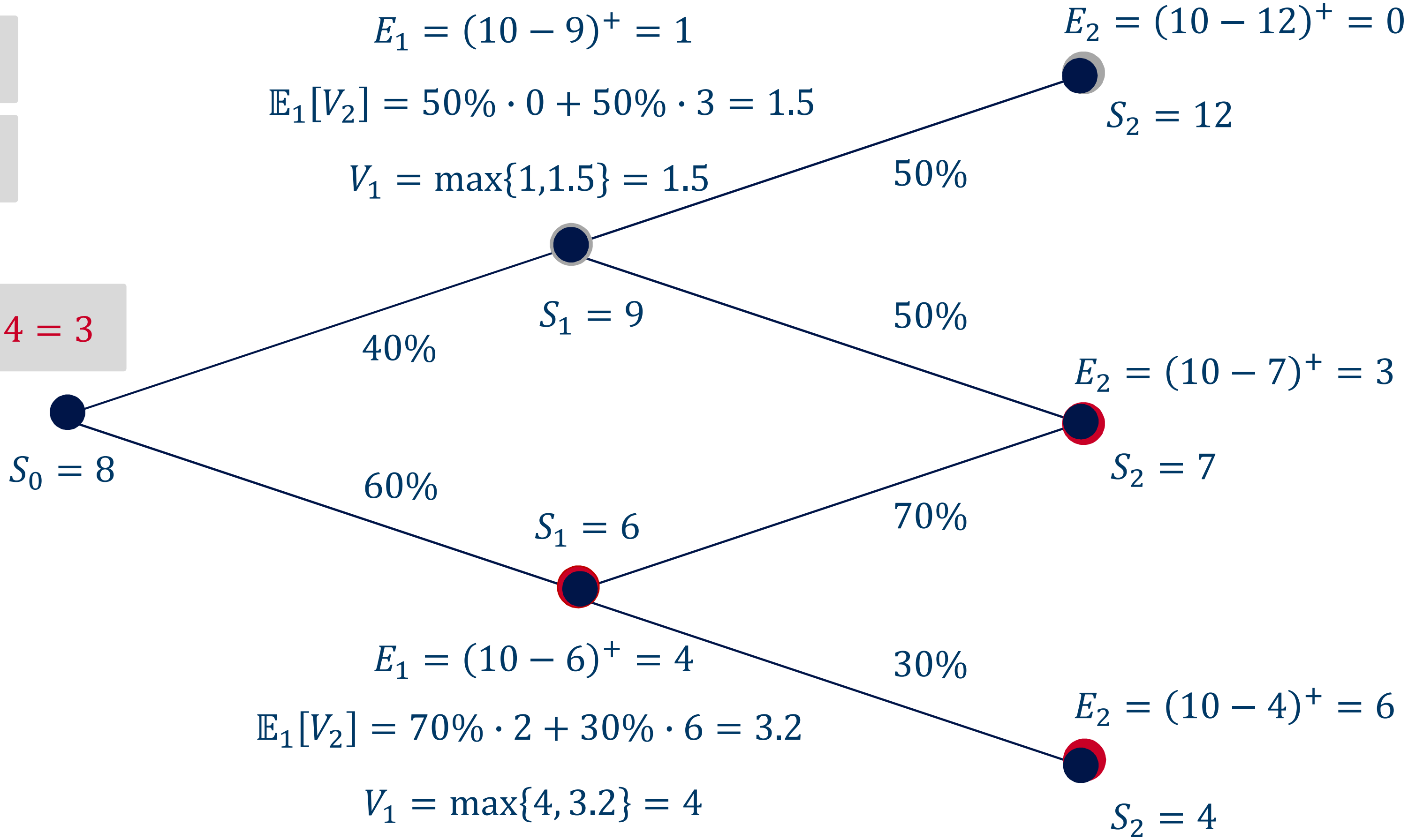
The classical pricing approach (2/3)

Example: Bermudan Put $(K\text{-}S)^{+}$

$r = 0\%$

$K = 10$

$V_0 = 40\% \cdot 1.5 + 60\% \cdot 4 = 3$



= No Exercise

= Exercise

The classical pricing approach (3/3)

