

Derivatives

Definition

A function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be differentiable in $x_0 \in \text{int}(D)$

($\text{int}(D)$ denotes the set of the interior points of D)

if:

$$\exists L = \lim_{h \rightarrow 0} \underbrace{\frac{f(x_0 + h) - f(x_0)}{h}}_{\text{difference quotient}}.$$

We call $L = f'(x_0)$. We define the left and right derivative of f in x_0 the two limits:

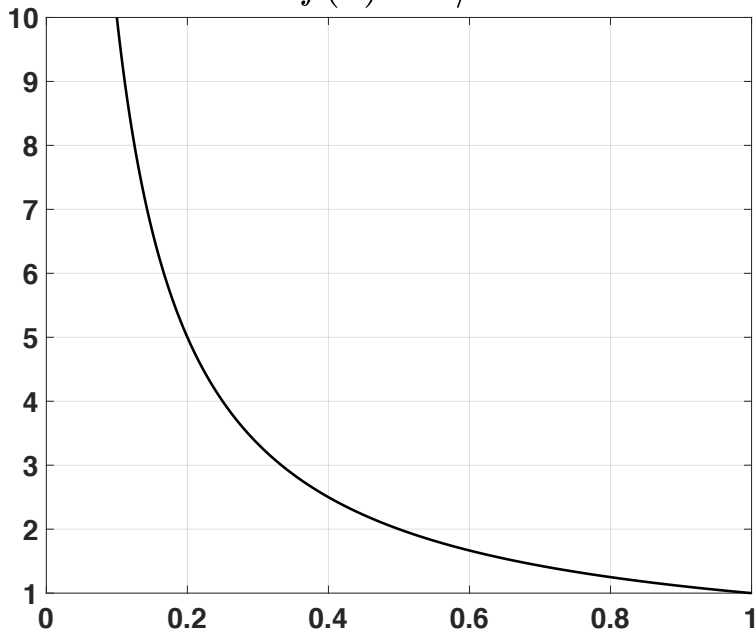
$$\lim_{h \rightarrow 0^-} \frac{f(x + h) - f(x)}{h}, \quad \lim_{h \rightarrow 0^+} \frac{f(x + h) - f(x)}{h},$$

when they exist and we call them $f'(x_0^-)$ and $f'(x_0^+)$ respectively.

Remark

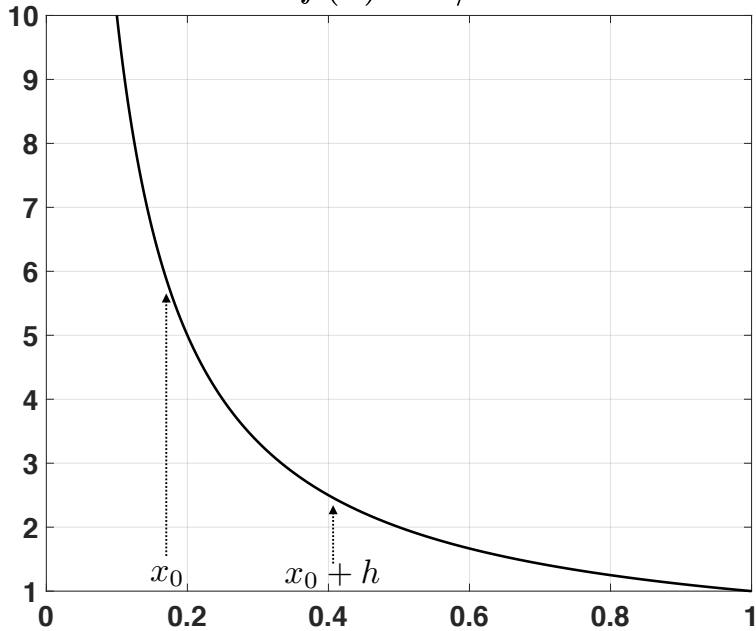
Note that the left and right derivatives can be defined even if x_0 belongs to the closure of D .

