SOLUTIONS TO SOME EXERCISES OF APRIL 10, 2017

Ex. 2 Study the following Integral Functions

$$F(x) = \int_0^x \frac{e^{-t^2}}{\sqrt[3]{t^2-9}}\,dt$$

a) DOMAIN: recall that the integral of a continuous function is well defined, hence let us see where the integrand is continuous $\Rightarrow$:

$\dfrac{e^{-t^2}}{\sqrt[3]{t^4-9}}$ is continuous where $t^4-9 \neq 0$ (the radical is cubic, so there are no restrictions on definiteness of $\sqrt[3]{\ }$) $\Rightarrow$

$$t \neq \pm\sqrt{3}$$

The domain although is the biggest interval containing all the points x for which the integral exists and is a finite number.

The integrand function is continuous for $-\sqrt{3} < t < \sqrt{3}$, this means then that in $-\sqrt{3} < x < \sqrt{3}$ the integral is finite as in this interval I can apply the FUNDAMENTAL THEOREM OF CALCULUS and say that $F(x)$ is continuous in $[0, x]$ with $0 < x < \sqrt{3}$ and in $[x, 0]$ with $-\sqrt{3} < x < 0$

Notice that the integrand function is an EVEN FUNCTION, i.e.

$$f(-t) = \frac{e^{-(-t)^2}}{\sqrt[3]{(-t)^4-9}} = \frac{e^{-t^2}}{\sqrt[3]{t^4-9}} = f(t) \quad \forall t \neq \pm\sqrt{3}$$

so it is sufficient to study the function $F(x)$ for $x > 0$ and then exploit this symmetry

Now we must analyze the integrability in the problematic points, $x=\sqrt{3}$, this implies dealing with an IMPROPER INTEGRAL

(2)

I will apply the comparison test to prove convergence of the improper integral $\int_0^{\sqrt{3}} \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt$:

$$\frac{e^{-t^2}}{\sqrt[3]{t^4-9}} < \frac{1}{\sqrt[3]{t^4-9}} = \frac{1}{\sqrt[3]{(t^2-3)(t^2+3)}} < \frac{1}{\sqrt[3]{t^2-3}} =$$

$e^{-t^2} = \frac{1}{e^{t^2}} < 1$

$t^2+3 \geq 3 > 1 \Rightarrow$ if you pass to the reciprocals the inequalities switch $\Rightarrow \frac{1}{t^2+3} \leq \frac{1}{3} < 1$ and $\frac{1}{\sqrt[4]{t^2+3}} < 1$

$$= \frac{1}{\sqrt[3]{(t-\sqrt{3})(t+\sqrt{3})}} < \frac{1}{\sqrt[3]{t-\sqrt{3}}}$$

we are working in a neighbourhood of $\sqrt{3}$ $\Rightarrow$ t is positive, hence as before $t+\sqrt{3} \geq \sqrt{3} > 1 \Rightarrow \frac{1}{t+\sqrt{3}} < 1$ and $\frac{1}{\sqrt[3]{t+\sqrt{3}}} < 1$

hence, if $\int_0^{\sqrt{3}} \frac{dt}{\sqrt[3]{t-\sqrt{3}}}$ is FINITE, by comparison, also $\int_0^{\sqrt{3}} \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt$ is FINITE:

$$\int \frac{dt}{\sqrt[3]{t-\sqrt{3}}} = \begin{bmatrix} u = t-\sqrt{3} \\ du = dt \end{bmatrix} = \int \frac{du}{\sqrt[3]{u}} = \frac{3}{2} u^{\frac{2}{3}} + c = \frac{3}{2}\sqrt[3]{(t-\sqrt{3})^2} + c$$

$$\rightarrow \int_0^{\sqrt{3}} \frac{dt}{\sqrt[3]{t-\sqrt{3}}} = \frac{3}{2}\left[\sqrt[3]{(t-\sqrt{3})^2}\right]_0^{\sqrt{3}} = -\frac{3}{2}\sqrt[3]{3} < +\infty$$

hence also the improper integral $\int_0^{\sqrt{3}} \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt$ is FINITE and the integral function is defined also in $x=\sqrt{3}$. By symmetry, the integral function is defined also in $x=-\sqrt{3}$

