Quantitative Methods – I

Practice 7

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THEME #1

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Random Variables & Probability Distribution

Random Variable

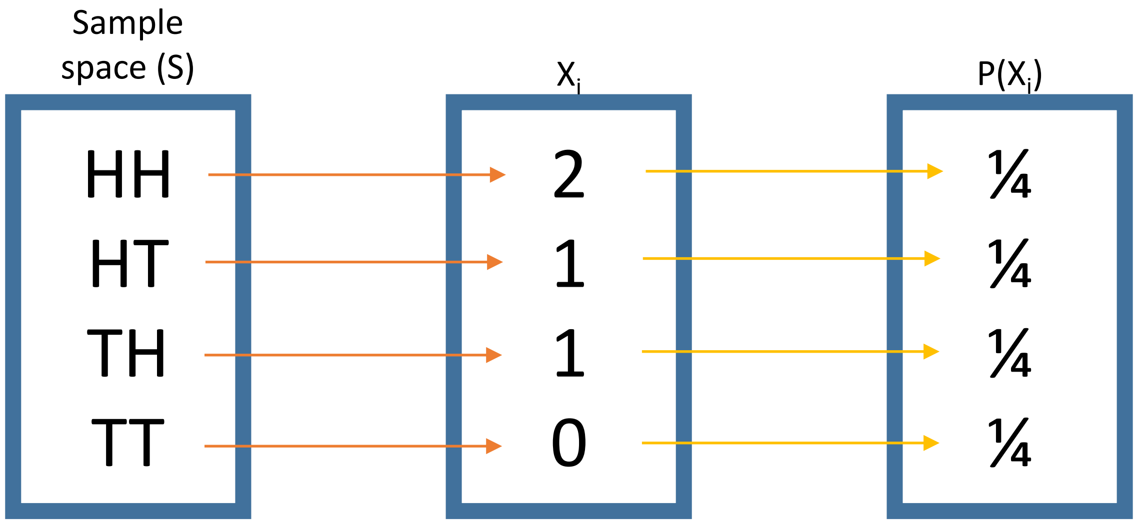
A variable whose value is determined by the outcome of an experiment.

Ex. Number of Head from the toss of two coins

Sample space (S)	Random Variable (X_i)	Probability $P(X_i)$
HH	2	¼
HT	1	¼
TH	1	¼
TT	0	¼

Probability distributions

Lists all the possible values that the random variable can assume and their corresponding probabilities.