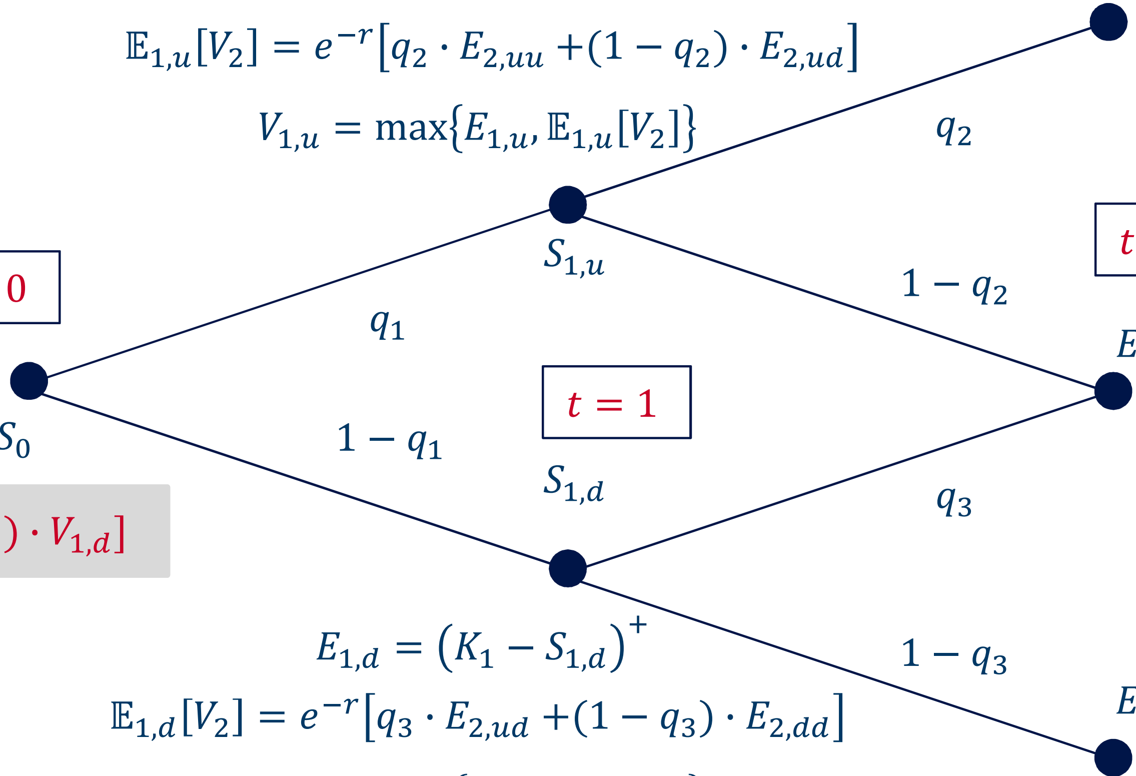
Algorithm: Bermudan Put (K-S)+

$$E_{1,u} = \left(K_1 - S_{1,u}\right)^+$$

$$\mathbb{E}_{1,u}[V_2] = e^{-r}\left[q_2 \cdot E_{2,uu} + (1-q_2)\cdot E_{2,ud}\right]$$

$$V_{1,u} = \max\left\{E_{1,u}, \mathbb{E}_{1,u}[V_2]\right\}$$

$$E_{2,uu} = \left(K_2 - S_{2,uu}\right)^+$$



$$E_{2,ud} = \left(K_2 - S_{2,ud}\right)^+$$

$$V_0 = e^{-r}\left[q_1 \cdot V_{1,u} + (1-q_1)\cdot V_{1,d}\right]$$

$$E_{1,d} = \left(K_1 - S_{1,d}\right)^+$$

$$\mathbb{E}_{1,d}[V_2] = e^{-r}\left[q_3 \cdot E_{2,ud} + (1-q_3)\cdot E_{2,dd}\right]$$

$$V_{1,d} = \max\left\{E_{1,d}, \mathbb{E}_{1,d}[V_2]\right\}$$

$$E_{2,dd} = \left(K_2 - S_{2,dd}\right)^+$$

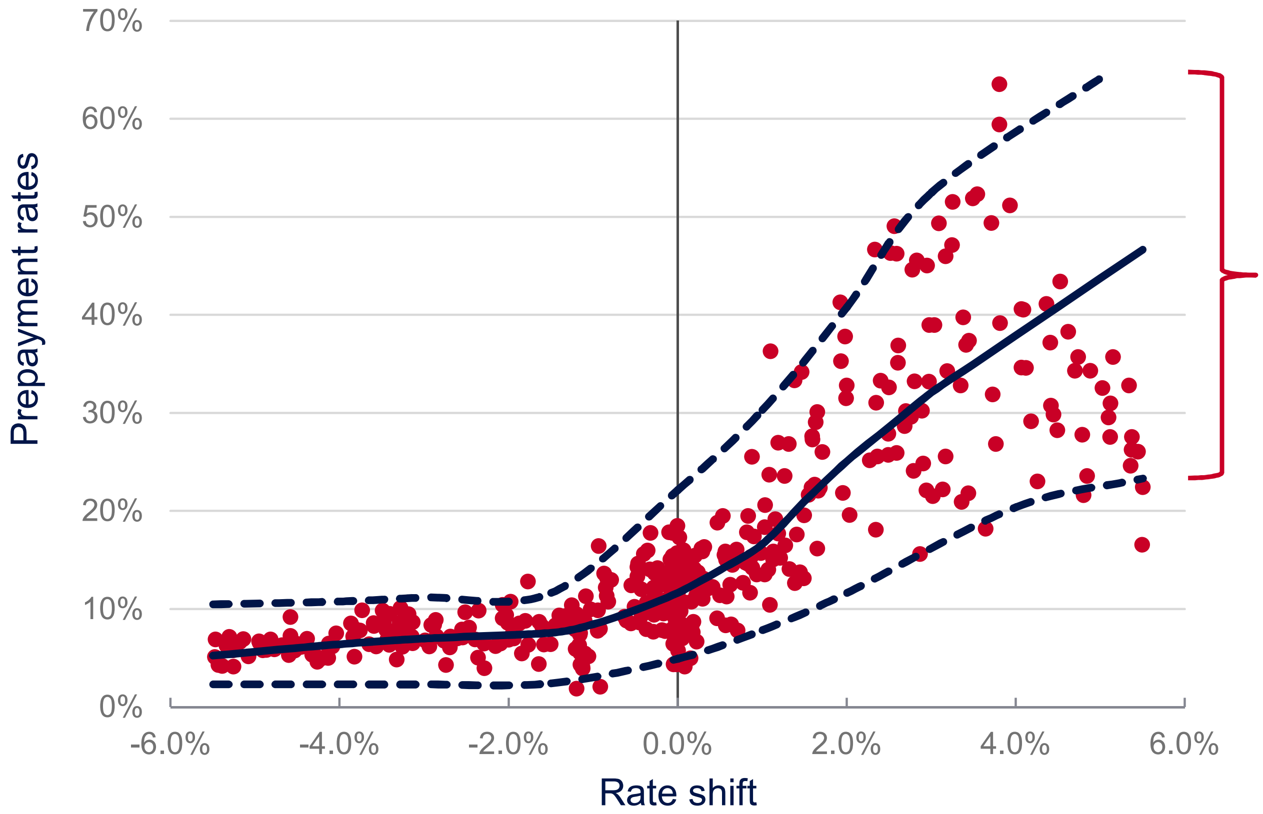
Early-Exercise Options

Real world

Empirical data of mortgage prepayments (1/2)

Observing the phenomenon…

- Some borrowers prepay when this is not convenient
- Some others do not prepay when this is convenient
- Prepayment rates exhibit **S-shaped dependence** on financial convenience measured by the rate shift
- **Residual variance** observed for the same market scenario
- Burnout effect as a function of loan age



Empirical data of mortgage prepayments (2/2)