Hence the domain of $F(x)$ is all $\mathbb{R}$ i.e. (3)

$$D_{F(x)} = \mathbb{R}$$

b) INTERSECTION WITH AXES: the function $y = F(x)$ intersects the axis y in $x=0$ as the integration extremes are equal $F(0) = \int_0^0 \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt = 0$

c) LIMITS: we must study

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \int_0^x \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt = \int_0^{\infty} \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt$$

and this is an IMPROPER INTEGRAL. We have already seen the behavior of F in correspondence of $x = \pm\sqrt{3}$. So it is sufficient to see what happens to

$$\int_{\sqrt{3}}^{+\infty} \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt \qquad \text{Recall that } e^{-t^2} \le 1 \text{ and that } \sqrt[3]{t^4-9} \underset{\text{for } t\to\infty}{\overset{\text{behaves}}{\sim}} t^{4/3}$$

hence we will study:

$$\int_{\sqrt{3}}^{+\infty} \frac{e^{-t^2}}{t^{4/3}}\,dt \le \int_{\sqrt{3}}^{\infty} \frac{1}{t^{4/3}}\,dt = \lim_{x\to+\infty} \int_{\sqrt{3}}^{x} \frac{1}{t^{4/3}}\,dt =$$

$$= \lim_{x\to+\infty} \frac{-3}{t^{1/3}}\Bigg|_{\sqrt{3}}^{x} = \lim_{x\to+\infty}\left[\underbrace{\frac{-3}{\sqrt[3]{x}}}_{\to 0} - \frac{-3}{(\sqrt{3})^{1/3}}\right] = \frac{3}{3^{1/6}} = 3^{5/6} < +\infty$$

$\Rightarrow$ it is CONVERGENT $\Rightarrow$ putting all together

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \int_0^x \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt = \lim_{x\to+\infty}\left[\int_0^{\sqrt{3}} \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt + \int_{\sqrt{3}}^{x} \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt\right] = L$$

where $L \in \mathbb{R}$ is a finite Real number

NOTICE that this L is negative

I don't know who L is but I know that it is

By the properties of the integral $\left(\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt\right)$ we have that the integral function $F(x)$ is an ODD function, i.e. $F(-x) = -F(x) \Longrightarrow$ (4)

$$\lim_{x\to-\infty}\int_0^x \frac{e^{-t^2}}{\sqrt[3]{t^4-9}}\,dt = -L > 0 \quad \text{as } L<0$$

the lines $y=L$ and $y=-L$ are HORIZONTAL ASYMPTOTES

d) FIRST DERIVATIVE: by the fundamental theorem of calculus:

$$F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x) = \frac{e^{-x^2}}{\sqrt[3]{x^4-9}}$$

for $x \neq \pm\sqrt{3}$; we must analyze the sign of this first derivative:

$$F'(x) > 0 \Longleftrightarrow \frac{e^{-x^2}}{\sqrt[3]{x^4-9}} > 0 \Rightarrow \begin{matrix} e^{-x^2} > 0 \ \forall x\in\mathbb{R} \\ \sqrt[3]{x^4-9} > 0 \end{matrix}$$

$$\Longleftrightarrow x^4-9>0 \Longleftrightarrow x<-\sqrt{3} \ \vee \ x>\sqrt{3}$$

Hence:

$F'(x) > 0$ if $x < \sqrt{3}$ or $x > \sqrt{3}$

$F'(x) < 0$ if $-\sqrt{3} < x < \sqrt{3}$

Hence:

$F(x)$ is increasing for $x<-\sqrt{3}$ or $x>\sqrt{3}$

$F(x)$ is decreasing for $-\sqrt{3}<x<\sqrt{3}$

The points $x=-\sqrt{3}$ and $x=\sqrt{3}$ are points of non differentiability. Let us study right and left limit of $F'(x)$ for $x=\sqrt{3}$

$$\lim_{x\to\sqrt{3}^-} F'(x) = \lim_{x\to\sqrt{3}^-} \frac{e^{-x^2}}{\sqrt[3]{x^4-9}} = \left[\frac{e^{-1}}{0^-}\right] = -\infty$$ 5

$$\lim_{x\to\sqrt{3}^+} F'(x) = \lim_{x\to\sqrt{3}^+} \frac{e^{-x^2}}{\sqrt[3]{x^4-9}} = \frac{e^{-1}}{0^+} = +\infty$$