$$f(x) = 1/x$$



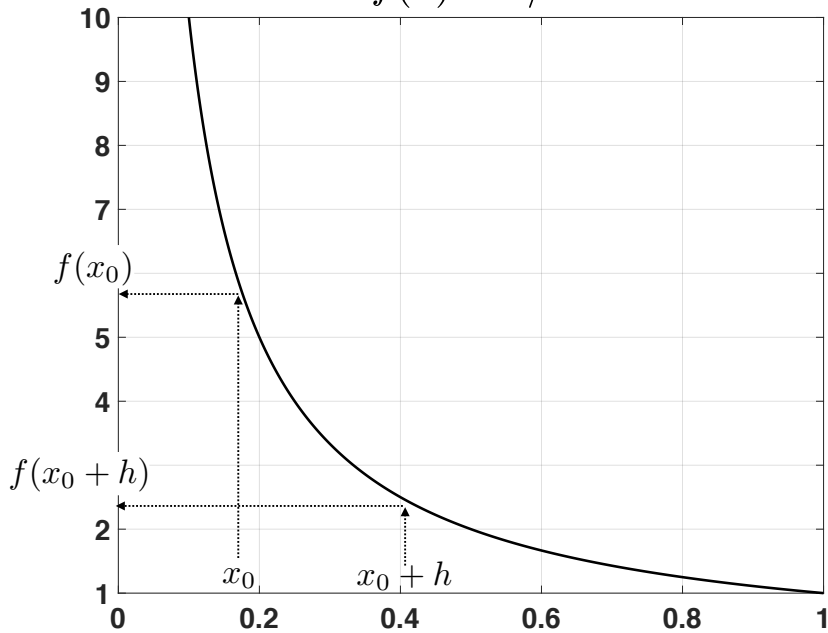
x

$$f(x) = 1/x$$



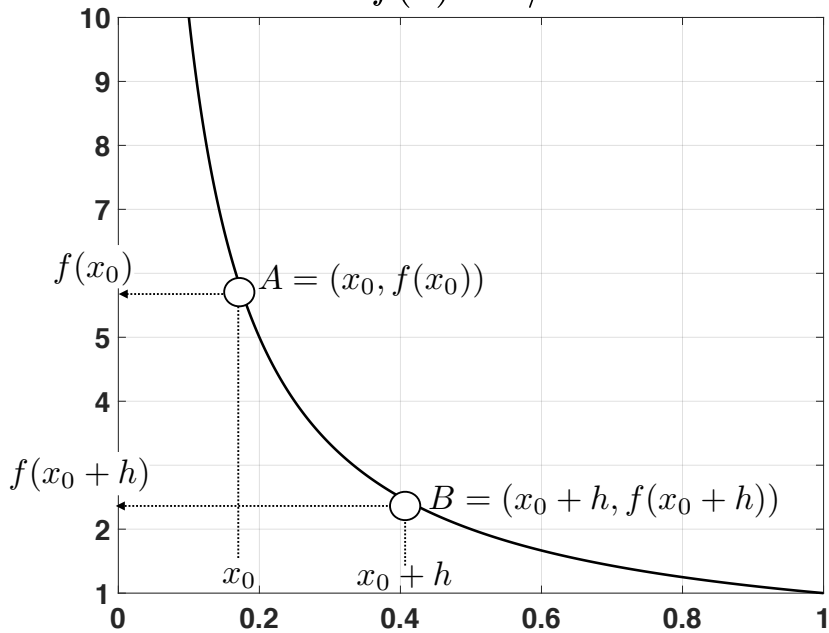
x

$$f(x) = 1/x$$



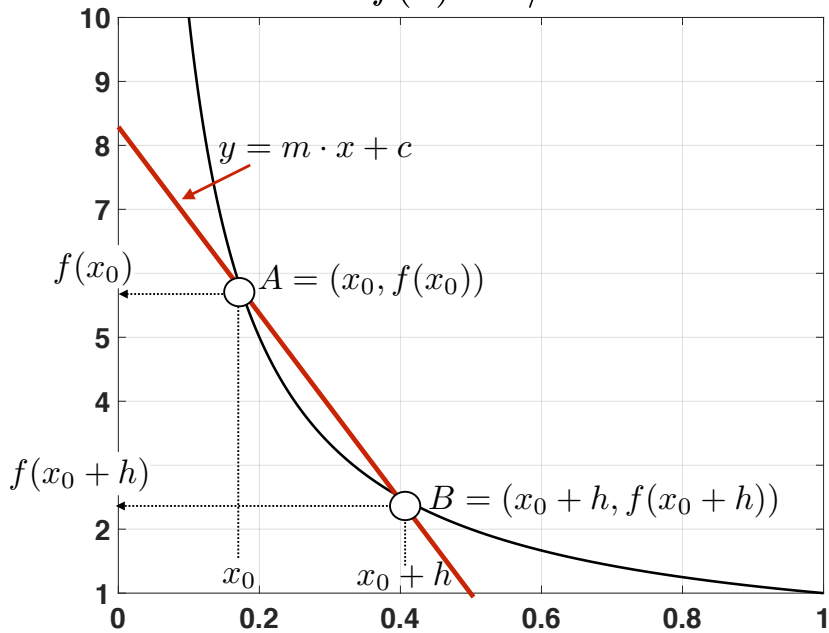
x

$$f(x) = 1/x$$



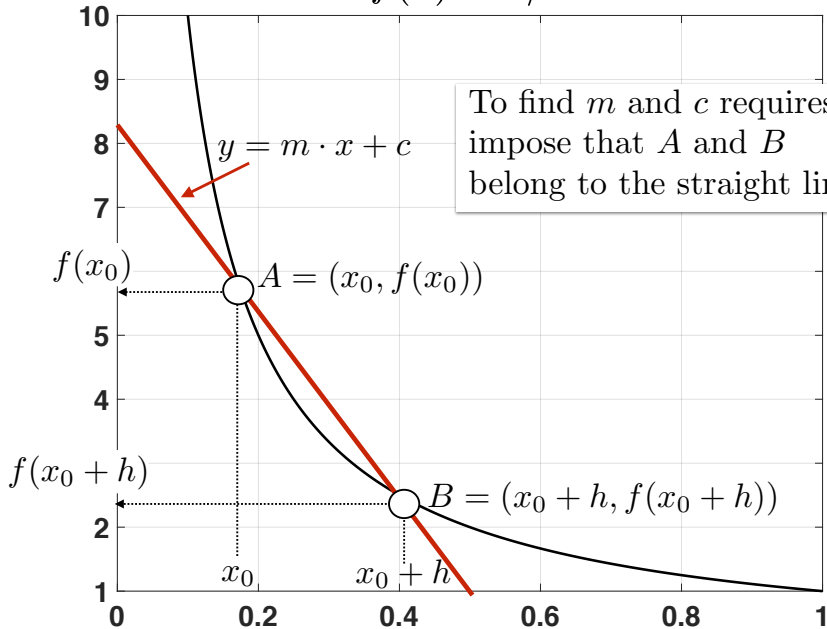
x

$$f(x) = 1/x$$



x

$$f(x) = 1/x$$



To find m and c requires to impose that A and B belong to the straight line...

Derivatives

Geometric interpretation of the derivative

Let f be differentiable in x_0 . Find the equation of the straight line passing through $A = (x_0, f(x_0))$ and $B = (x_0 + h, f(x_0 + h))$.

- Generic equation of the straight line $y = mx + c$
- A belongs to the line $\Leftrightarrow f(x_0) = m_h x_0 + c_h$.
- B belongs to the line $\Leftrightarrow f(x_0 + h) = m_h (x_0 + h) + c_h$.