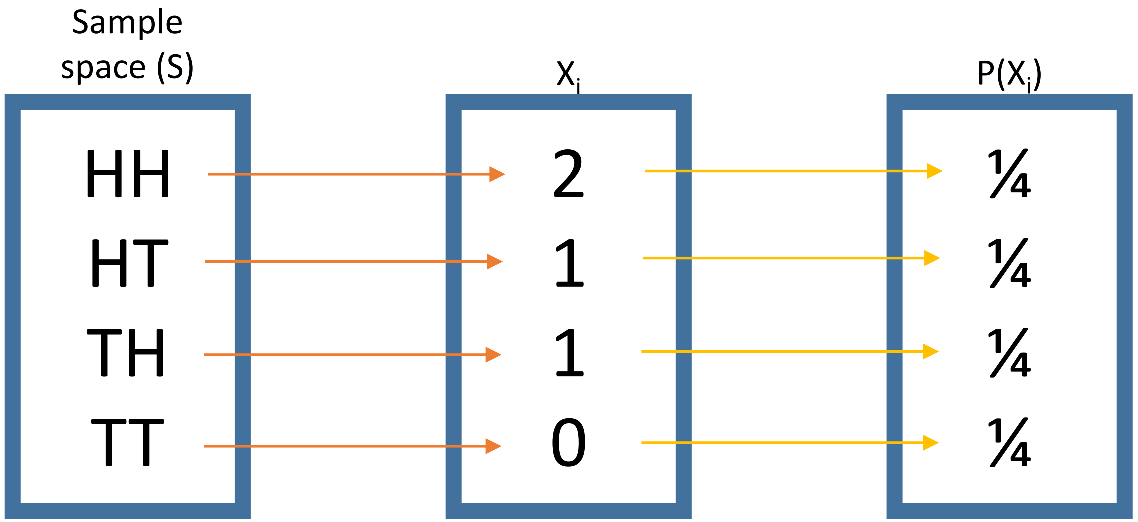


X_i	$P(X_i)$
0	¼=0.25
1	2/4=0.5
2	¼=0.25
Tot.	1.0

Probability distributions

2 conditions:

1. $0 \leq P(X_i) \leq 1$
2. $\Sigma\, P(X_i) = 1$



X_i	$P(X_i)$
0	¼=0.25
1	2/4=0.5
2	¼=0.25
Tot.	1.0

Exercise 1

Each of the following tables lists certain values of x and their probabilities. Determine whether or not each table represents a valid probability distribution.

a)

x_i	$P(x_i)$
0	0.18
1	0.01
2	0.29
3	0.37

b)

x_i	$P(x_i)$
2	0.35
3	0.24
4	0.18
5	0.23

c)

x_i	$P(x_i)$
7	0.65
8	0.50
9	-0.15

Solution

a) Each probability listed in this table is in the range 0 to 1, it satisfies the first condition of a probability distribution.

But, the sum of all probabilities is not equal to 1.0 because $\Sigma P(x_i)=0.18+0.01+0.29+0.37=0.85$ and the second condition is not satisfied.

This table **does not represent** a valid probability distribution.

Exercise 1

Each of the following tables lists certain values of x and their probabilities. Determine whether or not each table represents a valid probability distribution.

a)

x_i	$P(x_i)$
0	0.18
1	0.01
2	0.29
3	0.37

b)

x_i	$P(x_i)$
2	0.35
3	0.24
4	0.18
5	0.23

c)

x_i	$P(x_i)$
7	0.65
8	0.50
9	-0.15

Solution

b) Each probability listed in this table is in the range 0 to 1.

Also, $\Sigma P(x_i)=0.35+0.24+0.18+0.23=1.0$

Consequently, this table **represents** a valid probability distribution.

Exercise 1

Each of the following tables lists certain values of x and their probabilities. Determine whether or not each table represents a valid probability distribution.

a)

x_i	$P(x_i)$
0	0.18
1	0.01
2	0.29
3	0.37

b)

x_i	$P(x_i)$
2	0.35
3	0.24
4	0.18
5	0.23

c)

x_i	$P(x_i)$
7	0.65
8	0.50
9	-0.15

Solution

c) The sum of all probabilities listed in this table is equal to 1.0, **but** one of the probabilities is negative.

This violates the first condition of a probability distribution.

Therefore, this table **does not represent** a valid probability distribution.

Exercise 2

We toss 3 coins. Let X be a random variable that counts the number of heads. Obtain the probability distribution of X.

Also calculate the probability of having :

1. two heads;
2. no head;
3. more than one head;
4. at least one head;
5. less than three heads;
6. at most one head.

Solution

The **8** possible **outcomes**, w, of the experiment and the probabilities with which they occur, P(w), are:

