## Review Problems

**Ex. 1** Consider the following function:

$$f(x) = e^{-x}(2x^2 + x + 1)$$

1. Calculate the domain, axis intercepts, sign of $f$
2. Calculate the limits for $x \to \pm\infty$, determine if the function has vertical and/or horizontal asymptotes
3. Calculate $f'(x)$. Determine where the function increases, decreases and its local extreme values.
4. Calculate $f''(x)$. Determine the intervals of convexity/concavity.
5. Draw the graph of the function.
6. Calculate $\inf f$, $\sup f$. Calculate $\max f$, $\min f$ (if they exist).
7. Determine the equation of the tangent to $f$ at $x = 0$.
8. (*) Consider $g$ the restriction of $f$ to the domain $D' = [0, +\infty)$. Determine $\inf g$, $\sup g$.Calculate $\max g$, $\min g$ (if they exist).

**Solution**

1. The function $f$ is the product of an exponential and a polynomial, we don't have fractions, thus the domain is $\mathbb{R}$.

   To calculate the intersection with the $x-$axis we put $f(x) = 0$ that is $e^{-x}(2x^2 + x + 1) = 0$. The first factor is never zero, so we just have to put $2x^2 + x + 1 = 0$, which never happens. So the graph of $f$ doesn't intersect the line $y = 0$.

   To calculate the intersection with the $y-$axis we put $x = 0$ and calculate $f$. $f(0) = e^0(1) = 1$ so we get the point $(0, 1)$.

   To study the sign of the function we have to find the intervals in which $f(x) \geq 0$. So $e^{-x}(2x^2 + x + 1) \geq 0$. The first factor is always positive. The second factor is a polynomial of degree 2. Since it doesn't have zeros it is always positive (it's a convex parabola which doesn't intersect the $x-$axis). So $f$ is always positive.

2. Since the domain is $\mathbb{R}$ we don't have vertical asymptotes.

   We have to calculate the limits for $x \to \pm\infty$.

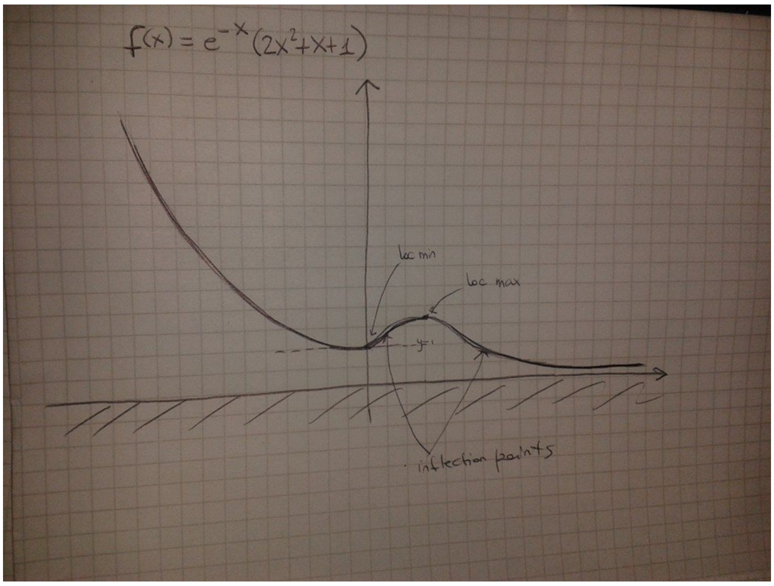
$$\lim_{x\to+\infty} e^{-x}(2x^2+x+1) = \lim_{x\to+\infty} \frac{2x^2 + x + 1}{e^x} = \lim_{x\to+\infty} \frac{4x + 1}{e^x} = \lim_{x\to+\infty} \frac{4}{e^x} = 0$$

where in the last two passages we used twice l'Hopital rule. So our function has a right horizontal asymptote $y = 0$.

$$\lim_{x \to +\infty} e^{-x}(2x^2 + x + 1) = +\infty$$

since both factors go to $+\infty$ as $x$ goes to $-\infty$. So we don't have a left horizontal asymptote.

3. $f'(x) = e^{-x}(-2x^2 + 3x)$. So the critical points are $x = 0$ and $x = 3/2$. Studying the sign of $f'$ we get that, since $e^{-x}$ is always positive, $f$ decreases in $(-\infty, 0) \cup (3/2, +\infty)$ and increases in $(0, 3/2)$. $x = 0$ is a local minimum and $x = 3/2$ a local maximum.
4. $f''(x) = e^{-x}(2x^2 - 7x + 3)$. So the function is convex in $(-\infty, 1/2) \cup (3, +\infty)$ and concave in $(1/2, 3)$.
5. graph




6. From the data we have and from the graph we deduce that $\inf f = 0$, $\sup f = +\infty$ but $f$ doesn't have a global max or min.
7. We know that $x = 0$ is a critical point for $f$ so the tangent is a horizontal line. $f(0) = 1$ and $f'(0) = 0$, so the tangent is $y = 1$.
8. If we restrict the domain to $[0, \infty)$ we get a function $g$ that coincides with $f$ in that interval. For $g$ we have again that $\inf g = 0$ and it's not a minimum. But since $x = 3/2$ now is a local maximum and since for $g$ the first derivative is positive before $3/2$ and negative after $3/2$ we have that $\sup g = \max g = g(3/2) = f(3/2) = 7e^{-3/2}$.

**Ex. 2** Consider the following function:

$$f(x) = \frac{x-1}{x^2-4}$$

1. Calculate the domain $D$, axis intercepts, sign of $f$.
2. Determine if $D$ is open or closed, its interior and exterior points, accumulation points, isolated points.
3. Calculate the limits for $x \to \pm\infty$, determine if the function has vertical and/or horizontal asymptotes
4. Calculate $f'(x)$. Determine where the function increases, decreases and its local extreme values.
5. Calculate $f''(x)$. Determine the intervals of convexity/concavity.
6. Draw the graph of the function.
7. Calculate $\inf f$, $\sup f$. Calculate $\max f$, $\min f$ (if they exist).
8. Determine the Taylor approximation of order 2 at $x = 1$.