Opening Example

Are you a fan of people who work on Wall Street? Do you think that people who work on Wall Street are as honest and moral as the general public? In a Harris poll conducted in 2012, 28% of the U.S. adults polled agreed with the statement, "In general, people on Wall Street are as honest and moral as other people." Sixty-eight percent of the adults polled disagreed with this statement. (See Case Study 11–1.)

11.1 The Chi-Square Distribution

Definition

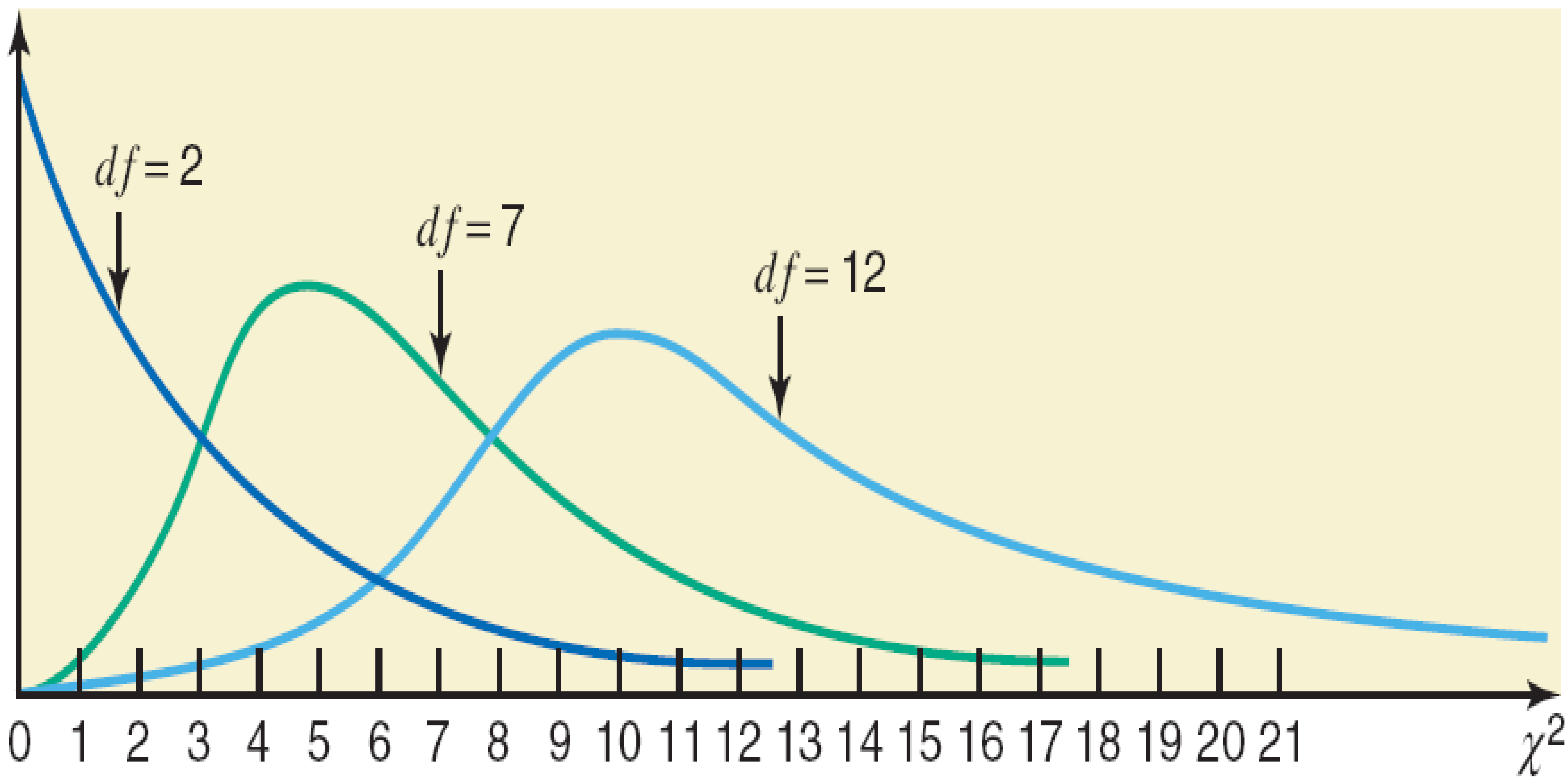
The ***chi-square distribution*** has only one parameter called the degrees of freedom.

The shape of a chi-square distribution curve is skewed to the right for small *df* and becomes symmetric for large *df*.

The entire chi-square distribution curve lies to the right of the vertical axis.

The chi-square distribution assumes nonnegative values only, and these are denoted by the symbol χ^2 (read as "chi-square").

Figure 11.1 Three Chi-Square Distribution Curves



Example 11-1

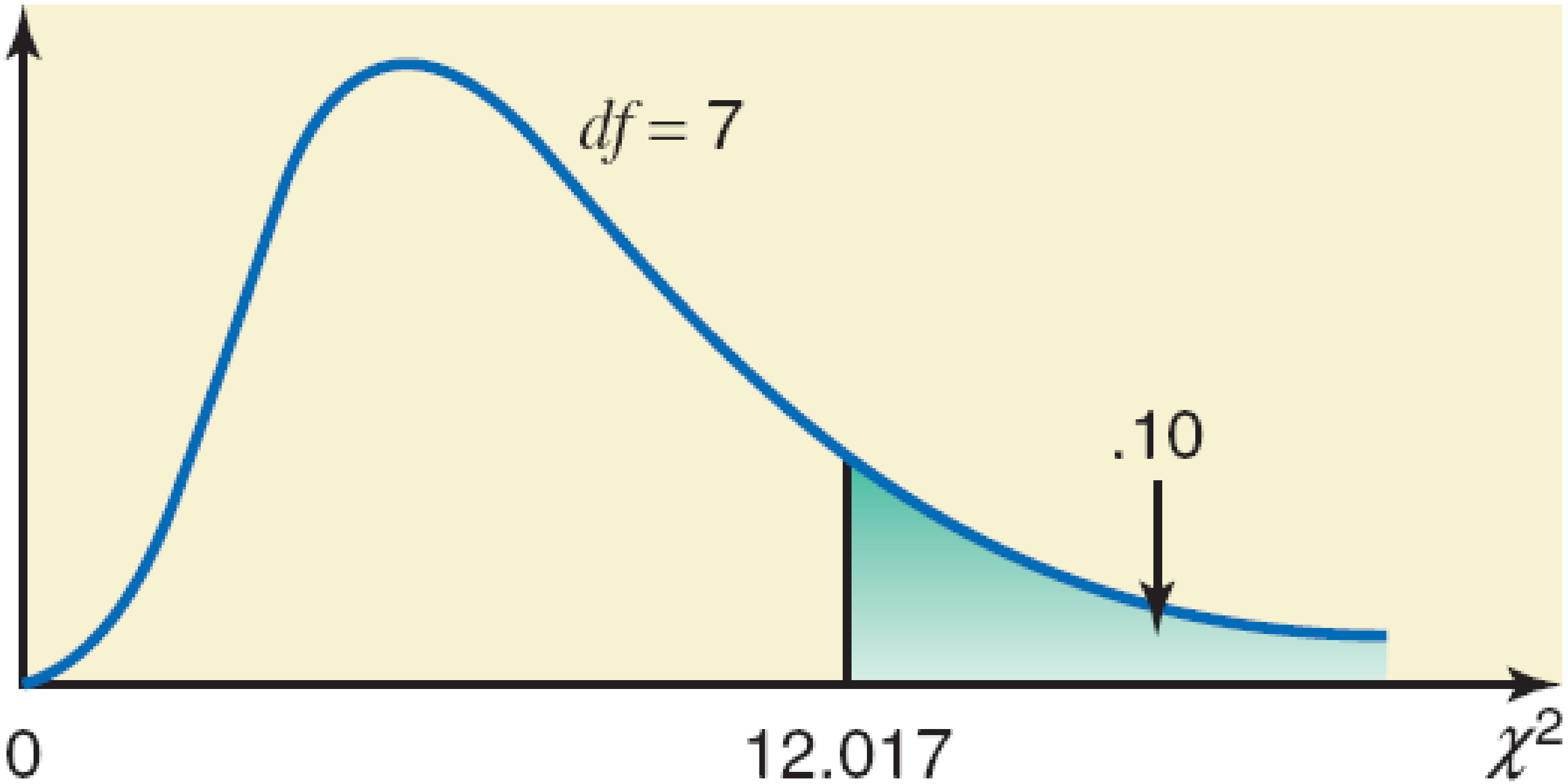
Find the value of χ^2 for 7 degrees of freedom and an area of .10 in the right tail of the chi-square distribution curve.