w	P (w)
HHH	1/8
HHT	1/8
HTH	1/8
HTT	1/8
THH	1/8
THT	1/8
TTH	1/8
TTT	1/8

Exercise 2

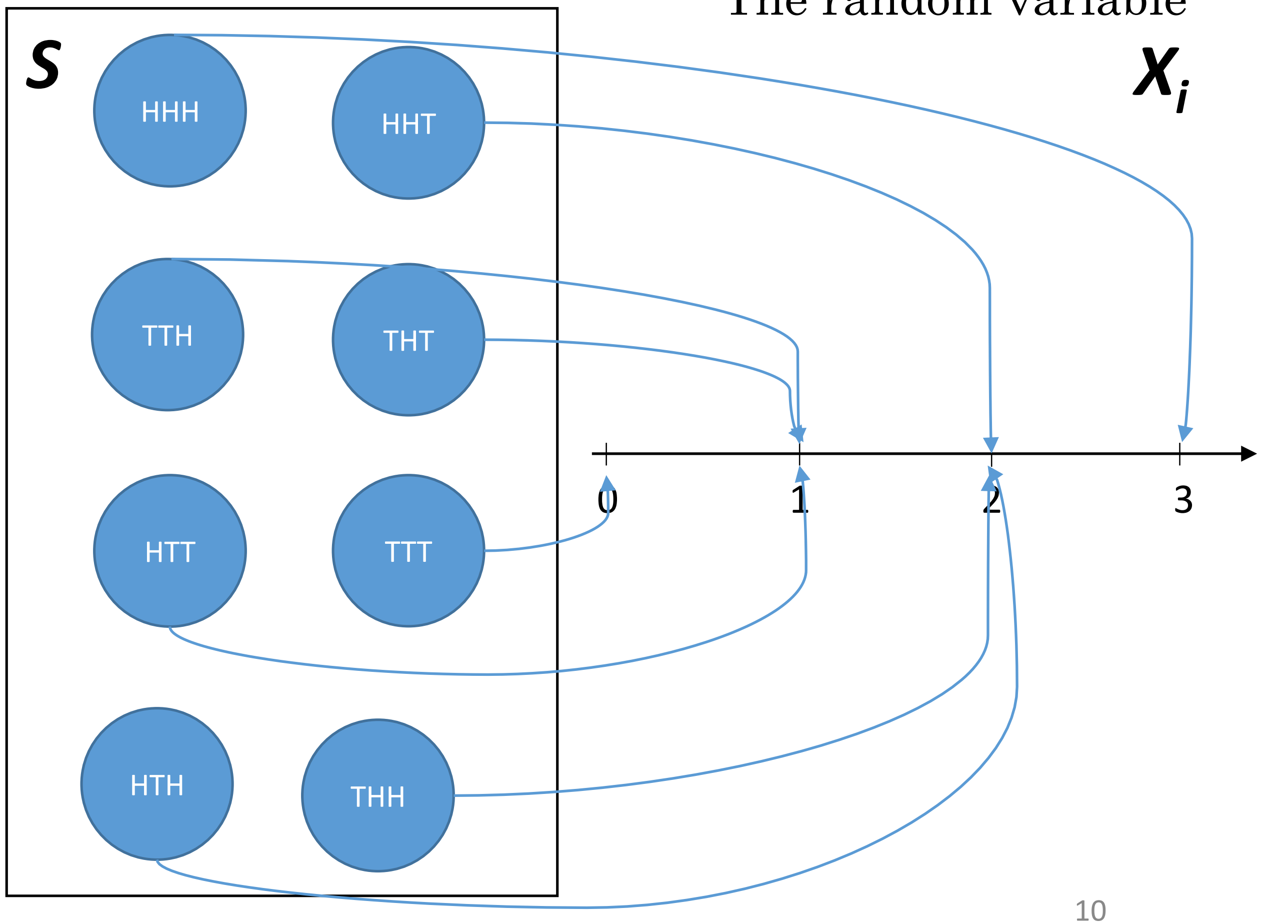
We toss 3 coins. Let X be random variable that counts the number of heads. Obtain the probability distribution of X.

Solution

X is the random variable that counts the number of heads.

w	P (w)	X_i
HHH	1/8	3
HHT	1/8	2
HTH	1/8	2
THH	1/8	2
HTT	1/8	1
THT	1/8	1
TTH	1/8	1
TTT	1/8	0

The random variable



Exercise 2

We toss 3 coins. Let X be random variable that counts the number of heads. Obtain the probability distribution of X.

Also calculate the probability of having :

1. two heads;
2. no head;
3. more than one head;
4. at least one head;
5. less than three heads;
6. at most one head.

