SOLUTIONS TO SOME EXERCISES OF MARCH 28, 2017

EX 8 $\int (x+1)^2 \cos x \, dx = (x+1)^2 \sin x - \int 2(x+1) \sin x \, dx$

INTEGRATION BY PARTS $f(x) = (x+1)^2 \Rightarrow f'(x) = 2(x+1)$

$g(x) = \sin x \Rightarrow g'(x) = \cos x$

with the same logic of we apply integration by parts also here

$= (x+1)^2 \sin x - 2\left[-(x+1)\cos x + \int \cos x \, dx\right]$

$= (x+1)^2 \sin x - 2\left[-(x+1)\cos x + \sin x\right] + c$

$= (x+1)^2 \sin x + 2(x+1)\cos x - 2\sin x + c$

EX. 9 $\int \cos x \log(\sin x) \, dx = \begin{bmatrix} u = \sin x \\ du = \cos x \, dx \end{bmatrix} = \int \log u \, du =$

SUBSTITUTION

BY PARTS

$= u \log u - \int \cancel{u} \cdot \frac{1}{\cancel{u}} \, du = u \log u - u + c = \sin x \log \sin x -$

$- \sin x + c = \sin x (\log \sin x - 1) + c$

EX. 10 $\int_0^1 \frac{x^3}{\sqrt{4+x^2}} dx = \int_0^1 \frac{x \cdot x^2}{\sqrt{4+x^2}} dx$ I suggest you solving the indefinite integral $\int \frac{x^3}{\sqrt{4+x^2}} dx$ and then apply the Fundamental theorem of Calculus

$\int \frac{x^3}{\sqrt{4+x^2}} dx = \int \frac{x^2 \cdot x}{\sqrt{4+x^2}} dx = \begin{bmatrix} 4+x^2 = u \Rightarrow x^2 = u-4 \\ 2x\,dx = du \Rightarrow x\,dx = \frac{1}{2} du \end{bmatrix}$

SUBSTITUTION

$= \frac{1}{2} \int \frac{u-4}{\sqrt{u}} du = \frac{1}{2}\left[\int \frac{u}{\sqrt{u}} du - 4 \int \frac{1}{\sqrt{u}} du\right] = \frac{1}{2}\left[\int \sqrt{u} \, du -\right.$

$\left. - 4 \int \frac{1}{\sqrt{u}} du\right] = \frac{1}{2}\left[\frac{2}{3} u^{3/2} - 4 \cdot \sqrt{u} \cdot 2\right] + c = \frac{1}{3} u^{3/2} - 4\sqrt{u} + c$

$= \frac{1}{3}(4+x^2)^{3/2} - 4\sqrt{4+x^2} + c \Rightarrow$

$\int_0^1 \frac{x^3}{\sqrt{4+x^2}} dx = \left[\frac{1}{3}(4+x^2)^{3/2} - 4\sqrt{4+x^2}\right]_0^1 = \frac{1}{3}\sqrt{5^3} - 4\sqrt{5} - \frac{1}{3}8 + 8$

EX. 11 $\int_1^2 x^4 \ln^2(x)\,dx$ solve before the indefinite integral

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$$\int x^4 \ln^2(x)\,dx = \frac{x^5}{5}\ln^2(x) - \int \frac{x^5}{5}\cdot 2\ln(x)\cdot\frac{1}{x}\,dx$$

BY PARTS

$f(x) = \ln^2(x) \to f'(x) = 2\ln(x)\cdot\frac{1}{x}$

$g(x) = \frac{x^5}{5} \to g'(x) = x^4$

$$= \frac{x^5}{5}\ln^2(x) - \frac{2}{5}\underbrace{\int x^4 \ln x\,dx}_{\text{BY PARTS}} = \frac{x^5}{5}\ln^2(x) - \frac{2}{5}\left[\frac{x^5}{5}\ln x - \int \frac{x^5}{5}\cdot\frac{1}{x}\,dx\right] = \frac{x^5}{5}\ln^2(x) - \frac{2}{25}x^5\ln x - \frac{2}{25}\int x^4\,dx =$$

$$= \frac{x^5}{5}\ln^2(x) - \frac{2}{25}x^5\ln x - \frac{2}{25}\frac{x^5}{5} + C$$

$$= \frac{x^5}{5}\ln^2(x) - \frac{2}{25}x^5\ln x - \frac{2}{125}x^5 + C \Rightarrow$$

$$\int_1^2 x^4\ln^2(x)\,dx = \left[\frac{x^5}{5}\ln^2(x) - \frac{2}{25}x^5\ln x - \frac{2}{125}x^5\right]_1^2$$

$$= \frac{32}{5}\ln^2(2) - \frac{64}{25}\ln 2 - \frac{64}{125} + \frac{2}{125}$$

EX. 12 $\int \cos(\sqrt{x})\,dx = \left[\begin{array}{l}\sqrt{x} = t \to x = t^2\\ dx = 2t\,dt\end{array}\right] = \int \cos(t)\cdot 2t\,dt$