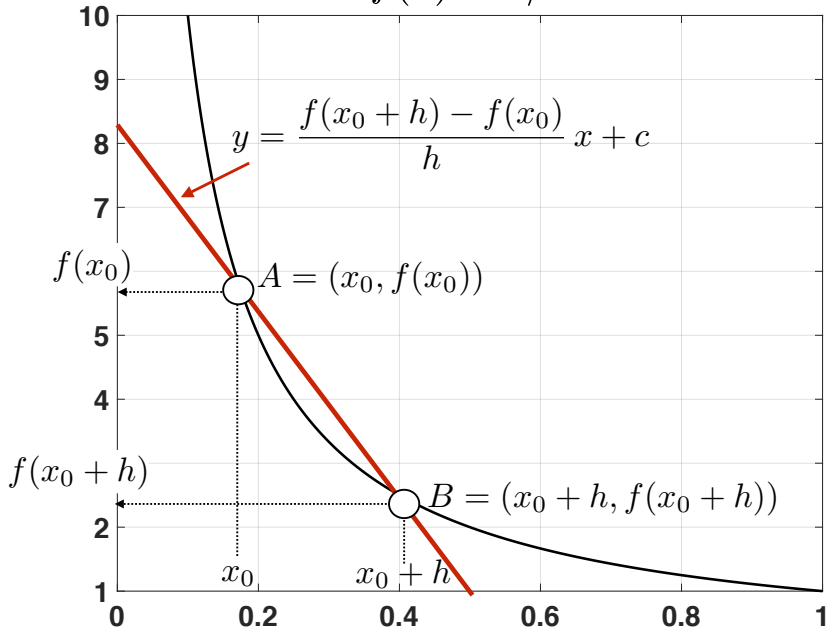
Whence

$$f(x_0 + h) - f(x_0) = m_h x_0 + m_h h + c_h - m_h x_0 - c_h = m_h h \Rightarrow$$

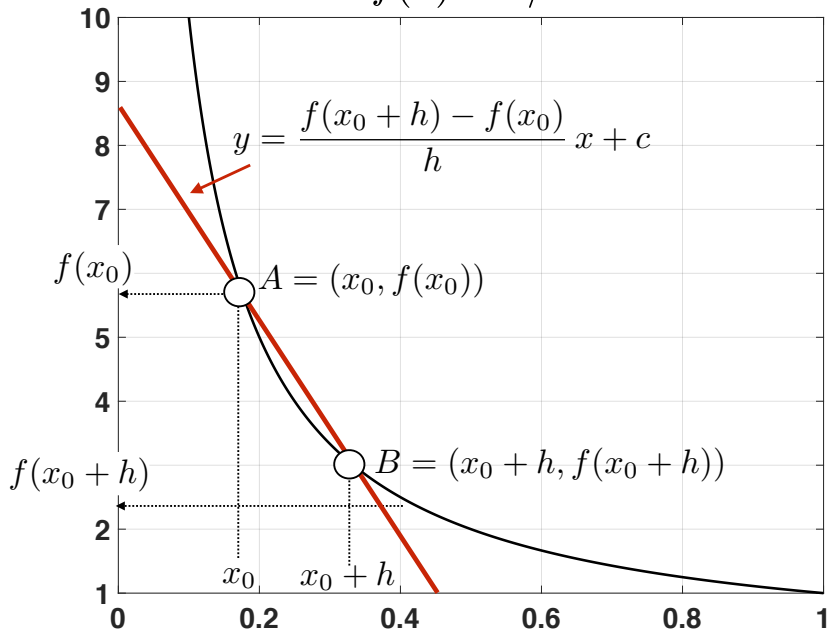
$$m_h = \frac{f(x_0 + h) - f(x_0)}{h}$$

If $h \rightarrow 0$ then $m_h \rightarrow f'(x_0) \Rightarrow$ The derivative is the angular coefficient of the line tangent to the graph.