Solution

w	P (w)	X_i
HHH	1/8	3
HHT	1/8	2
HTH	1/8	2
THH	1/8	2
HTT	1/8	1
THT	1/8	1
TTH	1/8	1
TTT	1/8	0

So, the probability distribution of X is:

x_i	$P(X = x_i)$
0	1/8
1	3/8
2	3/8
3	1/8

$$\Sigma P(X_i) = 1$$

Exercise 2

We toss 3 coins. Let X be random variable that counts the number of heads. Obtain the probability distribution of X.

Also calculate the probability of having :

1. two heads;
2. no head;
3. more than one head;
4. at least one head;
5. less than three heads;
6. at most one head.

Solution

From this probability distribution we can calculate the probability from 1. to 6.:

x_i	$P(X = x_i)$
0	1/8
1	3/8
2	3/8
3	1/8

1. $P(X = 2) = 3/8$
2. $P(X = 0) = 1/8$
3. $P(X > 1) = P(X = 2) + P(X = 3) = 3/8 + 1/8 = 4/8 = 1/2$
4. $P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) = 1/8 + 3/8 + 3/8 = 7/8$
5. $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = 7/8$

   or $P(X < 3) = 1 - P(X = 3)$ [2^{nd} condition of prob. distr.] $= 1 - 1/8 = 7/8$
6. $P(X \leq 1) = P(X = 0) + P(X = 1) = 1/8 + 3/8 = 4/8 = 1/2$

Exercise 3

The following table lists the probability distribution of the number of breakdowns per week for a machine based on past data.

Breakdowns per week	0	1	2	3
Probability	0.15	0.20	0.35	0.30

Find the probability that the number of breakdowns for this machine during a given week is:

a) exactly 2
b) less than 2
c) more than 1
d) at most 1

Solution

a) $P(X=2) = 0.35 = 35\%$

b) $P(X<2)=P(X=0)+P(X=1) = 0.15+0.20=0.35=35\%$

c) $P(X>1)=P(X=2)+P(X=3)=0.35+0.30=0.65=65\%$

d) $P(X\leq 1)=P(X=0)+P(X=1)=0.15+0.2=0.35=35\%$

THEME #2

Expected value and Variance of random variables

Expected Value

E(X) is the mean of the probability distribution

$$E(X) = \mu = \Sigma x_i \cdot P(x_i)$$

Variance and Standard Deviation

They measure the spread of the probability distribution:

$$Var\,(X) = \sigma^2 = E(X - E(X))^2 = E(X^2) - E(X)^2$$

For discrete, $V(X) = \sigma^2 = \Sigma x_i^2 \cdot P(x_i) - \mu^2$

Standard Deviation

$$SD(X) = \sigma = \sqrt{\sigma^2}$$

Exercise 4

Let X be the random variable indicating the number of textbooks required by a randomly chosen statistical course.

The r.v. X can assume only the values between 0 and 5.

The table below provides the probability distribution of the random variable:

X_i	0	1	2	3	4	5
$P(X = x_i)$	?	0.45	0.24	0.12	0.09	0.05

a. What is the of the missing probability?
b. What is the probability that a course will require 2 or more textbooks?
c. What is the probability that a course require 2 or 3 textbooks?
d. Find the expected value of the random variable X.
e. Find the variance and the standard deviation of the random variable X.

Exercise 4

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X_i	0	1	2	3	4	5
$P(X = x_i)$	?	0.45	0.24	0.12	0.09	0.05

Solution

a. What is the of the missing probability?

The probability distributions P(X = x) have 2 characteristics:

1. For each value of X, $0 \leq P(Xi) \leq 1$
2. $\Sigma P(Xi) = 1$

From the second characteristic we have:
$P(X = 0) + \sum P(X = x_i) = 1$ then $P(X = 0) = 1 - \sum P(X = x_i) = 1 - 0.95 = 0.05$

The probability of P(X=0) is 0.05

Exercise 4

Let X be the random variable indicating the number of textbooks required by a randomly chosen statistical course.