Table 11.1 χ^2 for df = 7 and .10 Area in the Right Tail

	Area in the Right Tail Under the Chi-Square Distribution Curve				
df	**.995**	**. . .**	**.100**	**. . .**	**.005**
1	0.000	. . .	2.706	. . .	7.879
2	0.010	. . .	4.605	. . .	10.597
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
7	0.989	. . .	**12.017**	. . .	20.278
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
100	67.328	. . .	118.498	. . .	140.169

Required value of χ^2

Figure 11.2 The χ^2 Value



Example 11-2

Find the value of χ^2 for 12 degrees of freedom and an area of .05 in the left tail of the chi-square distribution curve.

Example 11-2: Solution

Area in the right tail

= 1 – Area in the left tail

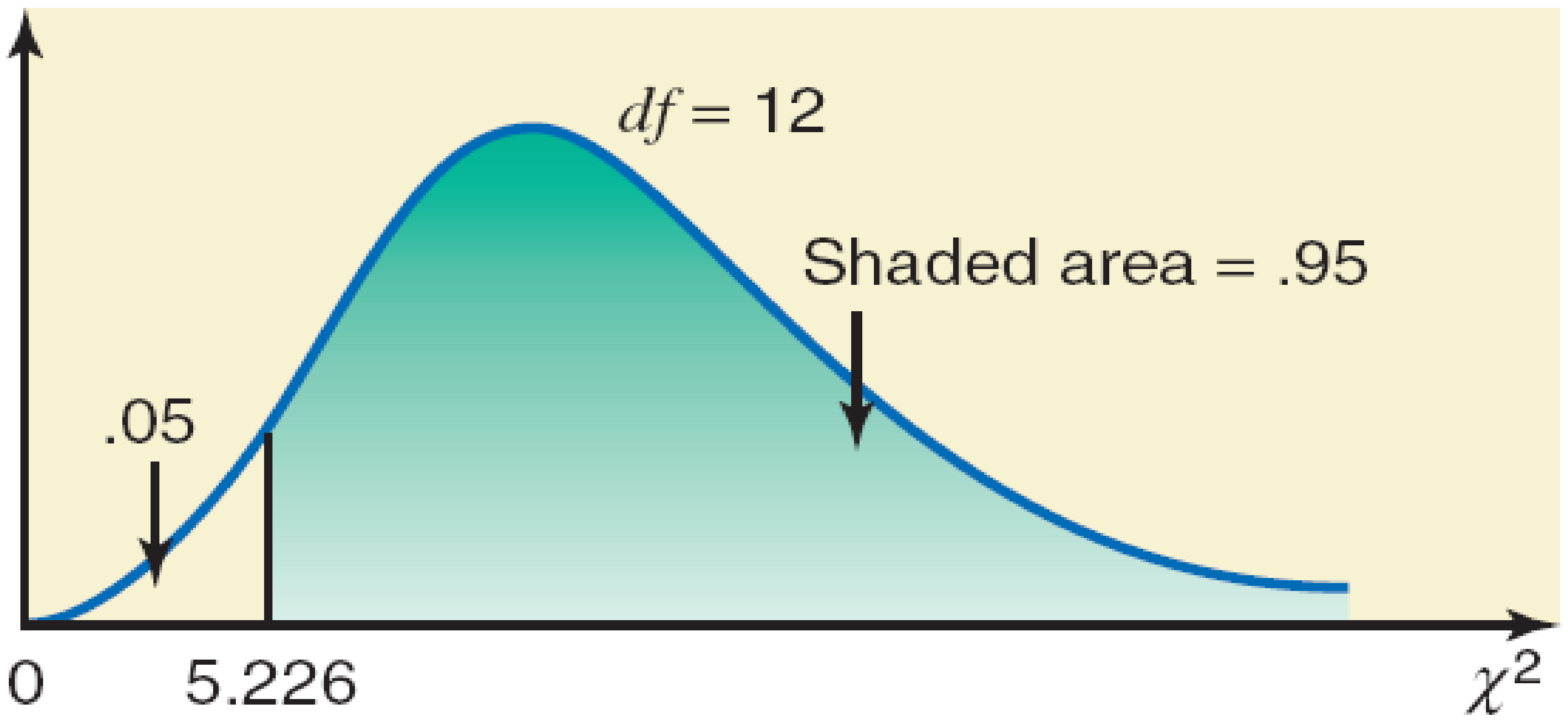
= 1 – .05 = .95

Table 11.2 χ^2 for df = 12 and .95 Area in the Right Tail

	Area in the Right Tail Under the Chi-Square Distribution Curve				
df	**.995**	**. . .**	$\boxed{\textbf{.950}}$	**. . .**	**.005**
1	0.000	. . .	0.004	. . .	7.879
2	0.010	. . .	0.103	. . .	10.597
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
$\boxed{\textbf{12}}$	3.074	. . .	$\boxed{\textbf{5.226}}$	. . .	28.300
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
100	67.328	. . .	77.929	. . .	140.169

Required value of χ^2

Figure 11.3 The χ^2 Value



11.2 A Goodness-of-Fit Test

Definition

An experiment with the following characteristics is called a ***multinomial experiment***.

1. The experiment consists of n identical trials (repetitions).
2. Each trial results in one of k possible outcomes (or categories), where $k > 2$.
3. The trials are independent.
4. The probabilities of the various outcomes remain constant for each trial.

Observed and Expected Frequencies

Definition

The frequencies obtained from the performance of an experiment are called the ***observed frequencies*** and are denoted by O.

The ***expected frequencies***, denoted by E, are the frequencies that we expect to obtain if the null hypothesis is true. The expected frequency for a category is obtained as

$$E = np$$

where n is the sample size and p is the probability that an element belongs to that category if the null hypothesis is true.

Degrees of Freedom for a Goodness of Fit Test

Degrees of Freedom for a Goodness-of-Fit Test

In a goodness-of-fit test, the ***degrees of freedom*** are

$$df = \boldsymbol{k} - 1$$

where $\boldsymbol{k}$ denotes the number of possible outcomes (or categories) for the experiment.

Test Statistic for a Goodness-of-Fit Test

The ***test statistic for a goodness-of-fit test*** is χ^2 and its value is calculated as

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

where

O = observed frequency for a category

E = expected frequency for a category = np

Remember that a ***chi-square goodness-of-fit test is always right-tailed test***.

Example 11-3

A bank has an ATM installed inside the bank, and it is available to its customers only from 7 AM to 6 PM Monday through Friday.

The manager of the bank wanted to investigate if the percentage of transactions made on this ATM is the same for each of the 5 days (Monday through Friday) of the week.

She randomly selected one week and counted the number of transactions made on this ATM on each of the 5 days during this week.

The information she obtained is given in the following table, where the number of users represents the number of transactions on this ATM on these days.

For convenience, we will refer to these transactions as "people" or "users."

Example 11-3

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Number of users	253	197	204	279	267

At the 1% level of significance, can we reject the null hypothesis that the number of people who use this ATM each of the 5 days of the week is the same?

Assume that this week is typical of all weeks in regard to the use of this ATM.

Example 11-3: Solution

Step 1:

$H_0 : p_1 = p_2 = p_3 = p_4 = p_5 = .20$

H_1 : At least two of the five proportions are not equal to .20

Example 11-3: Solution

Step 2:

There are 5 categories

- 5 days on which the ATM is used
- Multinomial experiment

We use the chi-square distribution to make this test.