hence $x=\sqrt{3}$ is a point with a CUSP and according to the sign of the first derivatives we may say that $x=\sqrt{3}$ is a non differentiable local min.
($x=-\sqrt{3}$ by ODD symmetry is a LOCAL MAX)

e) SECOND DERIVATIVE:

$$F''(x) = f'(x) = \frac{e^{-x^2}(-2x)\sqrt[3]{x^4-9} - e^{-x^2}\dfrac{4x^3}{3\sqrt[3]{(x^4-9)^2}}}{(\sqrt[3]{(x^4-1)^2}} =$$

$$= \frac{2e^{-x^2}x\,(-3(x^4-9)-2x^2)}{\sqrt[3]{(x^4-9)^4}} = \frac{2e^{-x^2}x\,(-3x^4+27-2x^2)}{\sqrt[3]{(x^4-9)^4}}$$

$$= \frac{2e^{-x^2}x(-3x^4-2x^2+27)}{\sqrt[3]{(x^4-9)^4}} \qquad x \neq \pm\sqrt{3}$$

Hence the sign of the second derivative is given by the combination of signs of x and $(-3x^4-2x^2+27)$

$\rightarrow x>0$

$\rightarrow -3x^4-2x^2+27>0 \iff 3x^4+2x^2-27<0$ set $t=x^2$

$\iff 3t^2+2t-27<0 \qquad t_{1,2} = \frac{-1\pm\sqrt{1+81}}{3} = \frac{-1\pm\sqrt{82}}{3} \rightarrow$

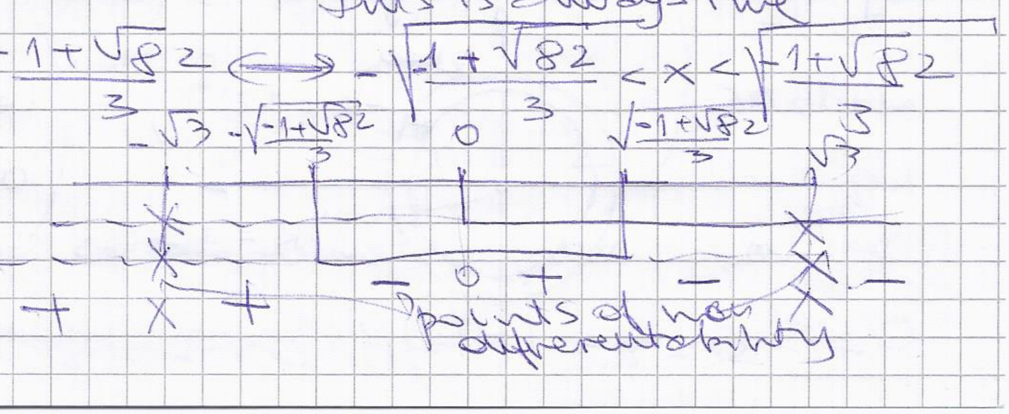
$\frac{-1-\sqrt{82}}{3} < t < \frac{-1+\sqrt{82}}{3} \iff \frac{-1-\sqrt{82}}{3} < x^2 < \frac{-1+\sqrt{82}}{3}$

this is always true

and $\rightarrow \quad x^2 < \frac{-1+\sqrt{82}}{3} \iff -\sqrt{\frac{-1+\sqrt{82}}{3}} < x < \sqrt{\frac{-1+\sqrt{82}}{3}}$