The r.v. X can assume only the values between 0 and 5.

The table below provides the probability distribution of the random variable:

X_i	0	1	2	3	4	5
$P(X = x_i)$	0.05	0.45	0.24	0.12	0.09	0.05

Solution

b. What is the probability that a course will require **2** or **more** textbooks?

$P(X \geq 2)$ is a sum of probability:

$P(X \geq 2)= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = 0.24 + 0.12 + 0.09 + 0.05 = 0.5 = 50\%$

or the probability of the complementary event: $1 - P(X < 2) = 1 – P(X=0) – P(X=1) = 1 – 0.45 – 0.05 = 0.5$

c. What is the probability that a course require **2 or 3** textbooks?

The two event are <u>disjoint</u> so:

$P(X=2 \text{ or } 3) = P(X=2 \cup X=3) = P(X=2) + P(X=3) =0.24+0.12 = 0.36 = 36\%$

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The table below provides the probability distribution of the random variable:

X_i	0	1	2	3	4	5
$P(X = x_i)$	0.05	0.45	0.24	0.12	0.09	0.05

Solution

d. Find the **expected value** of the random variable X.

For a discrete random variable the expected value E(X) is: $\mu = E(X) = \sum x_i \cdot P(X = x_i)$

X_i	$P(X = x_i)$	$x_i \cdot P(X = x_i)$
0	0.05	0.00
1	0.45	0.45
2	0.24	0.48
3	0.12	0.36
4	0.09	0.36
5	0.05	0.25
	1.00	1.90

So the expected value of X is:

$E(X) = \mu = 1.9$

Exercise 4

Let X be the random variable indicating the number of textbooks required by a randomly chosen statistical course.

The r.v. X can assume only the values between 0 and 5.

The table below provides the probability distribution of the random variable:

X_i	0	1	2	3	4	5
$P(X = x_i)$	0.05	0.45	0.24	0.12	0.09	0.05

Solution

e. Find the **variance** and the **standard deviation** of the random variable X

For a discrete random variable the variance is:

$$\sigma^2 = Var(X) = \sum (x_i - \mu)^2 \cdot P(X = x_i)$$

or

$$\sigma^2 = E(X^2) - \mu^2 = E(X^2) - (E(X))^2 = \sum x_i^2 \cdot P(X = x_i) - \mu^2$$

Use the second notation…

Exercise 4

Let X be the random variable indicating the number of textbooks required by a randomly chosen statistical course.

The r.v. X can assume only the values between 0 and 5.

The table below provides the probability distribution of the random variable:

X_i	0	1	2	3	4	5
$P(X = x_i)$	0.05	0.45	0.24	0.12	0.09	0.05

Solution

e. (cont.)

X_i	$P(X = x_i)$	$x_i^2 \cdot P(X = x_i)$
0	0.05	$0^2 \cdot 0.05$
1	0.45	$1^2 \cdot 0.45$
2	0.24	$2^2 \cdot 0.24$
3	0.12	$3^2 \cdot 0.12$
4	0.09	$4^2 \cdot 0.09$
5	0.05	$5^2 \cdot 0.05$
	1.00	5.18

$$\sigma^2 = E(X^2) - \mu^2 = \sum x_i^2 \cdot P(X = x_i) - \mu^2$$

$$\sigma^2 = 5.18 - 1.90^2 = 5.18 - 3.61 = 1.57$$

σ is the square root of σ^2

$$\sigma = \sqrt{1.57} = 1.25$$

THEME #3

—

Bernoulli, Binomial and Poisson probability distribution

Bernoulli Probability Distribution

Only 2 values (1="success", 0 = "failure"). 1 with probability p, 0 with probability $(1 - p)$

Probability function: $$P(x_i) = p^{x_i} \times (1-p)^{(1-x_i)}$$

$$E(X) = p \quad \text{and} \quad V(X) = p \times (1-p)$$

ex. Accept or decline an investment, vote yes or no on a ballot, etc.

