

# MATHEMATICS - SECOND EXERCISE LESSON

MONDAY, OCTOBER 5 2016

1) Given the following set

$$X = (1, 2) \cup \left\{ \frac{1}{n} \right\}_{n \geq 1}$$

find accumulation points, boundary points, interior and exterior points, isolated points, infimum, supremum, max and min.

$$\text{sol: Acc} = [1, 2] \cup \{0\}; \text{ Interior} = (1, 2); \text{ Exterior} = \bigcup_{n \in \mathbb{N}} \left( \frac{1}{n+1}, \frac{1}{n} \right) \bigcup (-\infty, 0) \bigcup (2, +\infty)$$

$$\text{Isolated} = \left\{ \frac{1}{n} \right\}_{n \geq 1}; \inf = 0; \sup = 2$$

2) Write the equations of the lines that contain the sides of the quadrilateral  $ABCD$ , where  $A(-3; 3)$ ,  $A(-3; -1)$ ,  $A(2; -2)$ ,  $A(2; 2)$ . Verify that the quadrilateral is a parallelogram.

$$\text{sol: } x + 3 = 0; x + 5y + 8 = 0; x = 2; x + 5y - 12 = 0$$

3) Determine for which values of  $k$  the line of equation  $kx + (k + 1)y + 2 = 0$  is

- a) parallel to the  $x$ -axis;
- b) parallel to the  $y$ -axis;
- c) parallel to the line of equation  $x - 2y = 0$ ;
- d) perpendicular to the line of equation  $4x - 2y + 1 = 0$ .

$$\text{sol: a) } k = 0; \text{ b) } k = -1; \text{ c) } k = -\frac{1}{3}; \text{ d) } k = 1.$$

4) Given the line of equation

$$x + (a + 2)y - 1 = 0$$

determine  $a$  such that the line

- a) parallel to the  $x$ -axis;
- b) parallel to the  $y$ -axis;

c) passes through the origin.

sol: a) doesn't exist; b)  $a = -2$ ; c) doesn't exist.

5) Find the domain of the functions below and express it in set notation

- $f(x) = \sqrt{x}$
- $f(x) = \frac{1}{3x-2}$
- $f(x) = x^3 - 1$

sol: a)  $\mathbb{R}_+$ ; b)  $\mathbb{R} \setminus \left\{\frac{2}{3}\right\}$  c)  $\mathbb{R}$

6) Let  $A = \{0, \pi, 5, 7\}$  and  $f : A \rightarrow \mathbb{R}$ . Determine the image of

- $f(x) = x + 3$
- $f(x) = x^2 + 2$

sol: a)  $\{3, \pi + 3, 8, 10\}$ ; b)  $\{2, \pi^2 + 2, 27, 51\}$

7) Find if the following functions are injective, surjective, bijective and invertible.

- $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x - 3$
- $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 - 3$

sol: a) Injective, surjective, Bijective, Invertible b) None of them

8) Given the functions  $f(x) = \frac{1}{1-x}$  and  $g(x) = \sqrt{x-1}$  find  $f \circ f(x)$ ,  $g \circ g(x)$ ,  $f \circ g(x)$ ,  $g \circ f(x)$ , and the correspondent domains.

$$\begin{aligned} \text{sol: } f \circ f(x) &= f(f(x)) = \frac{x-1}{x} & D_{f \circ f(x)} &= \mathbb{R} \setminus \{0, 1\} \\ g \circ g(x) &= g(g(x)) = \sqrt{\sqrt{x-1}-1} & D_{g \circ g(x)} &= [2, +\infty) \\ f \circ g(x) &= f(g(x)) = \frac{1}{1-\sqrt{x-1}} & D_{f \circ g(x)} &= [1, +\infty) \setminus \{2\} \\ f \circ f(x) &= f(f(x)) = \sqrt{\frac{x}{1-x}} & D_{f \circ f(x)} &= [0, 1) \end{aligned}$$

9) In each case replace the dots with the missing function. Determine the domain of each composite function

a)  $f(x) = \dots \quad g(x) = x + 4 \quad f \circ g(x) = x;$

b)  $f(x) = \sqrt{x} \quad g(x) = \dots \quad f \circ g(x) = |x|;$

c)  $f(x) = \frac{x+1}{x} \quad g(x) = \dots \quad f \circ g(x) = x.$

sol: a)  $f(x) = x - 4, D_{f \circ g(x)} = \mathbb{R};$  b)  $g(x) = x^2, D_{f \circ g(x)} = \mathbb{R};$  b)  $g(x) = \frac{1}{x-1}, D_{f \circ g(x)} = \mathbb{R} \setminus \{1\}$

10) Determine, if it exists, the inverse function  $f^{-1}$  of the following functions and the correspondent domains

a)  $y = f(x) = 2 - 3x;$

b)  $y = \frac{x}{x+3};$

c)  $y = \frac{3}{4x-4}$

d)  $y = \frac{x+1}{x+2}$

sol: a)  $f^{-1}(x) = \frac{2-x}{3}, D_{f^{-1}} = \mathbb{R} = Im f;$  b)  $f^{-1}(x) = \frac{3x}{1-x}, D_{f^{-1}} = \mathbb{R} \setminus \{1\} = Im f;$

c)  $f^{-1}(x) = \frac{3}{4}x + 1, D_{f^{-1}} = \mathbb{R};$  c)  $f^{-1}(x) = \frac{1-2x}{x-1}, D_{f^{-1}} = \mathbb{R} \setminus \{1\};$

11) Verify that the function defined by  $y = f(x) = x^2 - 4x + 9$  is not invertible. Identify a suitable restriction of the domain in which the function is invertible and identify the inverse function.

sol: in  $[2, +\infty)$   $f$  is injective;  $f^{-1}(x) = 2 + \sqrt{x-5}$

12) Prove that the functions

$$f(x) = x^2 - x + 1, x \geq \frac{1}{2} \quad \text{and} \quad g(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}, x \geq \frac{3}{4}$$

are inverse of each other.