Example 11-3: Solution

Step 3:

Area in the right tail = α = .01

$\boldsymbol{k}$ = number of categories = 5

$df = \boldsymbol{k} - 1 = 5 - 1 = 4$

The critical value of χ^2 = 13.277

Figure 11.4 Rejection and Nonrejection Regions

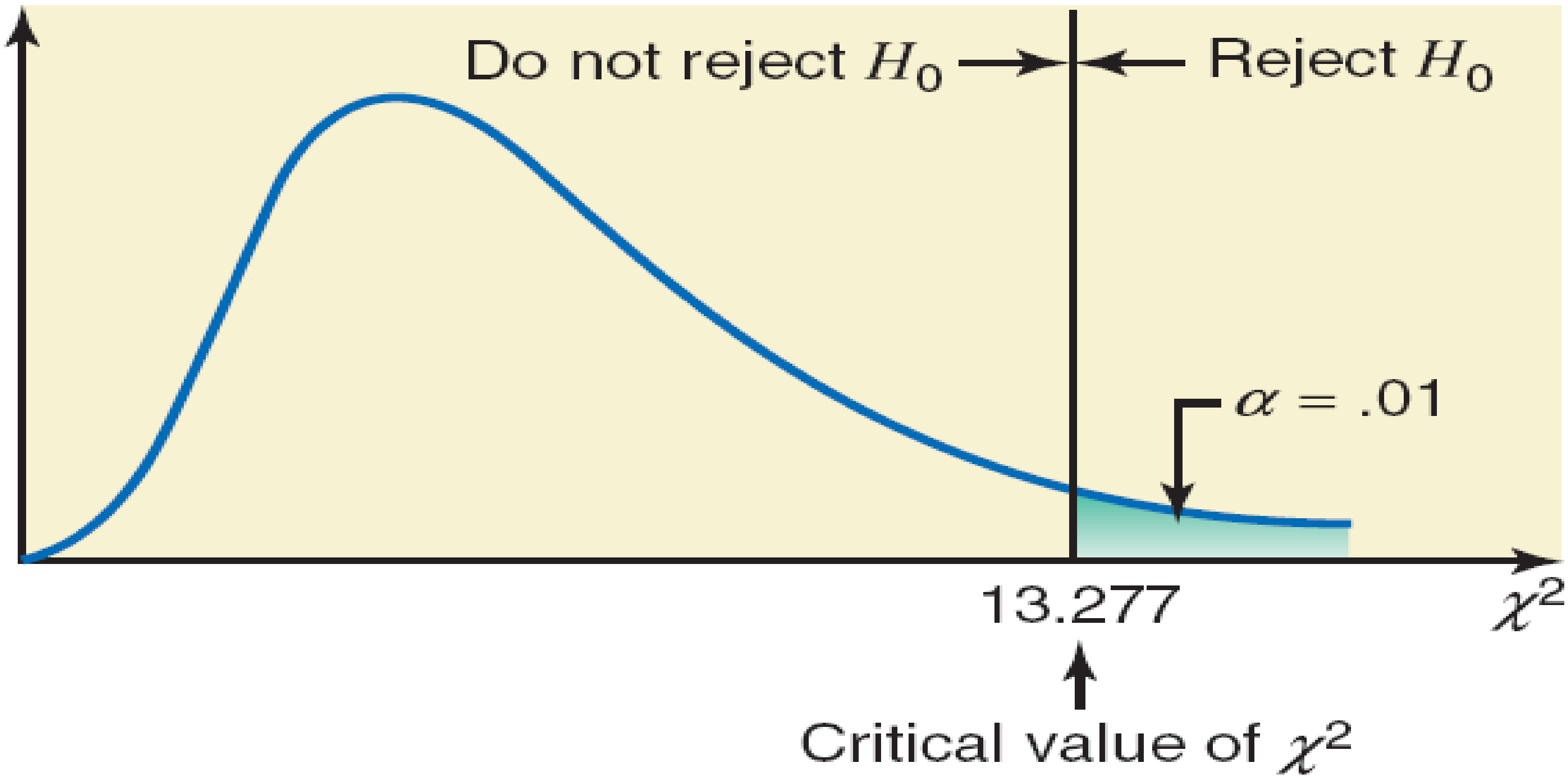


Table 11.3 Calculating the Value of the Test Statistic

Category (Day)	Observed Frequency O	p	Expected Frequency $E = np$	$(O - E)$	$(O - E)^2$	$\frac{(O - E)^2}{E}$
Monday	253	.20	$1200(.20) = 240$	13	169	.704
Tuesday	197	.20	$1200(.20) = 240$	−43	1849	7.704
Wednesday	204	.20	$1200(.20) = 240$	−36	1296	5.400
Thursday	279	.20	$1200(.20) = 240$	39	1521	6.338
Friday	267	.20	$1200(.20) = 240$	27	729	3.038
	$n = 1200$					Sum = 23.184

Example 11-3: Solution

Step 4:

All the required calculations to find the value of the test statistic χ^2 are shown in Table 11.3.

$$\chi^2 = \sum \frac{(O - E)^2}{E} = 23.184$$

Example 11-3: Solution

Step 5:

The value of the test statistic χ^2 = 23.184 is larger than the critical value of χ^2 = 13.277.

- It falls in the rejection region.

Hence, we reject the null hypothesis.

We state that the number of persons who use this ATM is not the same for the 5 days of the week.

Example 11-4

In a Gallup poll conducted April 3–6, 2014, Americans aged 18 and older were asked if upper-income people were "paying their fair share in federal taxes, paying too much or paying too little."

Of the respondents, 61% said too little, 24% said fair share, 13% said too much, and 2% had no opinion (www.gallup.com).

Assume that these percentages hold true for the 2014 population of Americans aged 18 and older. Recently, 1000 randomly selected Americans aged 18 and older were asked the same question.

The following table lists the number of Americans in this sample who belonged to each response.

Example 11-4

Response	Too Little	Fair Share	Too Much	No Opinion
Frequency	581	256	138	25

Test at a 2.5% level of significance whether the current distribution of opinions is different from that for 2014.

Example 11-4: Solution

Step 1:

H_0 : The current percentage distribution of opinions is the same as for 2014.

H_1 : The current percentage distribution of opinions is different from that for 2014.

Example 11-4: Solution

Step 2:

There are 4 categories

- Multinomial experiment

We use the chi-square distribution to make this test.

Example 11-4: Solution

Step 3:

Area in the right tail = α = .025

$\boldsymbol{k}$ = number of categories = 4

df = $\boldsymbol{k}$ – 1 = 4 – 1 = 3

The critical value of χ^2 = 9.348

Figure 11.5 Rejection and Nonrejection Regions

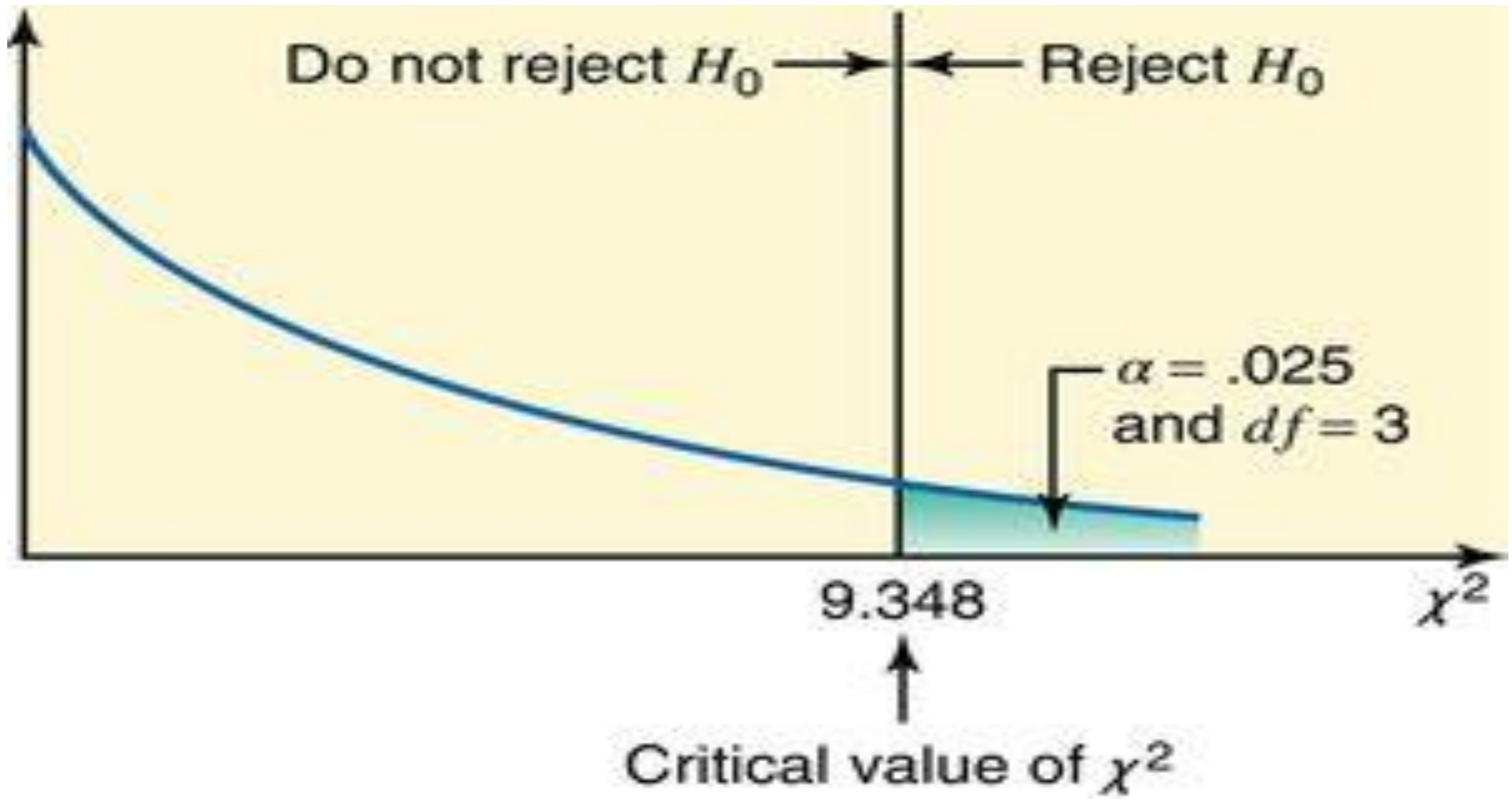


Table 11.4 Calculating the Value of the Test Statistic

Category (Response)	Observed Frequency O	p	Expected Frequency $E = np$	$(O - E)$	$(O - E)^2$	$\frac{(O - E)^2}{E}$
Too little	581	.61	1000(.61) = 610	−29	841	1.379
Fair share	256	.24	1000(.24) = 240	16	256	1.067
Too much	138	.13	1000(.13) = 130	8	64	.492
No opinion	25	.02	1000(.02) = 20	5	25	1.250
	$n = 1000$					Sum = 4.188