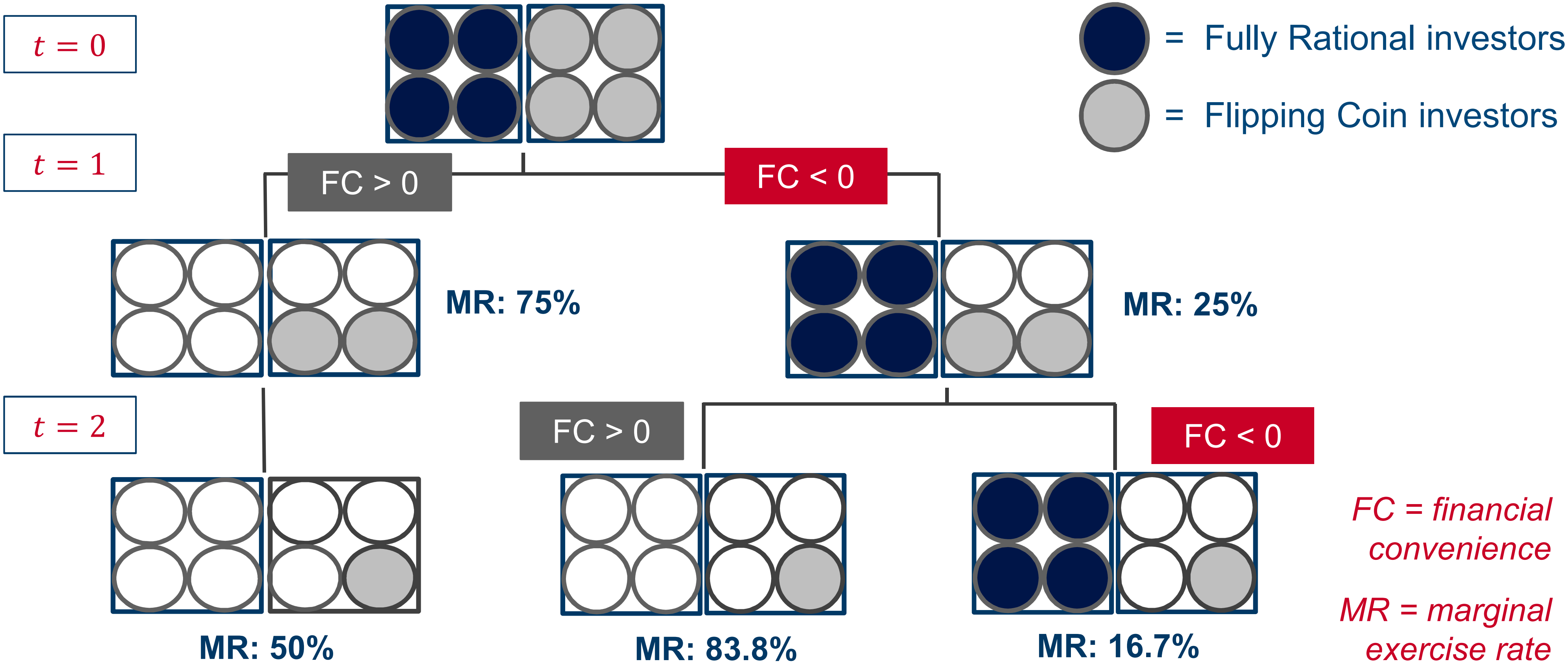
Trying to explain data…

Observed investor behavior
Early exercise rates are random variables
Early exercise rates depend on exogenous factors, besides financial reasons
Early exercise rates depend on the life of the contract
A pool of investors is likely to be heterogeneous (i.e. different exercise behavior), granular, …

Possible sources of sub-optimal exercise	
No interest in early exercise	
Different valuation of the contract	
Move to more valuable investments	
Regulatory constraints	
Large transaction costs	
Lack of information or sophistication	*Retail Investors*
Discontinuous monitoring of markets	*Retail Investors*
Life events	*Retail Investors*

Time dependency due to burnout effect

Example

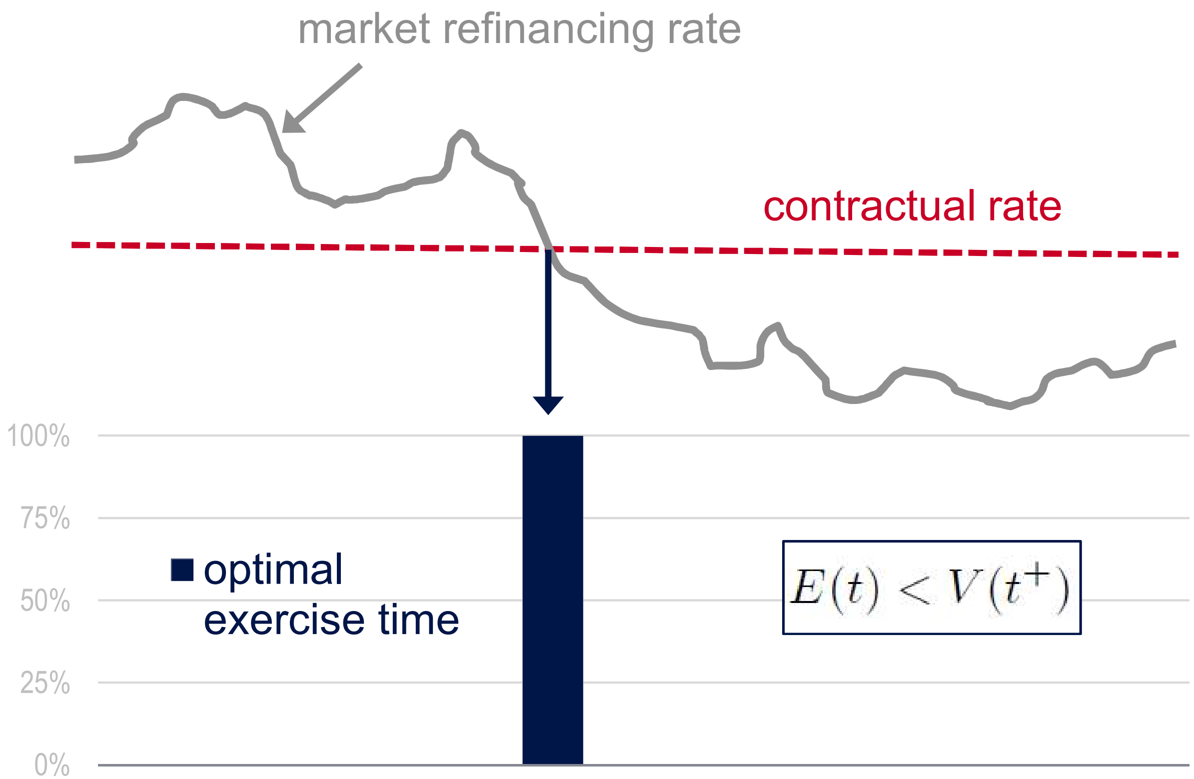


FC = financial convenience

MR = marginal exercise rate

Exercise times and cash flows (1/2)