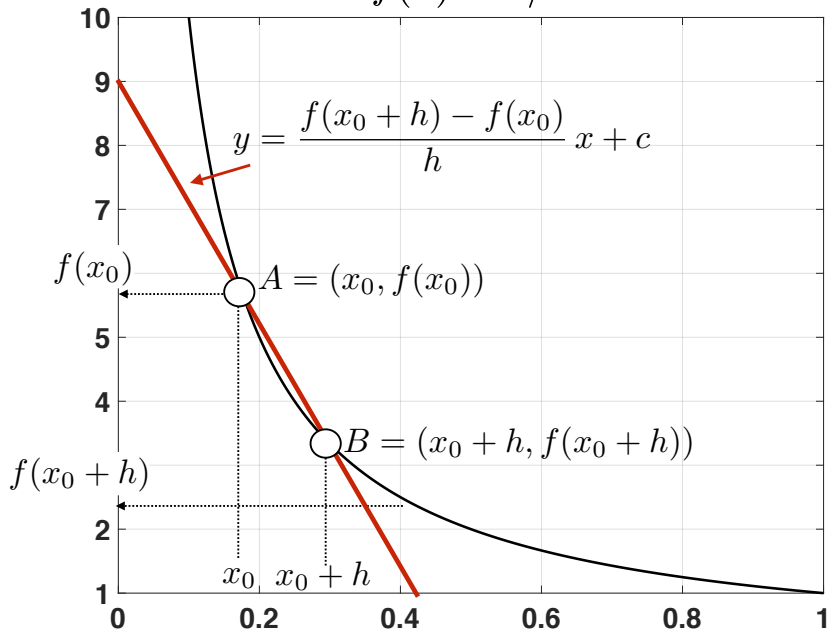
$$f(x) = 1/x$$



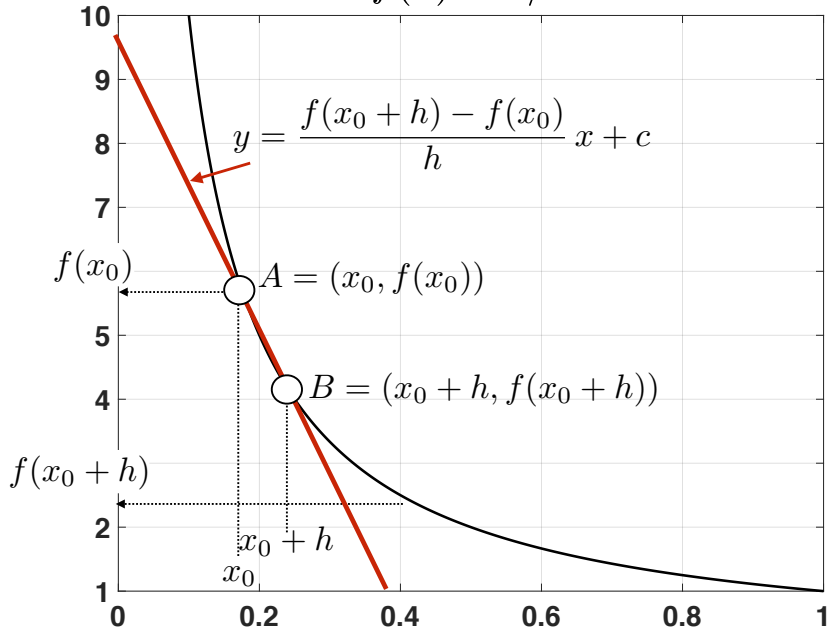
$$f(x) = 1/x$$



$$f(x) = 1/x$$



$$f(x) = 1/x$$



Derivatives

Theorem

If f is differentiable in x_0 then it is continuous in x_0 .

Proof. We know that the limit

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0),$$

exists and it is finite. Whence

$$\begin{aligned} \lim_{x \rightarrow x_0} (f(x) - f(x_0)) &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \\ &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \rightarrow x_0} (x - x_0) \\ &= f'(x_0) \cdot 0 = 0, \end{aligned}$$

which means

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

Derivatives

Problem

Is the converse true?

That is, if $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is continuous in x_0 , can we say that f is differentiable in x_0 ?

Answer

No! Differentiability is a condition much stronger than continuity.

As always, to prove our assertion, we need at least one counter-example:

$$f(x) = |x|,$$

is continuous in $x_0 = 0$, nevertheless

$$\lim_{h \rightarrow 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1, \quad \lim_{h \rightarrow 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1,$$

that is $f'(x_0^+) = +1 \neq f'(x_0^-) = -1$, whence $\nexists f'(0)$.

Definition

The point $x_0 = 0$ it's called an **angle point**. More generally, we say that a function $f(x)$ has an angle point in x_0 whenever the derivative has a jump discontinuity in x_0 .

Theorem

Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $g : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be two functions differentiable in x_0 .

For all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, the function $\alpha f + \beta g : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x), \quad \forall x \in D$$

is differentiable in x_0 and

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0).$$

Derivatives

Proof.

We have to prove the limit of the difference quotient

$$\lim_{h \rightarrow 0} \frac{(\alpha f + \beta g)(x_0 + h) - (\alpha f + \beta g)(x_0)}{h}$$

exists and it is finite. For this purpose note that

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{(\alpha f + \beta g)(x_0 + h) - (\alpha f + \beta g)(x_0)}{h} \\ = & \lim_{h \rightarrow 0} \frac{\alpha f(x_0 + h) + \beta g(x_0 + h) - \alpha f(x_0) - \beta g(x_0)}{h} \\ = & \lim_{h \rightarrow 0} \frac{\alpha f(x_0 + h) - \alpha f(x_0) + \beta g(x_0 + h) - \beta g(x_0)}{h} \\ = & \lim_{h \rightarrow 0} \left[\alpha \frac{f(x_0 + h) - f(x_0)}{h} + \beta \frac{g(x_0 + h) - g(x_0)}{h} \right] \\ = & \alpha f'(x_0) + \beta g'(x_0). \end{aligned} \tag{0.1}$$

Theorem

Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $g : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be two functions differentiable in x_0 .

The function $f \cdot g : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$(f \cdot g)(x) = f(x) \cdot g(x), \quad \forall x \in D$$

is differentiable in x_0 and

$$(f \cdot g)'(x_0) = f'(x_0) g(x_0) + f(x_0) g'(x_0).$$

Derivatives

Proof. We have to prove the limit of the difference quotient

$$\lim_{h \rightarrow 0} \frac{(f \cdot g)(x_0 + h) - (f \cdot g)(x_0)}{h}$$

exists and it is finite. For this purpose note that

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{(f \cdot g)(x_0 + h) - (f \cdot g)(x_0)}{h} \\ = & \lim_{h \rightarrow 0} \frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0)}{h} \\ = & \lim_{h \rightarrow 0} \frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0 + h) + f(x_0)g(x_0 + h) - f(x_0)g(x_0)}{h} \\ = & \lim_{h \rightarrow 0} \frac{g(x_0 + h)(f(x_0 + h) - f(x_0)) + f(x_0)(g(x_0 + h) - g(x_0))}{h} \\ = & \lim_{h \rightarrow 0} \left[g(x_0 + h) \frac{(f(x_0 + h) - f(x_0))}{h} + f(x_0) \frac{(g(x_0 + h) - g(x_0))}{h} \right]. \end{aligned}$$

g differentiable in $x_0 \Rightarrow g$ continuous in $x_0 \Rightarrow g(x_0 + h) \rightarrow g(x_0)$.

Whence

$$\lim_{h \rightarrow 0} \frac{(f \cdot g)(x_0 + h) - (f \cdot g)(x_0)}{h} = g(x_0)f'(x_0) + f(x_0)g'(x_0).$$

Theorem

Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $g : E \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with $f(D) \subseteq E$.

Hence it is possible to define $(g \circ f) : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ in the standard way

$$(g \circ f)(x) = g(f(x)), \quad \forall x \in D.$$

Assume f is differentiable in $x_0 \in D$ and g is differentiable in $f(x_0) \in E$.

Then $(g \circ f)$ is differentiable in x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0)) f'(x_0).$$

Proof. The proof is rather technical and we skip it.

- Compute the derivative of a constant function, $f(x) = c$ for all $x \in \mathbb{R}$.

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = 0 \Rightarrow f'(x) = 0.$$

- Compute the derivative of $f(x) = x$ for all $x \in \mathbb{R}$.