Example 11-4: Solution

Step 4:

All the required calculations to find the value of the test statistic χ^2 are shown in Table 11.4.

$$\chi^2 = \sum \frac{(O - E)^2}{E} = 4.188$$

Example 11-4: Solution

Step 5:

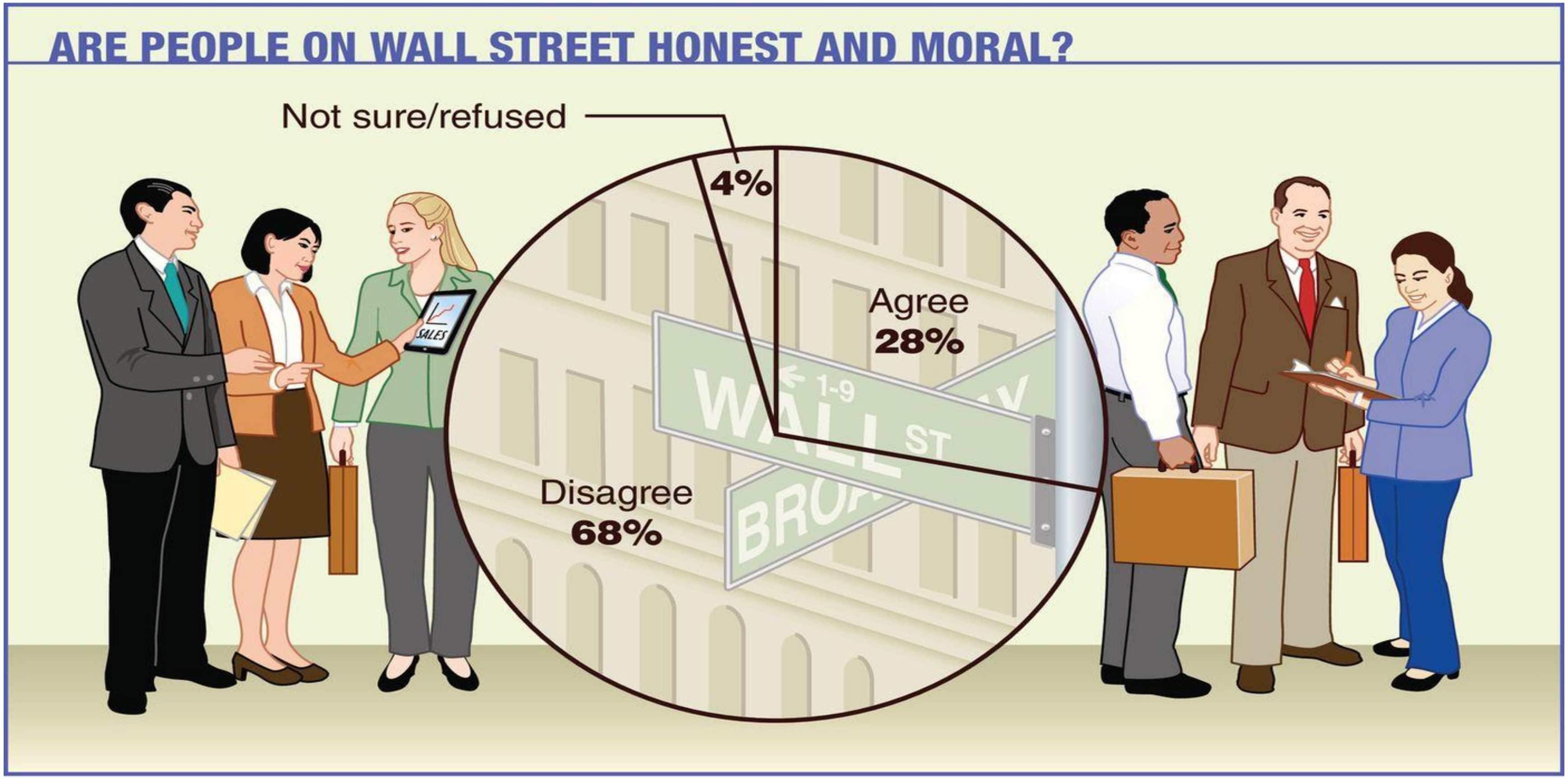
The value of the test statistic $\chi^2 = 4.188$ is smaller than the critical value of $\chi^2 = 9.348$

- It falls in the nonrejection region.

Hence, we fail to reject the null hypothesis.

We state that the current percentage distribution of opinions is the same as for 2014.

Case Study 11-1 Are People on Wall Street Honest and Moral?



Data source: Harris Interactive telephone poll of U.S. adults conducted April 10–17, 2012.

11.3 A Test of Independence or Homogeneity

Often we have information on more than one variable for each element.

Such information can be summarized and presented using a two-way classification table, which is called a ***contingency table*** or ***cross-tabulation***.

	Full-Time	Part-Time
Male	6768	2615
Female	7658	3717

Students who are male and enrolled part-time

A Test of Independence or Homogeneity

- A Test of Independence
- A Test of Homogeneity

A Test of Independence

Definition

A test of independence involves a test of the null hypothesis that two attributes of a population are not related.

The ***degrees of freedom for a test of independence*** are

$$df = (R - 1)(C - 1)$$

where R and C are the number of rows and the number of columns, respectively, in the given contingency table.

A Test of Independence

Test Statistic for a Test of Independence

The value of the ***test statistic χ^2 for a test of independence*** is calculated as

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

where ***O*** and ***E*** are the observed and expected frequencies, respectively, for a cell.

Example 11-5

Lack of discipline have become major problems in schools in the United States.

A random sample of 300 adults was selected, and these adults were asked if they favor giving more freedom to schoolteachers to punish students for lack of discipline.

The two-way classification of the responses of these adults is represented in the following table.

Example 11-5

	In Favor (F)	Against (A)	No Opinion (N)
Men (M)	93	70	12
Women (W)	87	32	6

Calculate the expected frequencies for this table, assuming that the two attributes, gender and opinions on the issue, are independent.

Example 11-5: Solution

Table 11.6 Observed Frequencies

	In Favor (F)	Against (A)	No Opinion (N)	Row Totals
Men (M)	93	70	12	175
Women (W)	87	32	6	125
Column Totals	180	102	18	300

Expected Frequencies for a Test of Independence

The expected frequency E for a cell is calculated as

$$\boldsymbol{E} = \frac{\textbf{(Row total)(Column total)}}{\textbf{Sample size}}$$

Example 11-5: Solution

Table 11.7 Observed and Expected Frequencies

	In Favor (F)	Against (A)	No Opinion (N)	Row Totals
Men (M)	93 (105.00)	70 (59.50)	12 (10.50)	175
Women (W)	87 (75.00)	32 (42.50)	6 (7.50)	125
Column Totals	180	102	18	300

Example 11-6

Reconsider the two-way classification table given in Example 11-5.

In that example, a random sample of 300 adults was selected, and they were asked if they favor giving more freedom to schoolteachers to punish students for lack of discipline.

Based on the results of the survey, a two-way classification table was prepared and presented in Example 11-5.

Does the sample provide sufficient information to conclude that the two attributes, gender and opinions of adults, are dependent?

Use a 1% significance level.

Example 11-6: Solution

Step 1:

H_0: Gender and opinions of adults are independent

H_1: Gender and opinions of adults are dependent

Example 11-6: Solution

Step 2:

We use the chi-square distribution to make a test of independence for a contingency table.