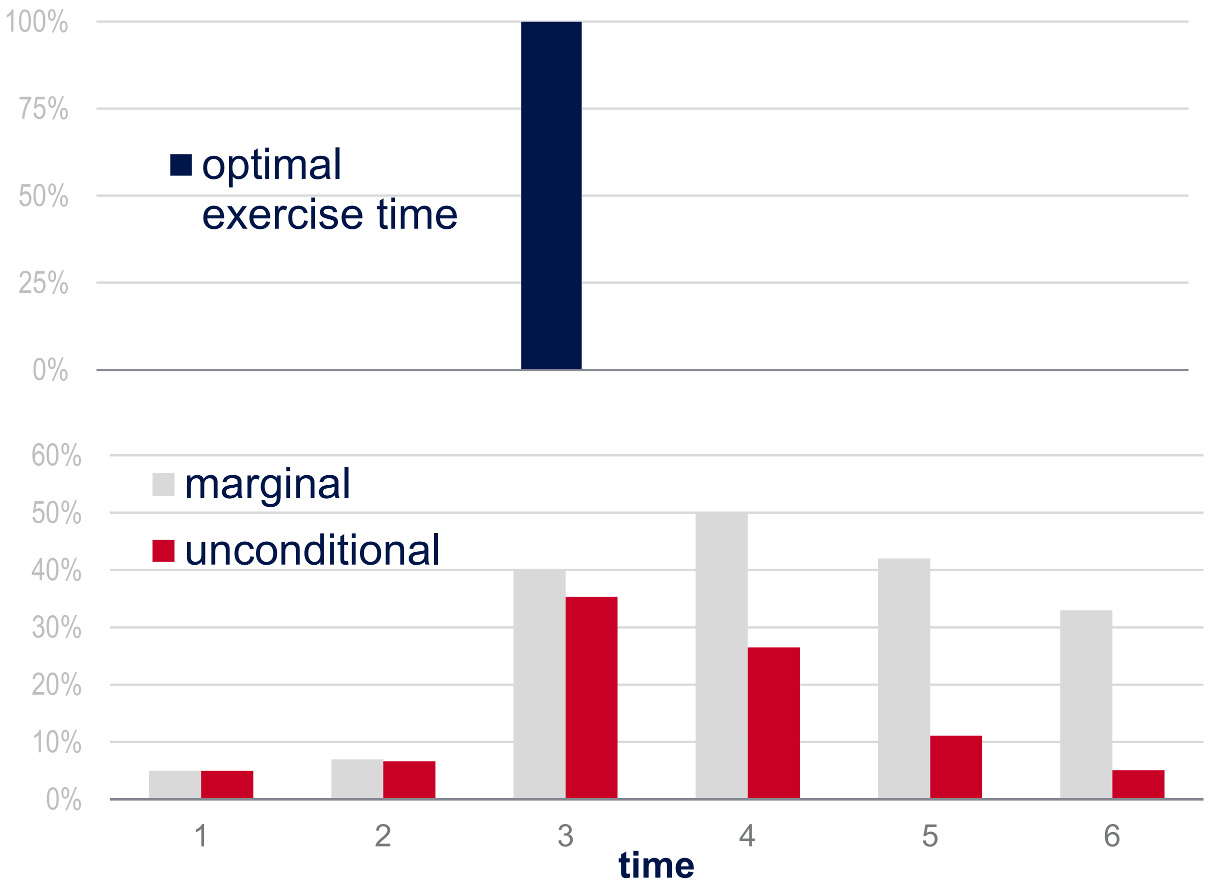
"Without" behavioral risk – back to the classical world



- Rational option holders follow an exercise strategy maximizing their return
- For a mortgage with no prepayment penalty, optimal exercise time τ^* occurs when the exercise value (i.e. the remaining balance) is less than the continuation value
- Conditionally upon a market scenario and given a model for market factors, the **optimal exercise time is univocally determined**

Exercise times and cash flows (2/2)

"With" behavioral risk – into the real world



- In the presence of behavioral risk the exercise time is random even when the market scenario is specified
- Only option exercise probabilities can be estimated
- We define behavioral risk by identifying the **additional cash flow variability** that it generates

Behavioral risk

Definition

Given a market scenario…

Classical option pricing

- The behavior of all market players is **fully predictable**
- Exercise if and only if $\{E_h \text{ vs } \mathbb{E}_h[V_{h+1}]\}$

Behavioral option pricing

- Investors' behavior is **not fully explained** by market
- Exercise probability is a function of $E_h, V_{h+1}, \ldots$

Behavioral risk is the additional source of uncertainty in the future cash flows of a contract, when the option holder does not follow an optimal exercise strategy as seen from the point of view of the option seller.

Behavioral risk

Modeling approach

Literature background

Three main categories

Econometric models

The prepayment rate depends on a set of explanatory variables describing the investor behavior. The model must be calibrated on historical data. Applications require the generation of future scenarios of all variables in a risk-neutral framework. Subject to the realization of a market scenario, the prepayment rate at each point in time is deterministically given.