Binomial Random Variable

Represents the number of successes in n Bernoulli experiments : a) independent b) with equal $\boldsymbol{p}$

Probability function: $$P(x) = \binom{n}{x} p^x \times (1-p)^{(n-x)}$$

$$E(X) = n \times p \quad \text{and} \quad V(X) = n \times p \times (1-p)$$

ex. Number of heads obtained tossing a coin 10 times.

Exercise 4

A club membership is renewed with **probability 0.65**. Let X be the random variable that represents the decision of each member to renew the subscription.

What is the distribution of X? Obtain E(X) and V(X).

Solution

The distribution is a Bernoulli probability distribution because the possible events are only 2:

“renew the membership” (event 1) or “not renew the membership” (event 0).

The probability distribution is:

$$P(X=x)=p^{x} \cdot (1-p)^{(1-x)} = 0.65^{x} \cdot (1-0.65)^{(1-x)}$$

The expected value of X for a Bernoulli distribution is:

$E(X) = p$

So, $E(X) = p = 0.65 = 65\%$

The variance of X for a Bernoulli distribution is:

$Var(X)= p \cdot (1-p)$

So, $Var(X)= p \cdot (1-p) = 0.65 \cdot (1-0.65) = 0.65 \cdot 0.35 = 0.2275 = 22.75\%$

Exercise 5

Based on his experience a travel agent believes that only 60% of the tickets booked then are bought.

If 8 customers have gone independently to the travel agency for a reservation, determine the probability that:

a. five buy a ticket;
b. everyone buys a ticket;
c. at least 2 buy a ticket.

Solution

The X is a Binomial because the customers can only BUY (event 1) or NOT BUY (event 0) a ticket.

The r.v. X is a Binomial distribution $X \sim Bin(n, p) = Bin(8; 0.6)$

where **n = 8** (the customers) and **p=0.6** (the probability of buying a ticket)

The probability function is:

$$P(X = x) = \binom{n}{x} p^x \cdot (1 - p)^{(n-x)}$$

a. The probability that **x = 5** customers buy a ticket is:

$$P(X = 5) = \binom{n}{x} p^x \cdot (1 - p)^{(n-x)} = \frac{n!}{(n-x)!\,x!} p^x \cdot (1 - p)^{(n-x)} = \binom{8}{5} 0.60^5 \cdot (1 - 0.60)^{(8-5)} =$$

$$= 8!/(5! \cdot 3!) \cdot 0.078 \cdot 0.064 = 56 \cdot 0.078 \cdot 0.064 = 0.28 = 28\%$$

Exercise 5

Based on his experience a travel agent believes that only 60% of the tickets booked then are bought.

If 8 customers have gone independently to the travel agency for a reservation, determine the probability that:

a. five buy a ticket;

b. everyone buys a ticket;

c. at least 2 buy a ticket.

Solution

b. The probability that **all customers** buy a ticket is:

$$P\,(\mathbf{X = 8}) = \binom{n}{x}\, p^{x} \cdot (1 - p)^{(n-x)} = \binom{8}{8}\, 0.60^{8} \cdot (1 - 0.60)^{(8-8)} =$$

$$= 8!/(8!\cdot 0!) \cdot\ 0.0168 \cdot 1 = 1 \cdot 0.0168 = 1.68\%$$

From the complementary statement of probability

c. The probability that **at least two customers** buy a ticket, $P(X \geq 2)$, is:

$$P\,(\mathbf{X \geq 2}) = 1 - P\,(X < 2) =$$

$$= 1 - P(X{=}1) - P(X{=}0) =$$

$$= 1 - \binom{8}{1}\, 0.60^{1} \cdot (1 - 0.60)^{(8-1)} - \binom{8}{0}\, 0.60^{0} \cdot (1 - 0.60)^{(8-0)} =$$

$$= 1 - 8!/(7!\cdot 1!) \cdot 0.6 \cdot 0.4^{7} - 8!/(0!8!) \cdot 0.6^{0} \cdot 0.4^{8} =$$

$$= 1 - 8 \cdot 0.6 \cdot 0.0016 - 1 \cdot 1 \cdot 0.0006 = 1 - 0.0077 - 0.0006 = 0.9917 = 99.17\%$$