Hence

$F''(x)>0$ in



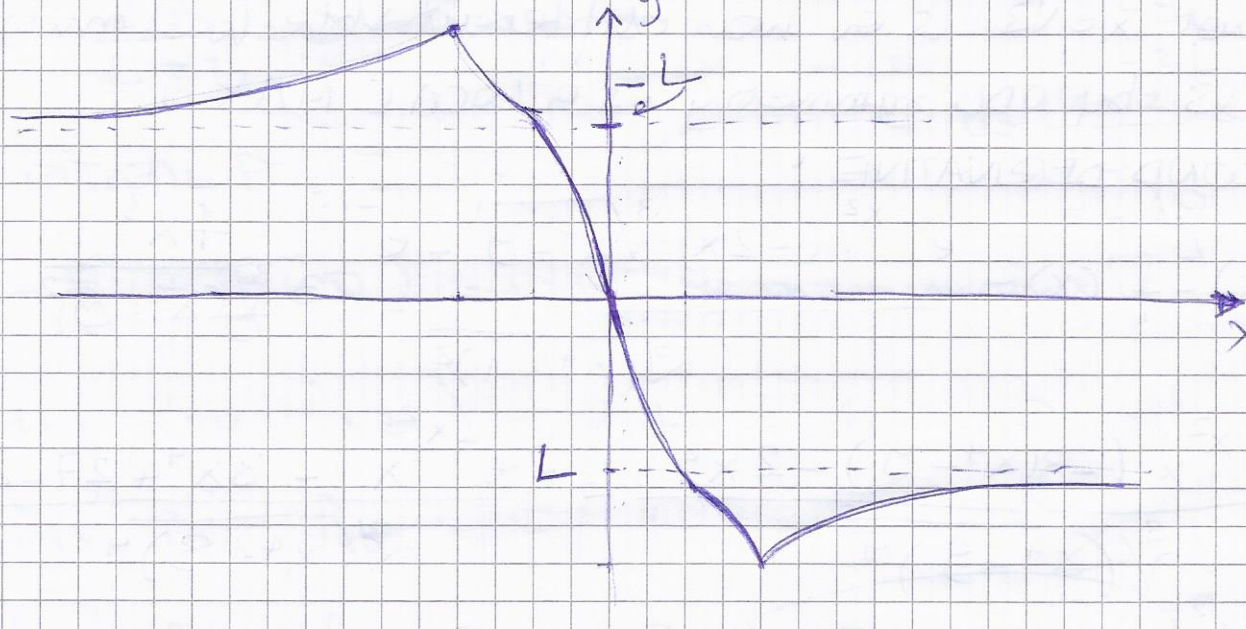
$F''(x)$ in $x < -\sqrt{3} \lor -\sqrt{3} < x < -\sqrt{\frac{-1+\sqrt{82}}{3}} \lor$

$\lor\ 0 < x < \sqrt{\frac{-1+\sqrt{82}}{3}} \longrightarrow \cup$

$F''(x) = 0$ in $x = 0$ → INFLECTION POINTS
in $x = \pm\sqrt{-1+\sqrt{82}}/3$

$F''(x)$ in the complementary of $\longrightarrow \cap$

f) GRAPH:



EX.5 Identify the LEVEL CURVES of the following functions

$f(x,y) = \sqrt{9 - x^2 - y^2} \longrightarrow \forall k \in \mathbb{R} \longrightarrow$

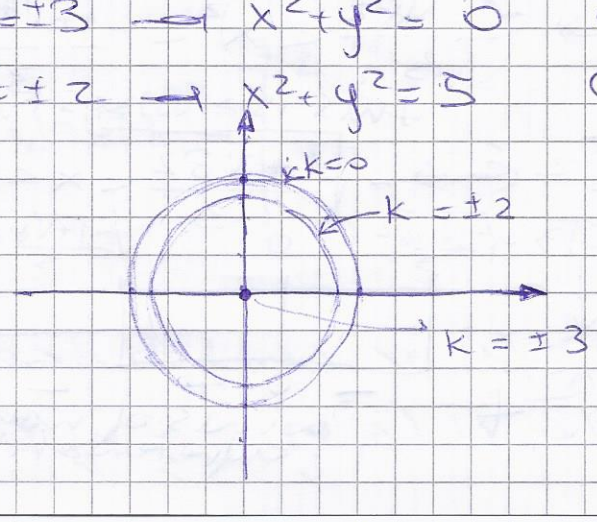
$\sqrt{9 - x^2 - y^2} = k \Longleftrightarrow 9 - x^2 - y^2 = k^2 \longrightarrow x^2 + y^2 = 9 - k^2$

and this equation represents a circumference of center $(0,0)$ and radius $r = \sqrt{9-k^2}$

if $k = 0 \rightarrow x^2 + y^2 = 9$ Center $(0,0)$ and $r = 3$

if $k = \pm 3 \rightarrow x^2 + y^2 = 0$ Center $(0,0)$ and $r = 0$

if $k = \pm 2 \rightarrow x^2 + y^2 = 5$ Center $(0,0)$ and $r = \sqrt{5}$



Level curves are all circumferences centered in $(0,0)$

6) $f(x,y) = e^{y/x} \longrightarrow e^{y/x} = k \;,\; k>0 \longrightarrow$ 7

$\frac{y}{x} = \ln k \longrightarrow y = (\ln k)\cdot x$ these are lines passing through the origin:

if $k=1 \rightarrow \ln 1 = 0 \rightarrow y = 0$

if $k=e \rightarrow \ln e = 1 \rightarrow y = x$

if $k=2 \rightarrow y = (\ln 2)\cdot x$

if $k=4 \rightarrow y = (\ln 4)\cdot x$

7) FIND CRITICAL POINTS OF THE FOLLOWING FUNCTIONS AND DETERMINE WHETHER THEY ARE MAX, MIN OR SADDLE PTS.