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{x+h - x}{h} = 1 \Rightarrow f'(x) = 1.$$

- For $n \in \mathbb{N}$, $n > 1$, compute the derivative of $f(x) = x^n$ for all $x \in \mathbb{R}$.

$$\begin{aligned}
 & \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\
 = & \lim_{h \rightarrow 0} \frac{\binom{n}{0} x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + \binom{n}{n} h^n - x^n}{h} \\
 = & \lim_{h \rightarrow 0} \frac{x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + h^n - x^n}{h} \\
 = & \lim_{h \rightarrow 0} \frac{\binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + h^n}{h} \\
 = & \lim_{h \rightarrow 0} \left(\binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \dots + h^{n-1} \right).
 \end{aligned}$$

Since $\lim_{h \rightarrow 0} h = \lim_{h \rightarrow 0} h^2 = \lim_{h \rightarrow 0} h^3 = \dots = \lim_{h \rightarrow 0} h^{n-1} = 0$ we have

$$f'(x) = \binom{n}{1} x^{n-1} = \frac{n!}{1! (n-1)!} x^{n-1} = n x^{n-1}.$$

- For $a > 0, a \neq 1$, compute the derivative of $f(x) = a^x$ for all $x \in \mathbb{R}$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = a^x \ln(a).$$

Since $\ln(e) = 1$ we have $(e^x)' = e^x$.

- Compute the derivative of $f(x) = \sin x$ for all $x \in \mathbb{R}$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \left[\sin x \underbrace{\frac{\cos h - 1}{h}}_{\downarrow 0} + \cos x \underbrace{\frac{\sin h}{h}}_{\downarrow 1} \right] = \cos x. \end{aligned}$$

- Compute the derivative of $f(x) = \cos x$ for all $x \in \mathbb{R}$. Remember that $\cos x = \sin(x + \frac{\pi}{2})$, hence by the rule of derivation of composite function

$$(\cos x)' = \left(\sin \left(x + \frac{\pi}{2} \right) \right)' = \cos \left(x + \frac{\pi}{2} \right) \left(x + \frac{\pi}{2} \right)' = \cos \left(x + \frac{\pi}{2} \right) = -\sin(x).$$

- For $a > 0, a \neq 1$, compute the derivative of $f(x) = \log_a(x)$ for all $x > 0$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\log_a(x+h) - \log_a x}{h} = \lim_{h \rightarrow 0} \frac{\log_a\left(\frac{x+h}{x}\right)}{h} = \lim_{h \rightarrow 0} \log_a\left(\frac{x+h}{x}\right)^{\frac{1}{h}} \\ &= \lim_{h \rightarrow 0} \log_a \left[\left(1 + \frac{h}{x}\right)^{\frac{x}{h}} \right]^{\frac{1}{x}} = \frac{1}{x} \lim_{h \rightarrow 0} \log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}}. \end{aligned}$$

Nevertheless, by changing variable, we get

$$\begin{aligned} \lim_{h \rightarrow 0} \log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}} &= \lim_{t \rightarrow 0} \log_a (1+t)^{\frac{1}{t}} = \lim_{q \rightarrow \infty} \log_a \left(1 + \frac{1}{q}\right)^q \\ &= \log_a \left(\lim_{q \rightarrow \infty} \left(1 + \frac{1}{q}\right)^q \right) = \log_a e. \end{aligned}$$

Whence

$$(\log_a(x))' = \frac{\log_a e}{x} = \frac{1}{x \ln a}.$$

In particular $(\ln(x))' = \frac{1}{x}$.

- For $\alpha \in \mathbb{R}, \alpha \neq 1$, compute the derivative of $f(x) = x^\alpha$ for all $x > 0$.

$$(x^\alpha)' = \left(e^{\ln(x^\alpha)}\right)' = \left(e^{\alpha \ln(x)}\right)' = e^{\alpha \ln(x)} (\alpha \ln(x))' = e^{\alpha \ln(x)} \frac{\alpha}{x} = x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

In addition to **angle points** we have ...

Definition

Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function and let x_0 be a point such that

$$\lim_{x \rightarrow x_0^+} f'(x) = +\infty \text{ and } \lim_{x \rightarrow x_0^-} f'(x) = -\infty,$$

or

$$\lim_{x \rightarrow x_0^+} f'(x) = -\infty \text{ and } \lim_{x \rightarrow x_0^-} f'(x) = +\infty,$$

In this case the point x_0 is called a **cuspid point**.

Definition

Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function and let x_0 be a point such that

$$\lim_{x \rightarrow x_0^+} f'(x) = +\infty \text{ and } \lim_{x \rightarrow x_0^-} f'(x) = +\infty,$$

or

$$\lim_{x \rightarrow x_0^+} f'(x) = -\infty \text{ and } \lim_{x \rightarrow x_0^-} f'(x) = -\infty,$$

In this case the point x_0 is called a **inflection point with vertical tangent**.

Example

$$f(x) = \sqrt{|x|} : \mathbb{R} \rightarrow \mathbb{R},$$

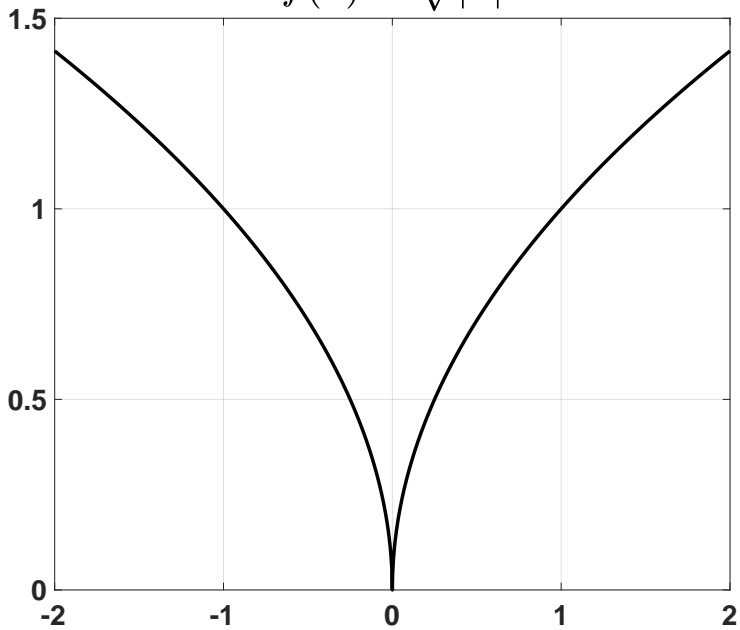
If $x > 0$ then $f(x) = \sqrt{x} = x^{1/2}$, which implies

$$f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \Rightarrow \lim_{x \rightarrow 0^+} f'(x) = +\infty.$$