Poisson Probability Distribution

Represents the number of occurrences in a given interval. Occurrences have to be random and independent

Probability function: $$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

with λ as average number of occurrences in a given interval and x number of occurrences (in the same interval)

$$E(X) = V(X) = \lambda$$

ex. Number of telemarketing phone calls received in a day

Exercise 6

On average a household receives 3 telemarketing phone calls per month. Find the probability that a randomly selected household receives:

a. exactly 2 calls during a given month;

b. exactly 1 call during a given week (assume 4 weeks each month);

c. Find the expected value

Solution

To calculate the points a. b. and c. we have to use the Poisson probability distribution

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

where λ is the number of telemarketing calls received <u>per month</u> ($\boldsymbol{\lambda} = \mathbf{3}$)

a. We have to calculate $P(\boldsymbol{X} = \mathbf{2})$.

Using the Poisson distribution, $P(X = 2) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{3^2 e^{-3}}{2!} =$ 9 · 0.049 / 2 = 0.2205 = 22.5%

b. If the calls per month are 3, the calls per week are:

$$\lambda = 3 \text{ calls per month } /4 \text{ weeks} = 0.75 \text{ calls per week}$$

So, $P(X = 1)$ = 0.75^1 · e^(-0.75) / 1! = 0.75 · 0.4724 = 0.3543 = 35.43%

c. For the Poisson distribution: E(X) = V(X) = λ = 3

Hypergeometric Distribution

The hypergeometric distribution is very similar to the binomial distribution.

Represents the number of successes in n **not independent** Bernoulli experiments

Values (integers): 0,1,2,…,n

Probability function: $$P(X = x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

with:

N = population size,

r= nr successes in population,

n = sample size,

x=nr successes in sample

Exercise 8

A firm has 12 employees, 7 females and 5 males. The company is planning to send 3 of them to a conference. Find the probability that all 3 of them are female.

Solution

Single employee is a Bernoulli experiment (1 if female, 0=otherwise), but in this case the event is not independent (each employer is different)

The hypergeometric distribution is:

$$P(X = x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

N=12, n=3, r=7, x=3

$$P(X = x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} = \frac{\binom{7}{3}\binom{12-7}{3-3}}{\binom{12}{3}} = \frac{7}{44} = 0.1591$$

Recap on Discrete Random Variables

For Binomial and Bernoulli sometimes you can find q instead of (1-p)

Bernoulli Probability Distribution

$P(x_i) = p^{x_i}\,(1-p)^{(1-x_i)}$ $E(X) = p$ and $Var(X) = p \cdot (1-p)$

Binomial Probability Distribution

$P(x) = \binom{n}{x} p^x\,(1-p)^{(n-x)}$ with n as total number of occurrences and x number of selected occurrences

$E(X) = n \cdot p$ and $Var(X) = n \cdot p \cdot (1-p)$

Poisson Probability Distribution

$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$ with λ as average number of occurrences in a given interval and x number of occurrences

$E(X) = Var(X) = \lambda$

Exercise 9

For each point, indicate first what probability distribution is used to solve the problem.

a. 60% of people who purchase sports cars are men. If 10 sports car owners are randomly selected, find the probability that exactly 7 are men.

b. The evaporation of water for a pool in California is (on average) 2 centimeters per month (30 days). Find the probability that the pool has lost 5 centimeters of water in August.

c. 1% of the product sold by Apple have some bugs. Find the probability that for 100 products sold at most 1 have bugs.

d. A company receives on average 3 returns of defective product per month. Find the probability that a randomly selected product sold in October could be defective.