$$f(x,y,z) = x^2 - y^2 + z^2 + 2xy - 4yz + x + y$$

$$\begin{cases} f_x = 2x + 2y + 1 = 0 \longrightarrow x = -\frac{(2y+1)}{2} = -\frac{\cancel{2}\cdot\frac{1}{\cancel{2}}z+1}{2} = -\frac{(z+1)}{2} \\ f_y = -2y + 2x - 4z + 1 = 0 \\ f_z = 2z - 4y = 0 \longrightarrow 4y = 2z \Rightarrow y = \frac{1}{2}z \end{cases}$$

$$\Rightarrow \begin{cases} x = -\frac{z+1}{2} \\ y = \frac{1}{2}z \\ -\cancel{2}\cdot\frac{1}{\cancel{2}}z + \cancel{2}\cdot\left(-\frac{z+1}{\cancel{2}}\right) - 4z + 1 = 0 \end{cases} \quad \begin{cases} x = -\frac{z+1}{2} \\ y = \frac{1}{2}z \\ -z - z - \cancel{1} - 4z + \cancel{1} = 0 \end{cases}$$

$$\begin{cases} x = -\frac{z+1}{2} \\ y = \frac{1}{2}z \\ -6z = 0 \end{cases} \quad \begin{cases} x = -\frac{1}{2} \\ y = 0 \\ z = 0 \end{cases}$$ CRITICAL POINT

Let us calculate the Hessian matrix $\longrightarrow$

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & -2 & -4 \\ 0 & -4 & 2 \end{pmatrix}$$

let us analyze the sign of the leading principal minors

$$|H_1| = 2 > 0 \quad |H_2| = \begin{vmatrix} 2 & 2 \\ 2 & -2 \end{vmatrix} = -8 < 0 \quad |H_3| = |H| = -8 - 32 - 8 < 0$$

The Hessian defines an INDEFINITE QUADRATIC FORM hence $(-\frac{1}{2}, 0, 0)$ is a SADDLE POINT

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EX 8 $\quad f(x,y,z) = 2x^2 - y^2 + xy + 2x - 2y + \ln(1+z^2)$

$$\begin{cases} f_x = 4x + y + 2 = 0 \\ f_y = -2y + x - 2 = 0 \\ f_z = \dfrac{2z}{1+z^2} = 0 \rightarrow z = 0 \end{cases} \qquad \begin{cases} 4x + y = -2 \\ x - 2y = 2 \end{cases}$$

$$x = \frac{\begin{vmatrix} -2 & 1 \\ 2 & -2 \end{vmatrix}}{\begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{4-2}{-8-1} = \frac{2}{-9} = -\frac{2}{9}$$

$$y = \frac{\begin{vmatrix} 4 & -2 \\ 1 & 2 \end{vmatrix}}{-9} = \frac{8+2}{-9} = -\frac{10}{9}$$

$\Rightarrow$ CRITICAL POINT IS

$(-\frac{2}{9}, -\frac{10}{9}, 0) \Rightarrow$ the Hessian matrix is

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & \dfrac{2(1+z^2) - 4z^2}{(1+z^2)^2} \end{pmatrix}$$

$$\Rightarrow H(-\tfrac{2}{9}, -\tfrac{10}{9}, 0) = \begin{pmatrix} 4 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ the leading principal minors are

$$|H_1| = 4 \qquad |H_2| = \begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix} = -8 - 1 = -9 < 0$$

$$|H_3| = |H| = \begin{vmatrix} 4 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix} = 2 \cdot (-8-1) = -18 < 0$$

The Hessian defines an INDEFINITE QUADRATIC FORM $\Rightarrow (-\frac{2}{9}, -\frac{10}{9}, 0)$ is a SADDLE POINT