Option-value models

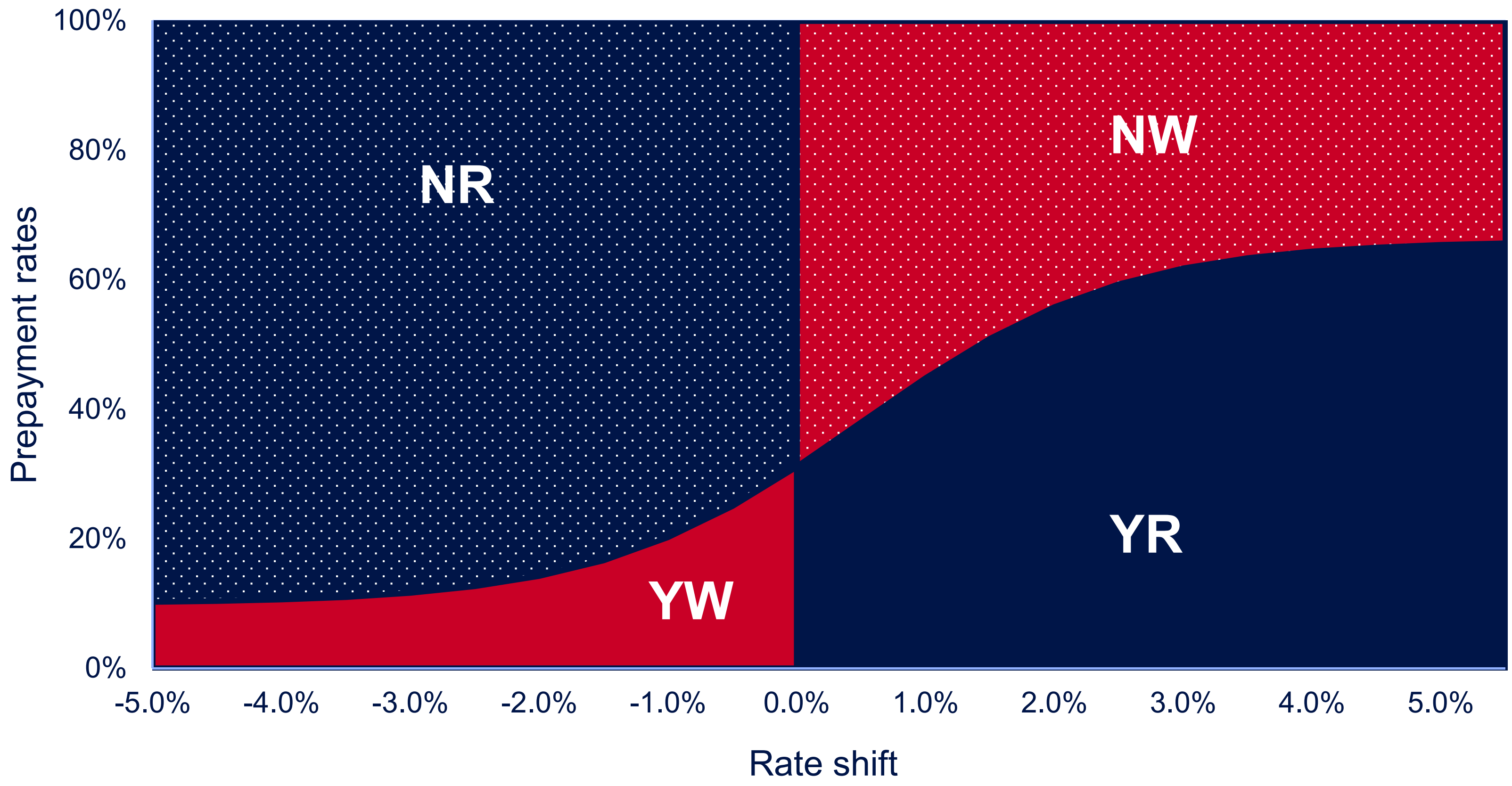
In this class of models early exercise occurs depending on the comparison between the redemption value of the contract and its continuation value. In order to take into account deviations from rationality, exercise constraints or frictions must be introduced. The existence of a baseline prepayment rate, due to exogenous reasons, is often assumed.

Intensity models

Factors possibly inducing the exercise of a prepayment option are not simulated directly: investor behavior is modeled by assuming a random prepayment rate and specifying the dynamics of a intensity function. In order to capture the dependency on market conditions, a non-zero correlation between the intensity process and other risk factors is introduced.

Rationality map

Sub-optimal exercise strategy



Exercise decision:

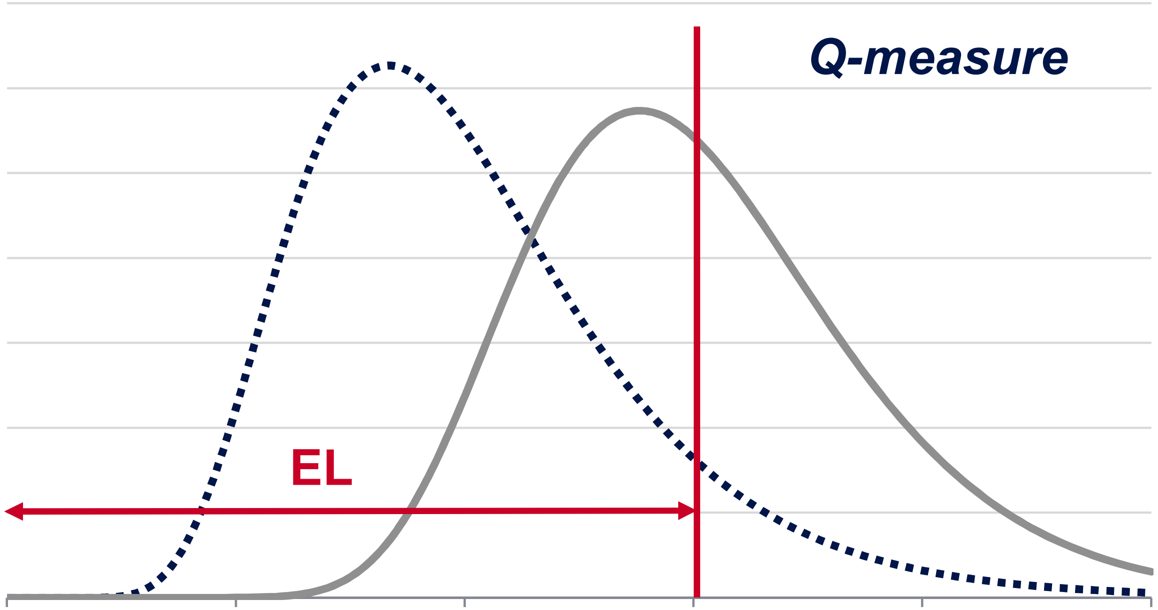
- Y (yes)
- N (no)

Financial convenience:

- R (right)
- W (wrong)