If $x < 0$ then $f(x) = \sqrt{-x} = (-x)^{1/2}$, which implies

$$f'(x) = \frac{1}{2} (-x)^{\frac{1}{2}-1} (-x)' = -\frac{1}{2} (-x)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{-x}} \Rightarrow \lim_{x \rightarrow 0^-} f'(x) = -\infty.$$

$$f(x) = \sqrt{|x|}$$



Example

$$f(x) = \begin{cases} \sqrt{|x|} & \text{if } x \geq 0 \\ -\sqrt{|x|} & \text{if } x < 0 \end{cases}$$

If $x > 0$ then $f(x) = \sqrt{x} = x^{1/2}$, which implies

$$f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \Rightarrow \lim_{x \rightarrow 0^+} f'(x) = +\infty.$$

If $x < 0$ then $f(x) = -\sqrt{-x} = -(-x)^{1/2}$, which implies

$$f'(x) = -\frac{1}{2} (-x)^{\frac{1}{2}-1} (-x)' = +\frac{1}{2} (-x)^{-\frac{1}{2}} = +\frac{1}{2\sqrt{-x}} \Rightarrow \lim_{x \rightarrow 0^-} f'(x) = +\infty$$

Derivatives: Rolle's Theorem

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) .
If $f(a) = f(b)$ then $\exists x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Proof. Continuity on $[a, b]$ and Weiestrass $\Rightarrow \exists m, M \in [a, b]$:

$$f(m) \leq f(x) \leq f(M), \quad \forall x \in \mathbb{R}.$$

If both $m \notin (a, b)$ and $M \notin (a, b)$ then

$$f(a) = f(b) \Rightarrow f(m) = f(M) \Rightarrow f(x) = f(m) = f(M) \Rightarrow f'(x_0) = 0 \forall x_0.$$

Assume, without loss of generality, that at least $M \in (a, b)$.

Derivatives: Rolle's Theorem

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = f(b)$ then $\exists x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Proof. $M \in (a, b)$ hence for $h > 0$ small $M + h \in (a, b)$. $f(M)$ is a maximum, so $f(M + h) - f(M) \leq 0$ and since $h > 0$ we get

$$\frac{f(M + h) - f(M)}{h} \leq 0 \Rightarrow f'(M^+) \leq 0 \quad (\triangle).$$

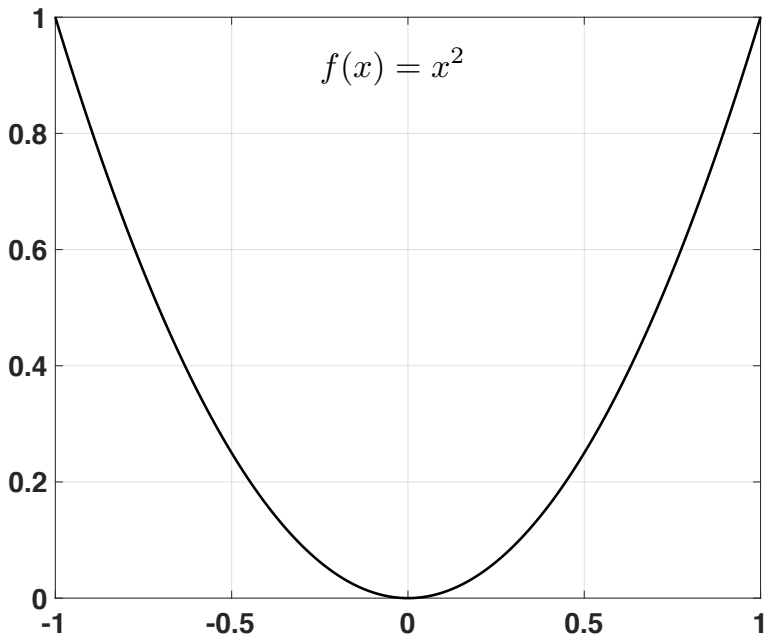
For $h < 0$ (small) the sign changes accordingly

$$\frac{f(M + h) - f(M)}{h} \geq 0 \Rightarrow f'(M^-) \geq 0 \quad (\square).$$

Since f is differentiable we have that

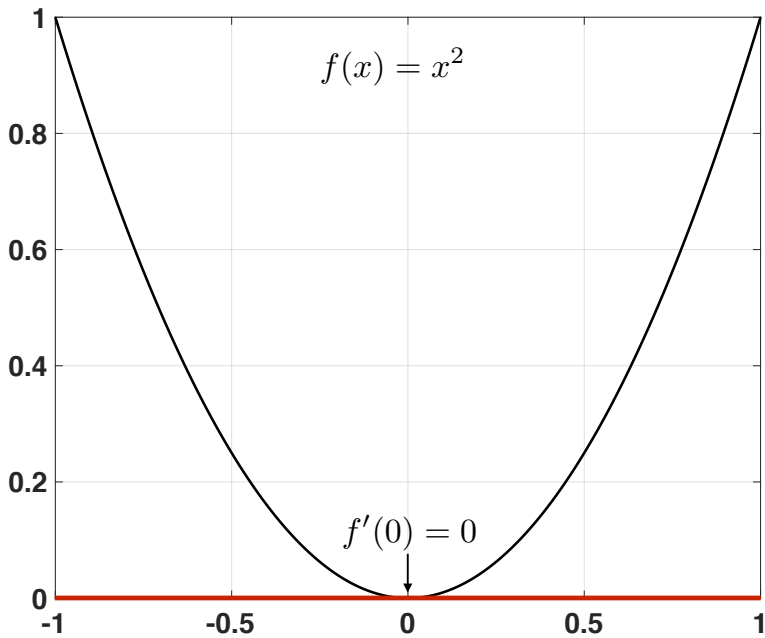
$$f'(M^-) = f'(M^+) = f'(M),$$

which combined with (\triangle) and (\square) gives $f'(M) = 0$.



$$f(x) = x^2$$

x



Derivatives: Lagrange's Mean Value Theorem.