Step 3:

α = .01

df = (R – 1)(C – 1) = (2 – 1)(3 – 1) = 2

The critical value of χ^2 = 9.210

Figure 11.6 Rejection and Nonrejection Regions

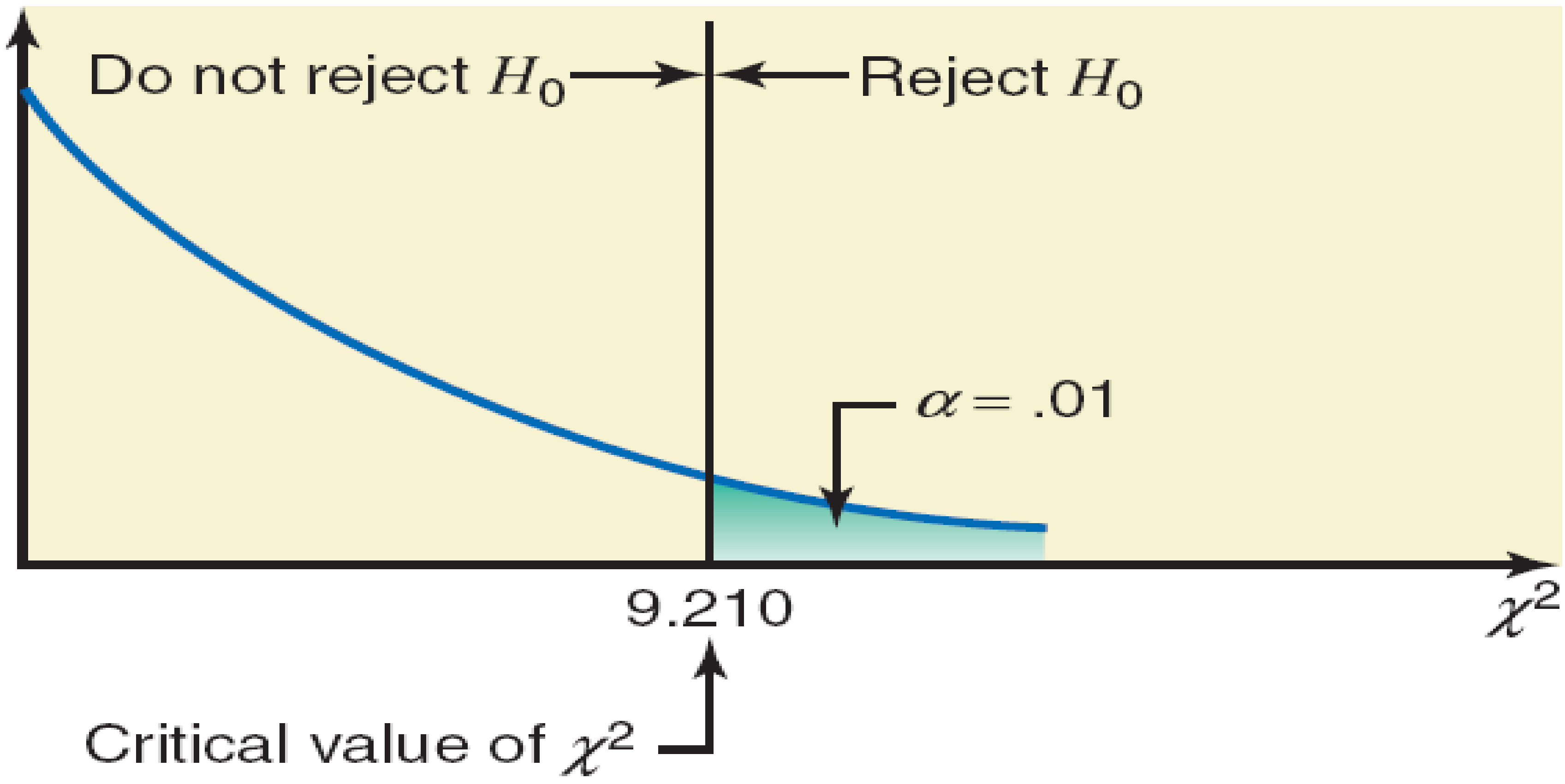


Table 11.8 Observed and Expected Frequencies

	In Favor (F)	Against (A)	No Opinion (N)	Row Totals
Men (M)	93 (105.00)	70 (59.50)	12 (10.50)	175
Women (W)	87 (75.00)	32 (42.50)	6 (7.50)	125
Column Totals	180	102	18	300

Example 11-6: Solution

Step 4:

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

$$= \frac{(93-105.00)^2}{105.00} + \frac{(70-59.50)^2}{59.50} + \frac{(12-10.50)^2}{10.50}$$

$$+ \frac{(87-75.00)^2}{75.00} + \frac{(32-42.50)^2}{42.50} + \frac{(6-7.50)^2}{7.50}$$

$$= 1.371 + 1.853 + .214 + 1.920 + 2.594 + .300 = 8.252$$

Example 11-6: Solution

Step 5:

The value of the test statistic $\chi^2 = 8.252$.

- It is less than the critical value of $\chi^2 = 9.210$.
- It falls in the nonrejection region.

Hence, we fail to reject the null hypothesis.

We state that there is not enough evidence from the sample to conclude that the two characteristics, *gender* and *opinions of adults*, are dependent for this issue.

Example 11-7

A researcher wanted to study the relationship between gender and owning smart phones among adults who have cell phones.

She took a sample of 2000 adults and obtained the information given in the following table.

	Own Cell Phones	Do Not Own Cell Phones
Men	640	450
Women	440	470

At a 5% level of significance, can you conclude that gender and owning a smart phone are related for all adults?

Example 11-7: Solution

Step 1:

H_0: Gender and owning a smart phone are not related

H_1: Gender and owning a smart phone are related

Example 11-7: Solution

Step 2:

We are performing a test of independence.

We use the chi-square distribution to make the test.

Step 3:

$\alpha = .05.$

$df = (R - 1)(C - 1) = (2 - 1)(2 - 1) = 1$

The critical value of $\chi^2 = 3.841$

Figure 11.7 Rejection and Nonrejection Regions

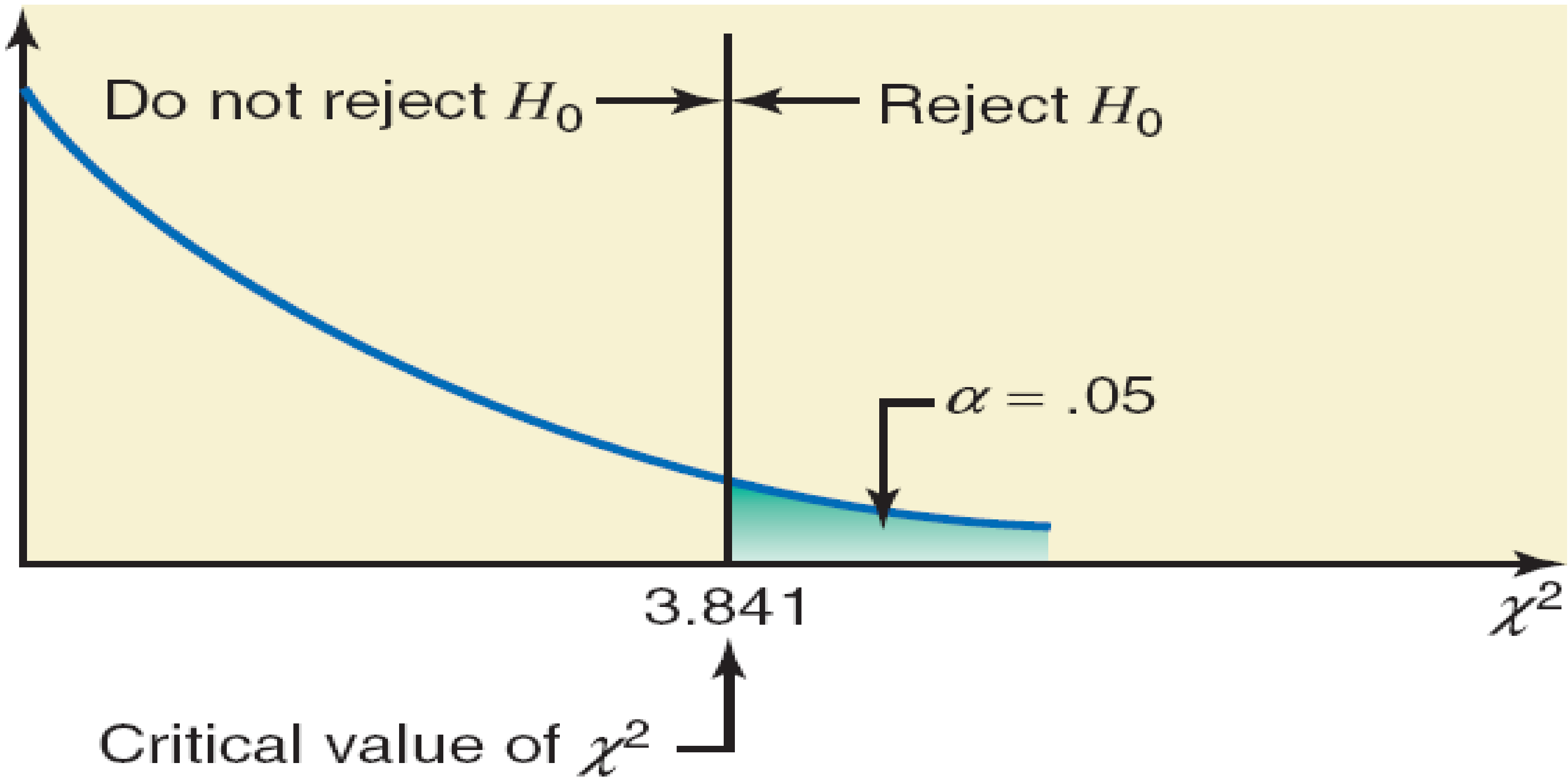


Table 11.9 Observed and Expected Frequencies

	Own Cell Phones (Y)	Do Not Own Cell Phones (N)	Row Totals
Men (M)	640 (588.60)	450 (501.40)	1090
Women (W)	440 (491.40)	470 (418.60)	910
Column Totals	1080	920	2000

Example 11-7: Solution

Step 4:

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

$$= \frac{(640-588.60)^2}{588.60} + \frac{(450-501.40)^2}{501.40}$$

$$+ \frac{(440-491.40)^2}{491.40} + \frac{(470-418.60)^2}{481.60}$$

$$= 4.489 + 5.269 + 5.376 + 6.311 = 21.445$$

Example 11-7: Solution

Step 5:

The value of the test statistic $\chi^2 = 21.445$.

- It is larger than the critical value of $\chi^2 = 3.841$.
- It falls in the rejection region.

Hence, we reject the null hypothesis.

We state that there is strong evidence from the sample to conclude that the two characteristics, gender and owning smart phones, are related for all adults.

A Test of Homogeneity

Definition

A ***test of homogeneity*** involves testing the null hypothesis that the proportions of elements with certain characteristics in two or more different populations are the same against the alternative hypothesis that these proportions are not the same.

Example 11-8

Consider the data on income distributions for households in California and Wisconsin given in Table 11.10.

Using a 2.5% significance level, test whether the distribution of households with regard to income levels is different (not homogeneous) for the two states.

Example 11-8

Table 11.10 Observed Frequencies

	California	Wisconsin	Row Totals
High income	70	34	104
Medium income	80	40	120
Low income	100	76	176
Column Totals	250	150	400

Example 11-8: Solution

Step 1:

H_0: The proportions of households that belong to different income groups are the same in both states

H_1: The proportions of households that belong to different income groups are not the same in both states

Example 11-8: Solution

Step 2:

We use the chi-square distribution to make a homogeneity test.

Step 3:

α = .025

$df = (R - 1)(C - 1) = (3 - 1)(2 - 1) = 2$

The critical value of χ^2 = 7.378

Figure 11.8 Rejection and Nonrejection Regions

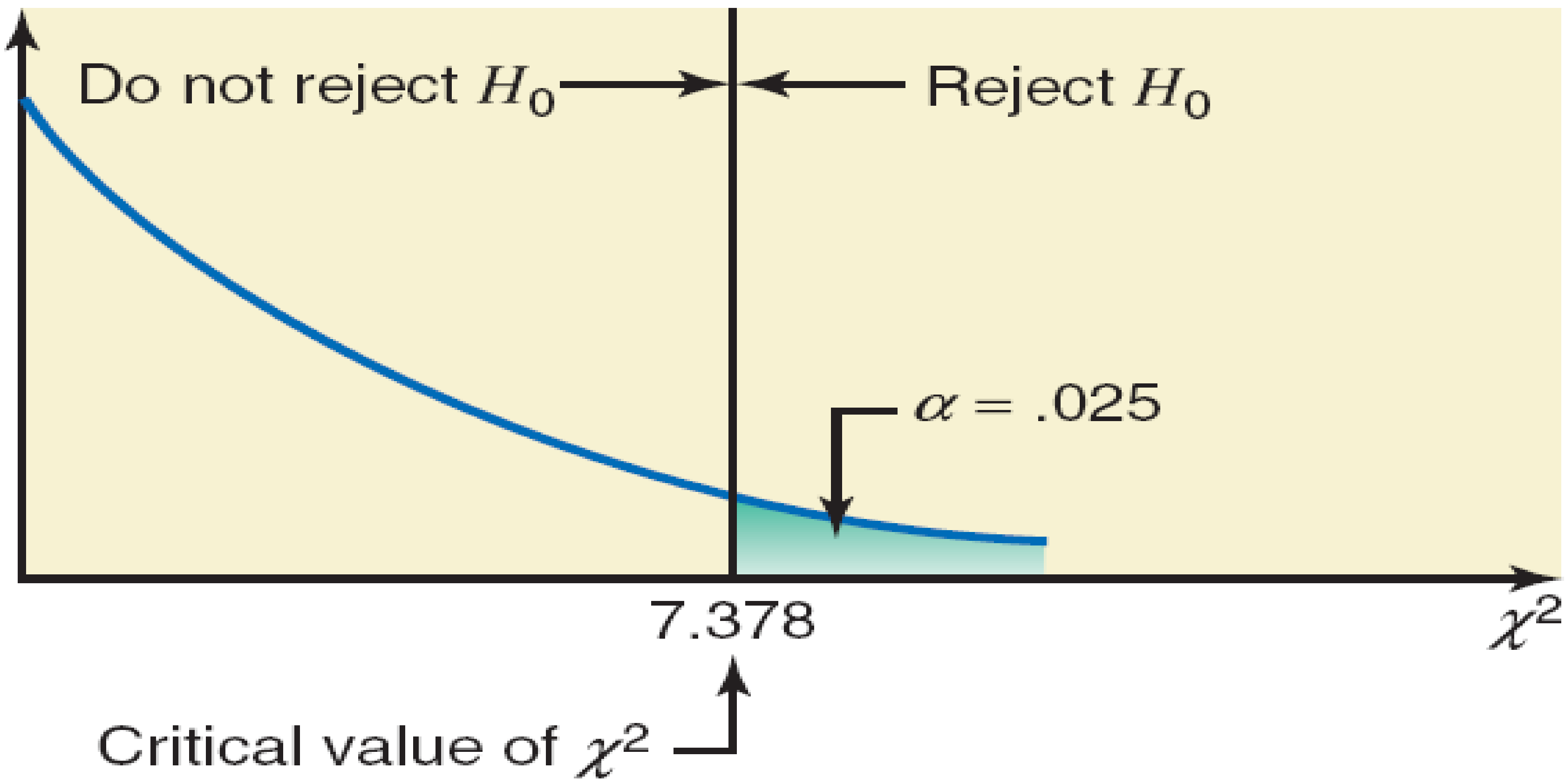


Table 11.11 Observed and Expected Frequencies

	California	Wisconsin	Row Totals
High income	70 (65)	34 (39)	104
Medium income	80 (75)	40 (45)	120
Low income	100 (110)	76 (66)	176
Column Totals	250	150	400

Example 11-8: Solution

Step 4:

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

$$= \frac{(70-65)^2}{65} + \frac{(34-39)^2}{39} + \frac{(80-75)^2}{75}$$

$$+ \frac{(40-45)^2}{45} + \frac{(100-110)^2}{110} + \frac{(76-66)^2}{66}$$

$$= .385 + .641 + .333 + .566 + .909 + 1.515 = 4.339$$

Example 11-8: Solution

Step 5:

The value of the test statistic $\chi^2 = 4.339$.

- It is less than the critical value of χ^2.
- It falls in the nonrejection region.

Hence, we fail to reject the null hypothesis.

We state that there is no evidence that the distributions of households with regard to income are different in California and Wisconsin.

11.4 Inferences About The Population Variance

- Estimation of the Population Variance
- Hypothesis Tests About the Population Variance

Inferences about the Population Variance

Sampling Distribution of $(n - 1)s^2 / \sigma^2$

If the population from which the sample is selected is (approximately) normally distributed, then

$$\frac{(n-1)s^2}{\sigma^2}$$

has a chi-square distribution with $n - 1$ degrees of freedom.

Estimation of the Population Variance

Confidence interval for the population variance σ^2

Assuming that the population from which the sample is selected is (approximately) normally distributed, we obtain the $(1 - \alpha)100\%$ ***confidence interval for the population variance σ^2*** as

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} \quad \text{to} \quad \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

Estimation of the Population Variance

where $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ are obtained from the chi-square distribution for $\alpha/2$ and $1-\alpha/2$ areas in the right tail of the chi-square distribution curve, respectively, and for $n - 1$ degrees of freedom.