Risk-neutral pricing

Replication approach

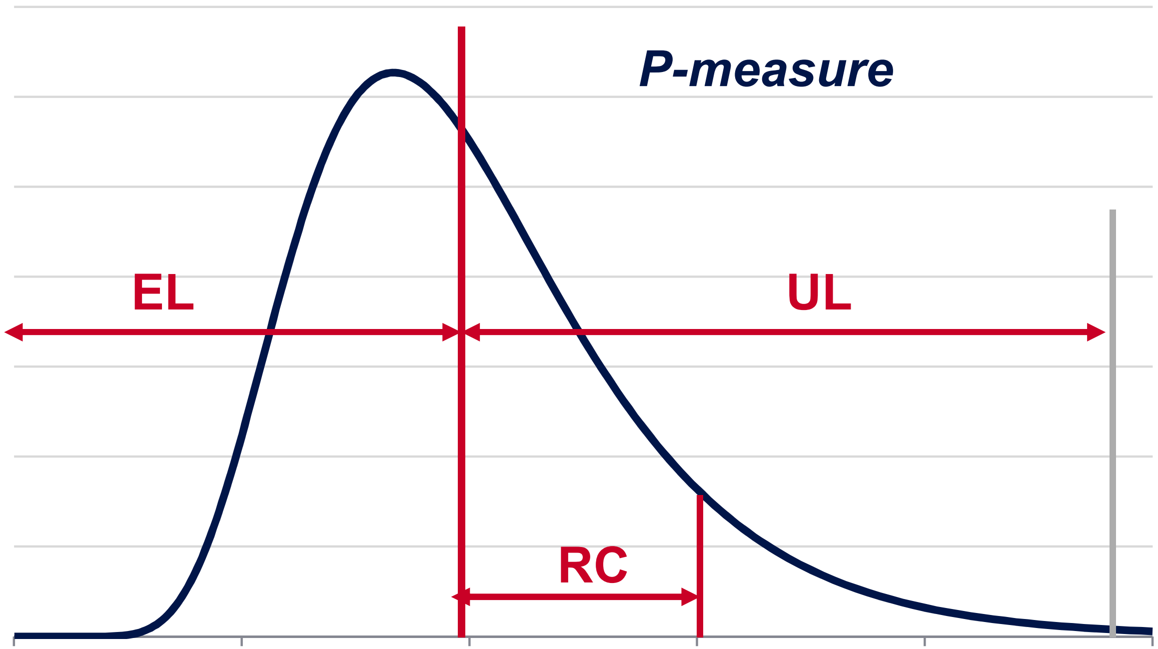


$$\boxed{V = \mathbb{E}_X^Q\left[\Psi\right]}$$

- The price of an instrument equals the cost of a self-financing hedging strategy
- **Risk premium is implied by market quotes** and prices computed by simply taking expectations under risk-neutral probabilities
- Hedging is often unfeasible since instruments in the replication portfolio are not traded or liquid
- Risk can only be diversified in a large and granular portfolio

Risk-adjusted pricing

Risk-charge approach



$$V = \mathbb{E}_X^P\left[\Psi\right] - \mathrm{RC}$$

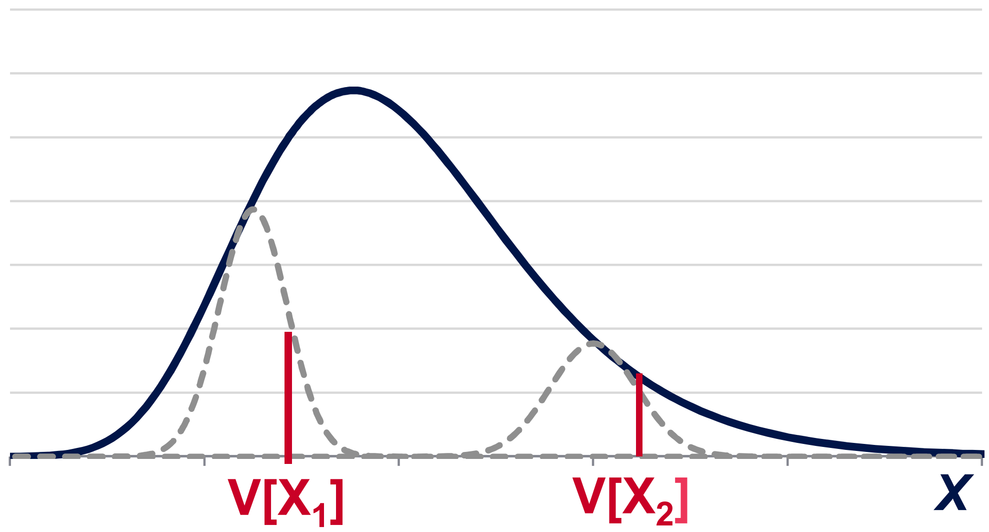
- The traditional risk-adjusted pricing approach consists in simulating the distribution of portfolio return under real-world probabilities
- Risk premium is the **cost of remunerating risky capital** needed to cover unexpected losses
- It depends on a target confidence level and hurdle rate for shareholders

Behavioral risk premium

Mixed approach

Since behavioral risk originates from a combination of market X and exogenous factors Z, we adopt a **mixed approach:**

- Risk neutral dynamics for market factors are calibrated from market quotes
- Exogenous factor dynamics are calibrated on historical basis



The price V(t) of a generic payoff Ψ is given by

$$V(t) = V_E(t) - V_U(t) = \mathbb{E}_X^Q \Big[\underbrace{\mathbb{E}_{Z,\tau}^P \big[\Psi | \vec{X} \big]}_{\text{expected}} \;\&\; \underbrace{- k \cdot \Phi_{Z,\tau}^P \big[\Psi | \vec{X} \big]}_{\text{conditional unexpected}} \Big] \quad \text{cash flows}$$

Behavioral risk

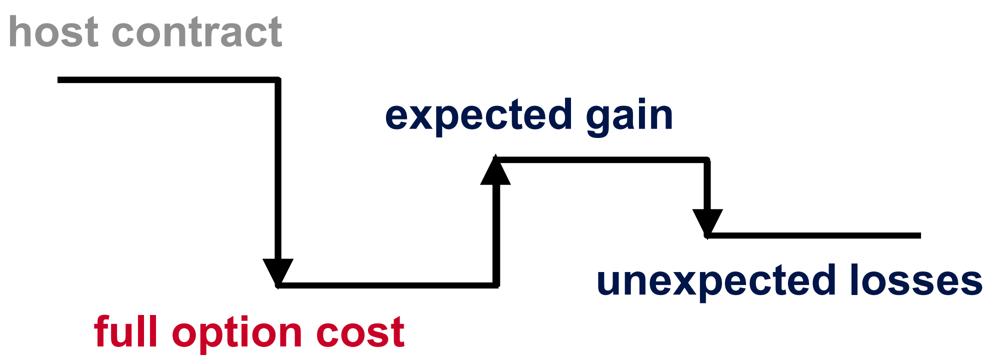
Mathematical framework

Behavioral risk adjustments (βVA)