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) .

There exists a point $x_0 \in (a, b)$ such that

$$f(b) - f(a) = (b - a) f'(x_0).$$

Proof. Consider the function $g(x) = f(x) - \alpha x$. Find α such that

$$\begin{aligned} g(a) = g(b) &\Leftrightarrow f(a) - \alpha a = f(b) - \alpha b \Leftrightarrow \alpha(b - a) = f(b) - f(a) \\ &\Leftrightarrow \alpha = \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

g , as f , is continuous in $[a, b]$ and differentiable in (a, b) and $g(a) = g(b)$
 \Rightarrow Rolle's Theorem on g .

$$\exists x_0 \in (a, b) : g'(x_0) = 0 = f'(x_0) - \alpha \Rightarrow f'(x_0) = \frac{f(b) - f(a)}{b - a}. \quad \square$$

Derivatives

Exercise

Use Lagrange's theorem to prove that

$$\ln(x) \leq x, \quad \forall x \in (0, \infty).$$

Solution.

- If $x \in (0, 1)$ we have $\ln(x) < 0 < x$, so the inequality is obvious.
- If $x = 1$ the inequality is $0 \leq 1$, which is true.
- If $x > 1$ Lagrange's theorem on the interval $[1, x]$ gives

$$\frac{\ln(x) - \ln(1)}{x - 1} = \frac{1}{c}$$

with $c \in (1, x)$ and where we have used $(\ln(x))' = 1/x$. In particular $c > 1$. Hence we get

$$\ln(x) = \frac{x - 1}{c} = \frac{x}{c} - \frac{1}{c} < \frac{x}{c} < x$$

where the last inequality follows exactly from $c > 1$.

Exercise

Use Lagrange's theorem to prove that

$$\sin(x) < x, \quad \forall x > 0.$$

Solution.

- If $x > \frac{\pi}{2}$ Then $x > 1$ and so $\sin(x) \leq 1 < \frac{\pi}{2} < x$.
- If $x \in (0, \frac{\pi}{2})$ apply Lagrange's theorem on the interval $[0, x]$

$$\frac{\sin(x) - \sin(0)}{x - 0} = \cos(c)$$

with $c \in (0, x)$. Since $0 < c < x$ we also have that $0 < c < \frac{\pi}{2}$ and hence $0 < \cos(c) < 1$, whence

$$0 < \frac{\sin(x)}{x} = \cos(c) < 1 \Rightarrow \sin(x) < x$$

Exercise

Let $f(t)$ be the GDP of a country. Assume also that $f(0) = 0$ and that f verifies the hypotheses of the Lagrange's theorem.

Assume that

$$f'(t) \leq 7\% \quad \forall t \geq 0.$$

Which is the maximum GDP at time t ?

Solution. Using the Lagrange's theorem on $[0, t]$ applied to the function f we know that

$$\frac{f(t) - f(0)}{t - 0} = f'(c) \leq 7\%$$

whence $f(t) \leq 7\% t$.

Exercise

Does there exist a continuous and differentiable function $f(x)$ such that $f(0) = -1$ and $f(2) = 4$ and $f'(x) \leq 2$ for all x ?

Solution. Since such a function would also verify the hypotheses of the Lagrange's theorem we would have

$$\frac{f(2) - f(0)}{2 - 0} = f'(c)$$

with $c \in (0, 2)$. This would imply

$$\frac{4 + 1}{2} = f'(c) \leq 2,$$

which is impossible.

Derivatives

Exercise

Let $f : [0, 1] \rightarrow \mathbb{R}$ be a function continuous on $[0, 1]$ and differentiable in $(0, 1)$ such that $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x \in (0, 1)$.

Prova that $f(x) = 0$ for all $x \in [0, 1]$.

Solution. Fix $x \in (0, 1)$. By Lagrange $\exists x_1$ such that $0 < x_1 < x$ and $f(x) = f'(x_1) x$, whence

$$|f(x)| = x |f'(x_1)| \leq x |f(x_1)|.$$

By Lagrange $\exists x_2$ such that $0 < x_2 < x_1$ and $f(x_1) = f'(x_2) x_1$, whence (using $x_1 < x$ and $|f'(x_2)| \leq |f(x_2)|$)

$$|f(x)| \leq x |f(x_1)| = x x_1 |f'(x_2)| \leq x x_1 |f(x_2)| = x^2 |f(x_2)|.$$

By iterating we get a sequence $x_n \in (0, 1)$ such that

$$|f(x)| \leq x^n |f(x_n)|.$$

By the Weierstrass theorem $|f(x)|$ is bounded in $[0, 1]$, whence $x^n |f(x_n)| \rightarrow 0$.

Derivatives: Cauchy's Mean Value Theorem

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ be cont. in $[a, b]$ and diff. in (a, b) .
Then there exists a $x_0 \in (a, b)$ such that

$$(f(b) - f(a)) g'(x_0) = (g(b) - g(a)) f'(x_0).$$

Proof. Suppose first that $g(b) \neq g(a)$. Define $h(x) = f(x) - \alpha g(x)$ and find the α such that

$$\begin{aligned} h(a) = h(b) &\Leftrightarrow f(a) - \alpha g(a) = f(b) - \alpha g(b) \\ &\Leftrightarrow \alpha (g(b) - g(a)) = f(b) - f(a) \Leftrightarrow \alpha = \frac{f(b) - f(a)}{g(b) - g(a)}. \end{aligned}$$

h , as f and g , is continuous in $[a, b]$ and differentiable in (a, b) . Apply Rolle's Theorem to h obtaining

$$\exists x_0 \in (a, b) : h'(\xi) = 0 = f'(x_0) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x_0)$$

whence the thesis.