Exercise 9

Solution

a. 60% of people who purchase sports cars are men. If 10 sports car owners are randomly selected, find the probability that exactly 7 are men.

a. We have to use the Binomial Distribution because the possible outcome are only 2:
men who **PURCHASE** sports car (event 1) and men who **NOT PURCHASE** sports car (event 0).
Identify 'n' and 'x' from the problem.
n is the number of randomly selected items, that in this case is sports car owners, and is **10**, and x, the number of men you are asked to "find the probability" for, that is 7
Find "p" the probability of success and "1-p" or "q" the probability of failure.
We are given **p = 60%,** or 0.6. therefore, the probability of failure is 1 – 0.6 = 0.4 (40%)

$$P(X=7)= \frac{n!}{(n-x)!\,x!} \times p^x \times (1-p)^{n-x} = \frac{10!}{3!\,7!} \times 0.6^7 \times 0.4^3 = 120 \times 0.0279936 \times 0.064 = 0.215 = 21.5\%$$

b. The evaporation of water for a pool in California is (on average) 2 centimeters per month (30 days).
Find the probability that the pool has lost 5 centimeters of water in August.

b. We have to use the Poisson Distribution because we have number of occurrence (cm of evaporation) in a given interval of time (a month).
Identify 'λ' and 'x' from the problem.
λ=2 (evaporation per month) and x=5 (evaporation in August)

$$P(X=5) = \frac{\lambda^x e^{-\lambda}}{x!} = 2^5 \times e^{-2} / 5! = 0.3609 = 3.6\%$$

Exercise 9

Solution

c. 1% of the product sold by Apple have some bugs. Find the probability that for 100 products sold at most 1 have bugs.

c. We have to use the Binomial Distribution because the possible outcome are only 2: the product **has a bug** (event 1) and the product **has NOT a bug** (event 0).
Identify 'n' and 'x' from the problem.

n=100 x=1

The probability are: **p = 0.01** and (1-p)=0.99

P(X=1)= $\frac{n!}{(n-x)!\,x!}$ × p^x × $(1\text{-}p)^{n-x}$ = $\frac{100!}{99!1!}$ × 0.01^1 × 0.99^{99} = 100 × 0.01 × 0.3697 = 0.3697= 36.97%

d. A company receives on average 3 returns of defective product per month. Find the probability that a randomly selected product sold in October could be defective.

d. We have to use the Poisson Distribution because we have number of occurrence (return of products) in a given interval of time (a month).
Identify 'λ' and 'x' from the problem.
λ=3 (return products per month) and x=1 (defective product sold in October)

$P(X = 1) = \frac{\lambda^x e^{-\lambda}}{x!}$ = $3^1 \times e^{-3}$ / 1! = 3 × 0,049 / 1 = 0,147 = 14,7%

Exercise 10

On average a household receives 3 telemarketing phone calls per month. Find the probability that a randomly selected household receives:

a. exactly 2 calls during a given month;

b. exactly 1 call during a given week (assume 4 weeks each month);

c. Find the expected value

Solution

To calculate the points a. b. and c. we have to use the Poisson probability distribution

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

where λ is the number of telemarketing calls received per month (λ =3)

a. We have to calculate $P(X = 2)$.

Using the Poisson distribution, $P(X = 2) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{3^2 e^{-3}}{2!} = 9 \cdot 0.049 / 2 = 0.2205 = 22.5\%$

b. If the calls per month are 3, the calls per week are:

$$\lambda = 3 \text{ calls per month } /4 \text{ weeks} = 0.75 \text{ calls per week}$$

So, $P(X = 1) = 0.75\text{^}1 \cdot e\text{^}(-0.75) / 1! = 0.75 \cdot 0.4724 = 0.3543 = 35.43\%$

c. For the Poisson distribution: E(X) = Var(X) = λ = 3