Adjusting price…

- We can define behavioral-value adjustment (βVA) as

$$V(t) = \underbrace{\overbrace{V_H(t)}^{\text{host contract}} - \overbrace{\mathrm{OVA}(t)}^{\text{option adj.}}}_{\text{rational price } V_{\text{sup}}} + \overbrace{\beta\mathrm{VA}(t)}^{\text{behavioral adj.}}$$



- Behavioral-value adjustments can be split into two components having opposite sign

$$\beta\mathrm{VA}(t) = \beta\mathrm{VA}_E(t) - \beta\mathrm{VA}_U(t) \qquad \begin{cases} \beta\mathrm{VA}_E(t) = V_E(t) - V_{\text{sup}}(t) \\ \beta\mathrm{VA}_U(t) = V_U(t) \end{cases}$$

Parallel with credit risk modeling

Comparable items

Credit Risk

- Defaultable asset
- Defaultable counterpart
- Default event
- Debtor creditworthiness

Behavioral Risk

- Early-exercise option
- Option holder
- Option exercise
- Investor behavior

Mathematical framework (1/2)

Microstructural approach

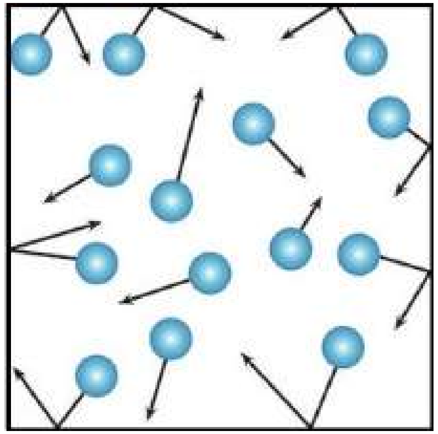
- **Microstructural approach**. For each *p*-th contract and *i*-th investor, the marginal probability of option exercise is a function of a set of market and exogenous factors (X, Z)

$$Q^{ip}(t) = R\left(\vec{X}(t), \vec{Z}(t), \vec{\theta}\right)$$

- **Market factors** affect both contractual payments and exercise decisions

- **Individual exogenous factors** are specified for all investors (each one having a systemic and an idiosyncratic component), like in the Vasicek model for credit risk

$$Z^i(t) = \rho \cdot \xi^0(t) + \sqrt{1-\rho^2} \cdot \ \xi^i(t)$$



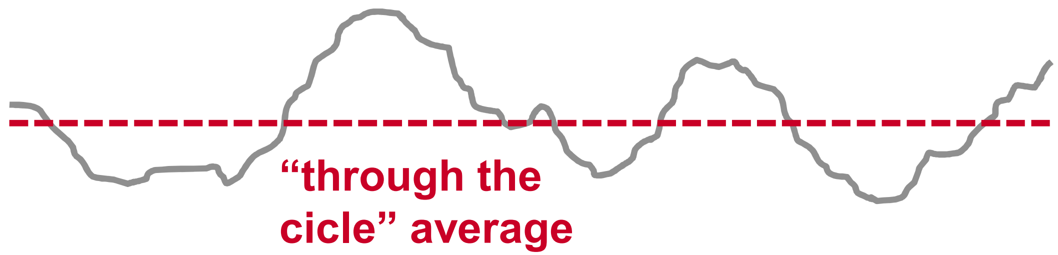
$$PV = nRT$$

Mathematical framework (2/2)

Additional hypothesis

- **Long term averaging**. We assume that the effect of exogenous factors tends to cancel out over a long period of time

$$\mathbb{E}^P\left[Z_\infty\right]=0, \qquad \mathbb{V}^P\left[Z_\infty\right]=1$$



- **Conditional independence**. Subject to the realization of a macro-scenario (X, Z), prepayment decisions are taken independently by different investors for each contract type.

- **Coherent risk measure** (such as Expected Shortfall), linked to the capital absorption needed to compensate for unexpected losses through the entire life of the contract. If the distribution is not excessively skewed we can choose

$$\Phi^P\left(\Psi|X\right)=\chi_q\cdot\sqrt{\mathbb{V}\left[\Psi|\,X\right]}$$

General payoffs

Main payoff

- Single contract discounted payoff, depending on exercise time τ

cash flows **no exercise** **&** **early exercise**

$$\Psi = \overbrace{\sum_{k=1}^{T} D_k \cdot C_k \cdot \mathbb{I}(\tau > t_k)}^{\text{no exercise}} + \overbrace{\sum_{k=1}^{T} D_k \cdot E_k \cdot \mathbb{I}(\tau = t_k)}^{\text{early exercise}}$$

- General formula for discounted portfolio payoff of instruments with embedded options

number of investors **number of contracts**

$$\Psi = \sum_{i=1}^{N} \sum_{p=1}^{M} N^{ip} \cdot \left(\sum_{k=0}^{T} D_k \cdot M_k^p \cdot \mathbb{I}(\tau^{ip} > t_k) \right) \qquad N^p = \sum_{i=1}^{N} N^{ip}$$

number of *p*-th contracts held by the *i*-th investor

Portfolio pricing

General result

General formula for portfolio pricing

$$V(0) = \mathbb{E}_X^Q \left[\Pi_0(X) - k \cdot \chi_q \cdot \sqrt{\Pi_1(X) + \Pi_2(X)} \right]$$

$$\Pi_0(X) = \mathbb{E}_Z^P \left[\mathbb{E}_\tau^P \left[\Psi \middle| X, Z \right] \middle| X \right] = \sum_{i=1}^{N} \sum_{p=1}^{M} \sum_{k=0}^{T} L_k^{ip} \cdot \mathbb{E}_Z^P \left[S_k^{ip} \middle| X \right]$$

revised cash flow expectation

$$\Pi_1(X) = \mathbb{E}_Z^P \left[\mathbb{V}_\tau^P \left[\Psi \middle| X, Z \right] \middle| X \right] = \sum_{i=1}^{N} \sum_{p=1}^{M} \sum_{k,h=0}^{T} L_k^{ip} L_h^{ip} \cdot \mathbb{E}_Z^P \left[S_{\max(k,h)}^{ip} - S_k^{ip} S_h^{ip} \middle| X \right]$$