BY SUBSTITUTION

$$= 2\int t\cos t\,dt = t\sin t - \int \sin t\,dt = t\sin t + \cos t + C$$

BY PARTS

$$= \sqrt{x}\sin\sqrt{x} - \cos\sqrt{x} + C$$

EX. 16 $\int \frac{\sqrt{x^2-1}}{x}\,dx = \left[\begin{array}{l}x = \sec t = \frac{1}{\cos t}\\ dx = -\frac{-\sin t}{\cos^2 t} = \tan t \sec t\end{array}\right] =$

TRIGONOMETRIC SUBSTITUTION

$$= \int \frac{\sqrt{\sec^2 t - 1}}{\cancel{\sec t}}\tan t\cdot\cancel{\sec t}\,dt = \int \tan^2 t\,dt = \int \frac{\sin^2 t}{\cos^2 t}\,dt$$

$$= \int \frac{1-\cos^2 t}{\cos^2 t}\,dt = \int \frac{1}{\cos^2 t}\,dt - \int dt = \tan t - t + C$$

$$= \sqrt{x^2-1} - \operatorname{arcsec} x + C$$

EX. 17 $\int \sqrt{e^x-1}\,dx = \left[\begin{array}{l} e^x-1=u \quad e^x=u+1 \\ x=\ln(u+1) \\ dx=\frac{du}{u+1} \end{array}\right] = \int \frac{\sqrt{u}}{u+1}\,du$

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$= \left[\begin{array}{l} u=t^2 \\ du=2t\,dt \end{array}\right] = \int \frac{2t^2}{t^2+1}\,dt = 2\left[\int \frac{t^2+1}{t^2+1}\,dt - \int \frac{dt}{t^2+1}\right]$

$= 2\left[t - \arctan t\right] + c = 2\left[\sqrt{e^x-1} - \arctan\sqrt{e^x-1}\right] + c$

EX. 18 $\int_1^2 \frac{\ln(x)}{\sqrt{x}}\,dx$ let us solve the indefinite integral

$\int \frac{\ln(x)}{\sqrt{x}}\,dx = 2\sqrt{x}\ln(x) - 2\int \sqrt{x}\cdot\frac{1}{x}\,dx = 2\sqrt{x}\ln x$

BY PARTS $\quad - 2\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\ln x - 4\sqrt{x} + c$

$\int_1^2 \frac{\ln(x)}{\sqrt{x}}\,dx = \left[2\sqrt{x}\ln x - 4\sqrt{x}\right]_1^2 = 2\sqrt{2}\ln 2 - 4\sqrt{2} + 4$

EX. 21 $\int_0^{\pi/2} \sin(2x)\,dx$ let us solve the indefinite integral

$\int \sin(2x)\,dx = \left[\begin{array}{l} 2x=u \\ x=\frac{u}{2} \quad dx=\frac{1}{2}du \end{array}\right] = \frac{1}{2}\int \sin u\,du =$

$= -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(2x) + c \rightarrow$

$\int_0^{\pi/2} \sin(2x)\,dx = \left[-\frac{1}{2}\cos(2x)\right]_0^{\pi/2} = -\frac{1}{2}\left(\cos\pi - \cos 0\right)$

$= -\frac{1}{2}(-1-1) = 1$

EX. 22 $\int \sin^2(\pi x)\cos^5(\pi x)\,dx = \int \sin^2(\pi x)\cos^4(\pi x)\cos x\,dx$

$= \int \sin^2(\pi x)(1-\sin^2\pi x)^2\cos x\,dx = \left[\begin{array}{l} u=\sin x \\ du=\cos x\,dx \end{array}\right]$

$= \int u^2(1-u^2)^2\,du = \int u^2(1-2u^2+u^4)\,du = \int (u^2-2u^4+u^6)\,du$

$= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + c = \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + c$