The confidence interval for the population standard deviation can be obtained by simply taking the positive square roots of the two limits of the confidence interval for the population variance.

Example 11-9

One type of cookie manufactured by Haddad Food Company is Cocoa Cookies.

The machine that fills packages of these cookies is set up in such a way that the average net weight of these packages is 32 ounces with a variance of .015 square ounce.

From time to time the quality control inspector at the company selects a sample of a few such packages, calculates the variance of the net weights of these packages, and constructs a 95% confidence interval for the population variance.

If either both or one of the two limits of this confidence interval is not in the interval .008 to .030, the machine is stopped and adjusted.

Example 11-9

A recently taken random sample of 25 packages from the production line gave a sample variance of .029 square ounce.

Based on this sample information, do you think the machine needs an adjustment?

Assume that the net weights of cookies in all packages are normally distributed.

Example 11-9: Solution

Step 1:

n = 25 and s^2 = .029

Step 2:

α = 1 - .95 = .05

$\alpha/2$ = .05/2 = .025

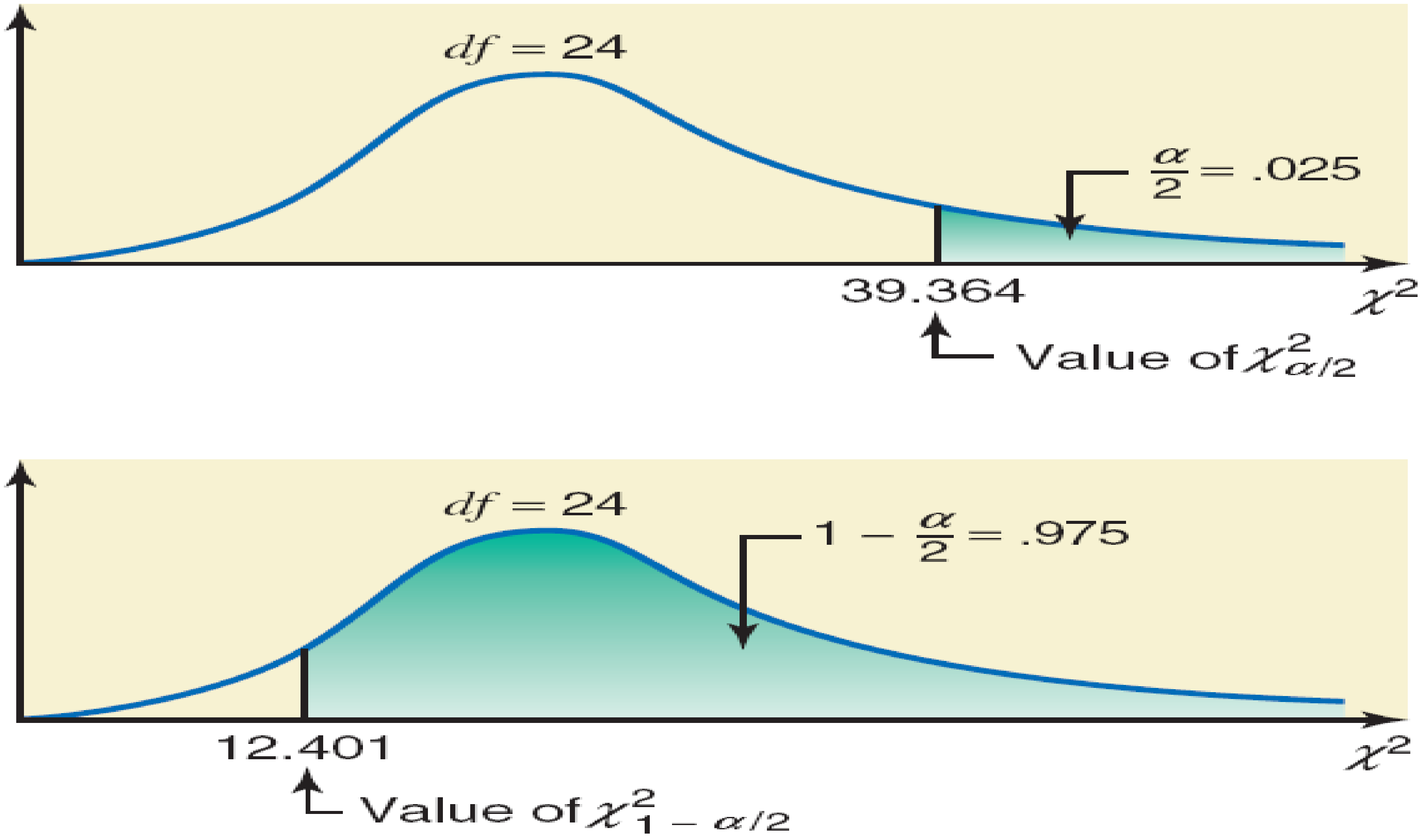
$1 - \alpha/2$ = 1 – .025 = .975

$df = \boldsymbol{n} - 1$ = 25 – 1 = 24

χ^2 for 24 df and .025 area in the right tail = 39.364

χ^2 for 24 df and .975 area in the right tail = 12.401

Figure 11.9 The Values of χ^2



Example 11-9: Solution

Step 3:

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} \quad \text{to} \quad \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

$$\frac{(25-1)(.029)}{39.364} \quad \text{to} \quad \frac{(25-1)(.029)}{12.401}$$

$$.0177 \quad \text{to} \quad .0561$$

Example 11-9: Solution

Thus, with 95% confidence, we can state that the variance for all packages of Cocoa Cookies lies between **.0177** and **.0561** square ounce.

We can obtain the confidence interval for the population standard deviation σ by taking the positive square roots of the two limits of the above confidence interval for the population variance.

Thus, a 95% confidence interval for the population standard deviation is **.133** to **.237**.

Hypothesis Tests about the Population Variance

Test statistic for a Test of Hypothesis About σ^2

The value of the ***test statistic*** χ^2 is calculated as

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

where s^2 is the sample variance, σ^2 is the hypothesized value of the population variance, and n – 1 represents the degrees of freedom.

The population from which the sample is selected is assumed to be (approximately) normally distributed.

Example 11-10

One type of cookie manufactured by Haddad Food Company is Cocoa Cookies.

The machine that fills packages of these cookies is set up in such a way that the average net weight of these packages is 32 ounces with a variance of .015 square ounce.

From time to time the quality control inspector at the company selects a sample of a few such packages, calculates the variance of the net weights of these packages, and makes a test of hypothesis about the population variance.

She always uses $\alpha = .01$. The acceptable value of the population variance is .015 square ounce or less. If the conclusion from the test of hypothesis is that the population variance is not within the acceptable limit, the machine is stopped and adjusted.

Example 11-10

A recently taken random sample of 25 packages from the production line gave a sample variance of .029 square ounce.

Based on this sample information, do you think the machine needs an adjustment?

Assume that the net weights of cookies in all packages are normally distributed.

Example 11-10: Solution

Step 1:

H_0: $\boldsymbol{\sigma^2} \le .015$

(The population variance is within the acceptable limit)

H_1: $\boldsymbol{\sigma^2} > .015$

(The population variance exceeds the acceptable limit)

Example 11-10: Solution

Step 2:

We use the chi-square distribution to test a hypothesis about σ^2

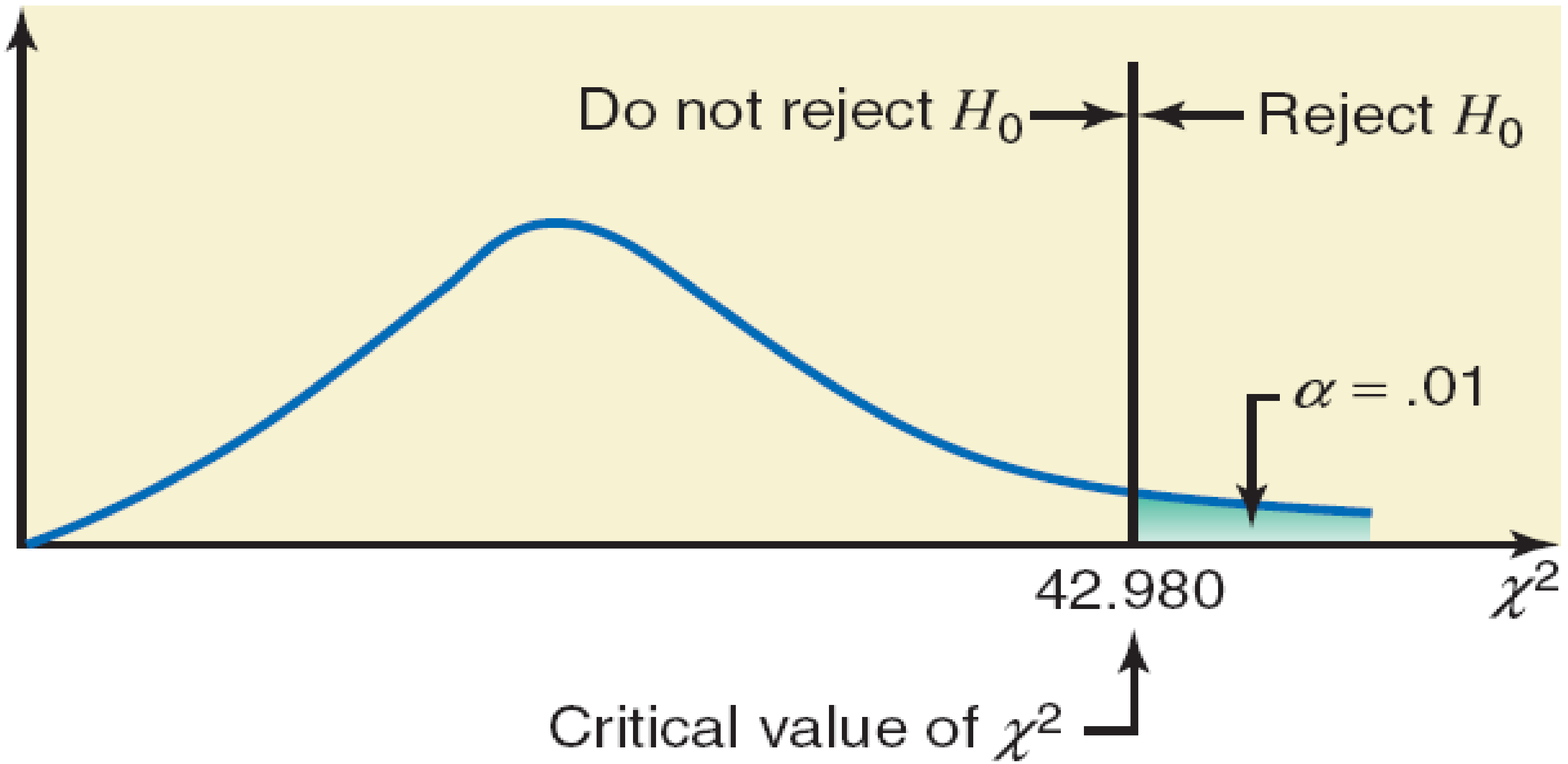
Step 3:

$\alpha = .01.$

$df = \boldsymbol{n} - 1 = 25 - 1 = 24$

The critical value of $\chi^2 = 42.980$

Figure 11.10 Rejection and Nonrejection Regions



Example 11-10: Solution

Step 4:

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(25-1)(.029)}{.015} = 46.400$$

From H_0

Example 11-10: Solution

Step 5:

The value of the test statistic $\chi^2 = 46.400$.

- It is greater than the critical value of χ^2
- It falls in the rejection region.

Hence, we reject the null hypothesis H_0.

We conclude that the population variance is not within the acceptable limit.

The machine should be stopped and adjusted.

Example 11-11

It is known that the variance of GPAs (with a maximum GPA of 4) of all students at a large university was .24 in 2014.

A professor wants to determine whether the variance of the current GPAs of students at this university is different from .24.

She took a random sample of 20 students and found that the variance of their GPAs is .27.

Using a 5% significance level, can you conclude that the current variance of the GPAs of students at this university is different from .24?

Assume that the GPAs of all current students at this university are (approximately) normally distributed.

Example 11-11: Solution

Step 1:

H_0: $\sigma^2 = .24$

(The population variance is not different from .24)

H_1: $\sigma^2 \neq .24$

(The population variance is different from .24)

Example 11-11: Solution

Step 2:

We use the chi-square distribution to test a hypothesis about σ^2

Step 3:

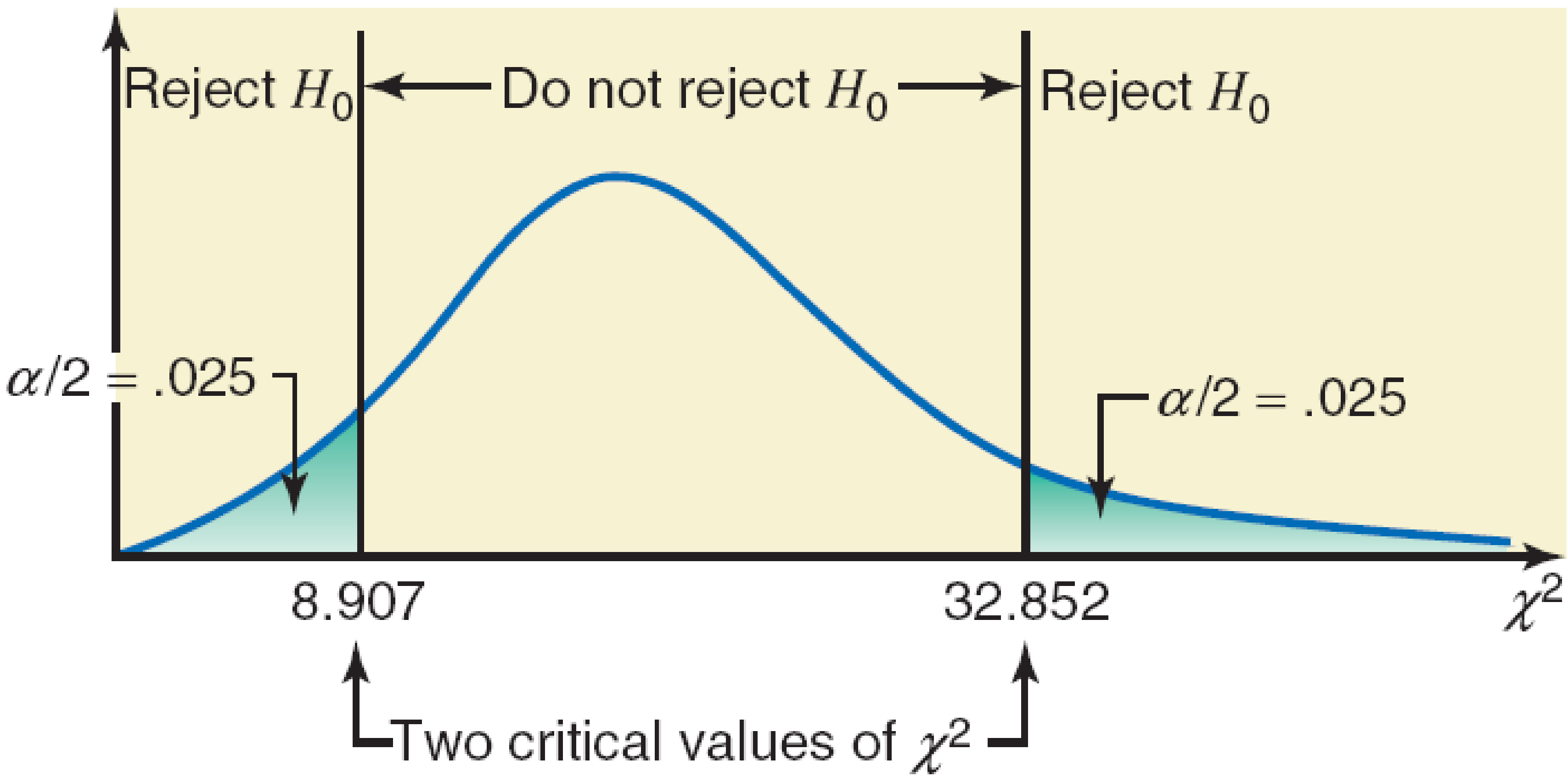
$\alpha = .05$

Area in each tail = .025

$df = \boldsymbol{n} - 1 = 20 - 1 = 19$

The critical values of $\chi^2 = 32.852$ and 8.907

Figure 11.11 Rejection and Nonrejection Regions



Example 11-11: Solution

Step 4:

$$\chi^2 = \frac{(n-1)\,S^2}{\sigma^2} = \frac{(20-1)\,(.27)}{.24} = 21.375$$

From H_0

Example 11-11: Solution

Step 5:

The value of the test statistic $\chi^2 = 21.375$.

- It is between the two critical values of χ^2
- It falls in the nonrejection region.

Consequently, we fail to reject H_0.

We conclude that the population variance of the current GPAs of students at this university does not appear to be different from .24.