granularity effect

$$\Pi_2(X) = \mathbb{V}_Z^P \left[\mathbb{E}_\tau^P \left[\Psi \middle| X, Z \right] \middle| X \right] = \sum_{i,j=1}^{N} \sum_{p,q=1}^{M} \sum_{k,h=0}^{T} L_k^{ip} L_h^{jq} \cdot \mathbb{V}_Z^P \left[S_k^{ip}, S_h^{jq} \middle| X \right]$$

variance induced by exogenous factors

with $L_k^{ip} = N^{ip} \cdot D_k \cdot M_k^p$

Granularity limit

Approximations

well-diversified portfolio

$$\begin{cases} N >> 1 \\ N^{ip} \approx \frac{N^p}{N} \\ \mathbb{V}_Z^P\left[S_k^{ip}, S_h^{jq}\middle| X\right] = 0 \quad \forall i \neq j \end{cases}$$

large number of counterparts

equally-sized contracts

purely idiosyncratic exogenous factors

behavioral risk is fully diversified and no βVA_U is needed

$$V(0) \approx \mathbb{E}_X^Q\left[\Pi_0(X)\right]$$

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granular portfolio

$$\begin{cases} N >> 1 \\ N^{ip} \approx \frac{N^p}{N} \end{cases}$$

$$V(0) \approx \mathbb{E}_X^Q\left[\Pi_0(X) - k \cdot \chi_q \cdot \sqrt{\Pi_2(X)}\right]$$

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granularity indicator

$$H^p = \sum_{i=1}^{N}\left(\frac{N^{ip}}{N^p}\right)^2$$

Herfindahl-Hirschman Index

$H^p \to 0$ in the granularity limit

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Thank you for your attention!

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Appendix

A specific behavioral intensity hybrid model (BIX)

- We assume the following response function R

$$\ln\left[1 - Q_k^{ip}\right] = \underbrace{A^{ip}\left(t_k, \vec{X}_k\right)}_{\text{average responsiveness to market factors (fit)}} + \underbrace{\overbrace{B^{ip}\left(t_k, \vec{X}_k\right)}^{\text{conditional standard deviation}} \cdot Z_k^i}_{\text{error process (residuals)}}$$

- Lognormal conditional survival probabilities

$$S_k^{ip}(X, Z) = \prod_{h=1}^{k}\left[1 - Q_h^{ip}(X, Z)\right] = e^{W_k^{ip}(X,Z)}$$

- Exogenous factors are modeled by AR(1) process with parameters $\left\{\rho, \alpha, \beta, \xi_0^0, \xi_0^i = 0 \quad \forall i > 0\right\}$

$$Z_k^i = \rho \cdot \xi_k^0 + \sqrt{1-\rho^2} \cdot \; \xi_k^i$$
$$\xi_k^i = \alpha \cdot \xi_{k-1}^i + \beta \cdot \varepsilon_k^i$$

$$\begin{cases} \mathbb{E}_Z^P\left[\xi_\infty | X\right] = 0 \quad \Rightarrow \quad |\alpha| < 1 \\ \mathbb{V}_Z^P\left[\xi_\infty | X\right] = 1 \quad \Rightarrow \quad \beta = \sqrt{1-\alpha^2} \end{cases}$$

Pricing of a homogeneous portfolio (1/2)

Homogeneous portfolio pricing

$$V(0) = \mathbb{E}_X^Q \left[\Pi_0(X) - k \cdot \chi_q \cdot \sqrt{\Pi_1(X) + \Pi_2(X)} \right]$$

$$A_k^{ip} = A_k^p, \;\; B_k^{ip} = B_k^p$$

- The first two terms Π_0(X) and Π_1(X) are linear with respect to the number of contracts

$$\Pi_0(X) = \sum_{p=1}^{M} \left[L_0^p(X) + \sum_{k=1}^{T} L_k^p(X) \cdot E_k^p(X) \right]$$

$$\Pi_1(X) = \sum_{p=1}^{M} \left[H^p \cdot \sum_{k=0}^{T} L_k^p(X) \cdot I_k^p(X) \cdot E_k^p(X) \right]$$

with

$$L_k^p(X) = N^p \cdot D_k \cdot M_k^p \qquad E_k^p(X) = e^{\mu_k^p(X) + \frac{1}{2}\sigma_k^{p\,2}(X)}$$

$$I_k^p(X) = L_k^p(X) \cdot \left(1 - e^{\mu_k^p(X) + \frac{3}{2}\sigma_k^{p\,2}(X)}\right) + 2 \cdot \sum_{h=0}^{k-1} L_h^p(X) \cdot \left(1 - e^{\mu_h^p(X) + \frac{3}{2}\sigma_h^{p\,2}(X)}\right)$$

Pricing of a homogeneous portfolio (2/2)

- Since the last term Π_2 corresponds to the variance of a weighted sum of lognormal variables, we rely on Gentle's approximation which was originally developed for the pricing of Asian options within BS framework

$$\Pi_2(X) = \mathbb{V}_Z^P\left[\sum_{p=1}^{M}\sum_{k=1}^{T}\sum_{i=1}^{N} L_k^{ip}(X)\cdot S_k^{ip}(X,Z)\middle| X\right]$$
$$\approx \mathbb{V}_Z^P\left[\prod_{p=1}^{M}\prod_{k=1}^{T}\prod_{i=1}^{N} e^{L_k^{ip}(X)\cdot W_k^{ip}(X,Z)}\middle| X\right] = \mathbb{V}_Z^P\left[e^{\Omega(X,Z)}\middle| X\right]$$

$$\Pi_2(X) \approx e^{2\cdot M_\Omega(X)+\Sigma_\Omega^2(X)}\cdot\left(e^{\Sigma_\Omega^2(X)}-1\right)$$

In practice, one can simulate several market scenarios X, compute conditional values Π_0(X), Π_1(X), Π_2(X), and then apply

$$V(0) = \mathbb{E}_X^Q\left[\Pi_0(X) - k\cdot\chi_q\cdot\sqrt{\Pi_1(X)+\Pi_2(X)}\right]$$