EX. 23 $\displaystyle\int \frac{3\sin x}{9\cos^2 x - 18\cos x + 10}\,dx = \left[\begin{array}{l} 3\cos x = u \\ du = -3\sin x\,dx \end{array}\right]$

$\displaystyle = -\int \frac{du}{u^2-6u+10} = -\int \frac{du}{(u-3)^2+1} = \left[\begin{array}{l} u-3=t \\ du = dt \end{array}\right] = -\int \frac{dt}{t^2+1}$

the polynomial
$u^2-6u+10$ has $\Delta < 0$
→ this quadratic expression
is irreducible. We will complete the
square with u^2-6u →;

$$u^2-6u+9-9+10 =$$
$$=(u-3)^2+1$$

$\displaystyle = -\arctan t + c = -\arctan(u-3)+c = -\arctan(3\cos x - 3) + c$

EX. 24 $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}}\,dx =$

$\displaystyle = \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x}{\sqrt{1-x^2}}\,dx = \text{arcsen } x$

$\displaystyle - \int \frac{x}{\sqrt{1-x^2}}\,dx = \left[\begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \end{array}\right] = \arcsin x + \frac{1}{2}\int \frac{du}{\sqrt{u}} =$

$= \arcsin x + \sqrt{u} + c = \arcsin x + \sqrt{1-x^2} + c$

EX. 25 $\displaystyle\int_2^3 \frac{1}{x^2-1}\,dx$ solve the indefinite integral

$\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x-1)(x+1)}\,dx =$

PARTIAL FRACTIONS →:

$\displaystyle\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} =$

$\displaystyle = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} = \frac{(A+B)x + A - B}{(x-1)(x+1)}$

$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \rightarrow \begin{cases} A=-B \\ -2B=1 \end{cases} \quad A=\frac{1}{2} \quad B=-\frac{1}{2}$

$\displaystyle = \frac{1}{2}\left[\int \frac{dx}{x-1} - \int \frac{dx}{x+1}\right] = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$

EX. 26 $\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \frac{x-1}{(x+1)(x+2)}\,dx$

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let us solve the indefinite integral first

$\int \frac{x-1}{(x+1)(x+2)}\,dx =$ [PARTIAL FRACTION $\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$

$= \frac{(A+B)x+2A+B}{(x+1)(x+2)}$ $\begin{cases} A+B=1 \rightarrow A=1-B \\ 2A+B=-1 \quad 2(1-B)+B=-1 \end{cases}$

$\Rightarrow 2-2B+B=-1 \Rightarrow -B=-3 \quad B=3 \Rightarrow A=-2$]

$= -\int \frac{2\,dx}{x+1} + \int \frac{3\,dx}{x+2} = -2\log|x+1| + 3\log|x+2| + c$

$\longrightarrow \int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \log\frac{27}{4} - \log 8$

$= \log\left(\frac{|x+2|^3}{(x+1)^2}\right) + c$

EX 27 $\int \frac{x+4}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx$

this quadratic polynomial is IRREDUCIBLE as $\Delta < 0$

SUBSTITUTION $u = x^2+2x+5$ $du = (2x+2)\,dx$

complete the square with $x^2+2x \rightarrow (x^2+2x+1)-1+5$

$= \frac{1}{2}\log(x^2+2x+5) + \int \frac{3}{(x^2+2x+1)-1+5}\,dx$

$= \frac{1}{2}\log(x^2+2x+5) + \int \frac{3}{(x+1)^2+4}\,dx = \begin{bmatrix} u = x+1 \\ du = dx \end{bmatrix} =$

$= \frac{1}{2}\log(x^2+2x+5) + 3\int \frac{du}{u^2+4} = \frac{1}{2}\log(x^2+2x+5) + \frac{3}{4}\int \frac{du}{\left(\frac{u}{2}\right)^2+1}$

$= \begin{bmatrix} t = \frac{u}{2} \Rightarrow du = 2\,dt \\ dt = \frac{1}{2}\,du \end{bmatrix} = \frac{1}{2}\log(x^2+2x+5) + \frac{3}{2}\int \frac{dt}{t^2+1}$

$= \frac{1}{2}\log(x^2+2x+5) + \frac{3}{2}\arctan t + c = \frac{1}{2}\log(x^2+2x+5) +$

$+ \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + c$

EX. 28 $\displaystyle\int_3^4 \frac{x^3-2x-4}{x^3-2x}dx = \int_3^4 \frac{x^3-2x}{x^3-2x}dx - \int_3^4 \frac{4}{x^3-2x}dx$

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$\Rightarrow$ let us solve first the indefinite integral

$$\int \frac{x^3-2x-4}{x^3-2x}dx = \int dx - \int \frac{4}{x(x-\sqrt{2})(x+\sqrt{2})}dx \stackrel{*}{}$$

PARTIAL FRACTION

$$-\frac{4}{x(x-\sqrt{2})(x+\sqrt{2})} = \frac{A}{x} + \frac{B}{x-\sqrt{2}} + \frac{C}{x+\sqrt{2}} =$$

$$= \frac{A(x^2-2)+B(x^2+\sqrt{2}x)+C(x^2-\sqrt{2}x)}{x(x-\sqrt{2})(x+\sqrt{2})} =$$

$$= \frac{(A+B+C)x^2+\sqrt{2}(B-C)x-2A}{x(x-\sqrt{2})(x+\sqrt{2})} \Rightarrow \begin{cases} A+B+C=0 \\ \sqrt{2}(B-C)=0 \\ -2A=-4 \end{cases}$$

$$\begin{cases} 2+2B=0 \\ B=C \\ A=2 \end{cases} \quad \begin{cases} B=-1 \\ C=-1 \\ A=2 \end{cases} \rightarrow$$

$$\stackrel{*}{=} x + \int \frac{2}{x}dx - \int \frac{dx}{x-\sqrt{2}} - \int \frac{dx}{x+\sqrt{2}} = x + 2\log|x|$$

$$-\log|x-\sqrt{2}| - \log|x+\sqrt{2}| + c = x + \log x^2 -$$

$$-\log|x^2-2| + c = x + \log\left|\frac{x^2}{x^2-2}\right| + c \Rightarrow$$

$$\Rightarrow \int_3^4 \frac{x^3-2x-4}{x^3-2x}dx = \left[x + \log\left|\frac{x^2}{x^2-2}\right|\right]_3^4$$

$$= 4 + \log\left(\frac{\cancel{16}^{\,8}}{\cancel{14}_{\,7}}\right) - 3 - \log\left(\frac{9}{7}\right) = 1 + \log\left(\frac{8}{7}\right) - \log\left(\frac{9}{7}\right)$$

$$= 1 + \log\left[\frac{8}{\cancel{7}} \cdot \frac{\cancel{7}}{9}\right] = 1 + \log\left(\frac{8}{9}\right)$$

EX. 29 $\displaystyle\int \frac{e^x}{(e^x-2)(e^{2x}+1)}\,dx = \left[\begin{array}{l} u=e^x \\ du=e^x dx \end{array}\right] =$

$$= \int \frac{du}{(u-2)\underbrace{(u^2+1)}_{\text{IRREDUCIBLE}}} \quad (*)$$

PARTIAL FRACTIONS

$$\frac{1}{(u-2)(u^2+1)} = \frac{A}{u-2} + \frac{Bu+C}{u^2+1}$$

$$= \frac{Au^2+A+(Bu+C)(u-2)}{(u-2)(u^2+1)} = Au^2+A+Bu^2+2Bu+Cu-2C$$

$$= \frac{(A+B)u^2+(2B+C)u+A-2C}{(u-2)(u^2+1)} \qquad \begin{cases} A+B=0 \\ 2B+C=0 \\ A-2C=1 \end{cases} \qquad \begin{cases} A=-B \\ C=-2B \\ -B+4B=1 \end{cases}$$

$$\begin{cases} B=\frac{1}{3} \\ A=-\frac{1}{3} \\ C=-\frac{2}{3} \end{cases} \Longrightarrow (*) \; -\frac{1}{3}\int\frac{du}{u-2} + \frac{1}{3}\int\frac{u-2}{u^2+1}\,du =$$

$$= -\frac{1}{3}\log|u-2| + \underbrace{\frac{1}{3}\int\frac{u}{u^2+1}\,du}_{\text{SUBSTITUTION}} - \frac{2}{3}\int\frac{du}{u^2+1}$$

$t=u^2+1$

$dt=2u\,du$

$u\,du=\frac{1}{2}\,dt$

$$= -\frac{1}{3}\log|u-2| + \frac{1}{6}\log(u^2+1) - \frac{2}{3}\arctan u + c =$$

$$= -\frac{1}{3}\log|e^x-2| + \frac{1}{6}\log(e^{2x}+1) - \frac{2}{3}\arctan e^x + c$$