Derivatives: Cauchy's Mean Value Theorem

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ be cont. in $[a, b]$ and diff. in (a, b) .
Then there exists a $x_0 \in (a, b)$ such that

$$(f(b) - f(a)) g'(x_0) = (g(b) - g(a)) f'(x_0).$$

Proof. Suppose now that $g(b) = g(a)$. Apply Rolle's Theorem to g obtaining

$$\exists x_0 \in (a, b) : g'(x_0) = 0.$$

whence the claimed identity ($0 = 0$) is verified.

Derivatives

Exercise

Use Cauchy's Theorem to prove that

$$0 < 1 - \cos(x) < \frac{x^2}{2}, \quad \forall x > 0.$$

Solution. Apply Cauchy's theorem to

$$f(x) = 1 - \cos(x), \quad g(x) = \frac{x^2}{2}.$$

on the interval $[0, x]$.

$$\exists x_0 \in (0, x) : \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(x_0)}{g'(x_0)} \Rightarrow \frac{1 - \cos(x)}{\frac{x^2}{2}} = \frac{\sin(x_0)}{x_0}.$$

Nevertheless we have proved that $\frac{\sin(x)}{x} < 1$ for $x > 0$, whence

$$0 < \frac{1 - \cos(x)}{\frac{x^2}{2}} < 1 \Rightarrow 1 - \cos(x) < \frac{x^2}{2}.$$

Derivatives

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable in (a, b) , then f' cannot have any jump discontinuity on (a, b) .

Proof. Let $x_0 \in (a, b)$. We know that the limit

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists and is finite. Suppose that the limits $\lim_{x \rightarrow x_0^+} f'(x) = A$ and $\lim_{x \rightarrow x_0^-} f'(x) = B$ exist and are finite. If $x > x_0$ we have

$$\exists x_1 \in (x_0, x) : \frac{f(x) - f(x_0)}{x - x_0} = f'(x_1).$$

As $x \rightarrow x_0^+$ also $x_1 \rightarrow x_0^+$ then

$$f'(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x_1 \rightarrow x_0^+} f'(x_1) = A$$

Similarly by considering $x \rightarrow x_0^-$ we can show that $B = f'(x_0)$ and then $A = B$.

Derivatives

Remark

The derivative of a function could, however, have other types of discontinuities.

Example

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

f is continuous everywhere since $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$. If $x \neq 0$ the derivative is

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Nevertheless since

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0,$$

the function is differentiable in $x_0 = 0$ and $f'(0) = 0$. Nevertheless

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right] = \nexists$$

whence f' has an essential discontinuity in $x_0 = 0$.

Increasing and Decreasing functions: a reminder

Definition

Let $D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function and let $I \subset D$ be an open interval $I = (a, b)$, subset of the domain. We say that the function f is strictly increasing in I if

$$\forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) < f(x_2),$$

we say that the function f is increasing in I if

$$\forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2).$$

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable in (a, b) .

If $f'(x) \geq 0$ (resp. $f'(x) \leq 0$) for all x in (a, b) then f is increasing (resp. decreasing) in (a, b) .

Proof. Consider $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Mean value theorem implies

$$\exists x_0 \in (x_1, x_2) : f(x_2) - f(x_1) = f'(x_0)(x_2 - x_1).$$

If $f'(x_0) \geq 0$ it follows that $f(x_1) \leq f(x_2)$, that is f is increasing.

If $f'(x_0) \leq 0$ it follows that $f(x_2) \leq f(x_1)$, that is f is decreasing.

Derivatives and monotonicity

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable in (a, b) .

If f is increasing (resp. decreasing) in (a, b) then $f'(x) \geq 0$ (resp. $f'(x) \leq 0$) for all x in (a, b) .

Proof. Assume that f is increasing and let x_0 be in (a, b) . For $h > 0$ we have $x_0 \leq x_0 + h$, whence

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0 \Rightarrow f'(x_0^+) \geq 0.$$

Similarly, for $h < 0$ we have $h = -|h|$ and hence

$$\frac{f(x_0 - |h|) - f(x_0)}{-|h|} = \frac{f(x_0) - f(x_0 - |h|)}{|h|} \geq 0 \Rightarrow f'(x_0^-) \geq 0$$

whence $f'(x_0) \geq 0$ (the case f decreasing is identical).

Local minima and local maxima

Definition

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function.

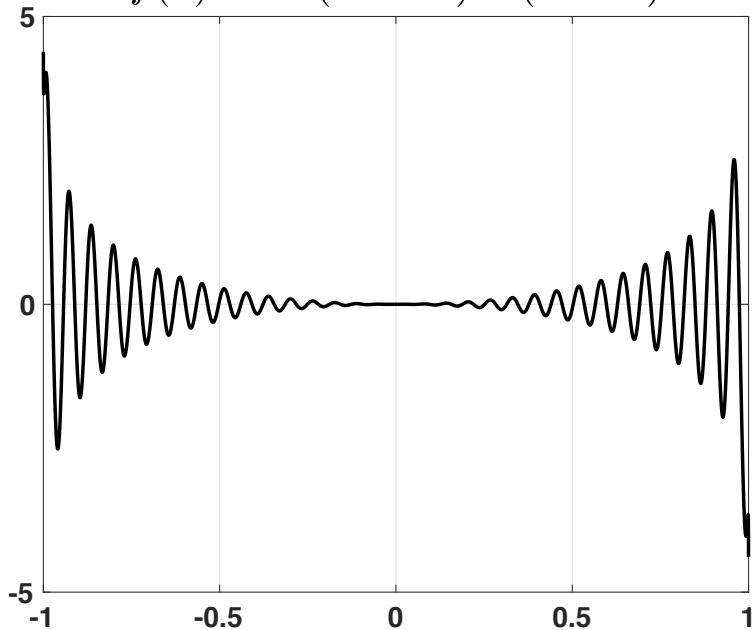
A point $x_0 \in (a, b)$ is a **local minimum** if there exists a sufficiently small $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subseteq (a, b) \text{ and } f(x) \geq f(x_0), \forall x \in (x_0 - \delta, x_0 + \delta).$$

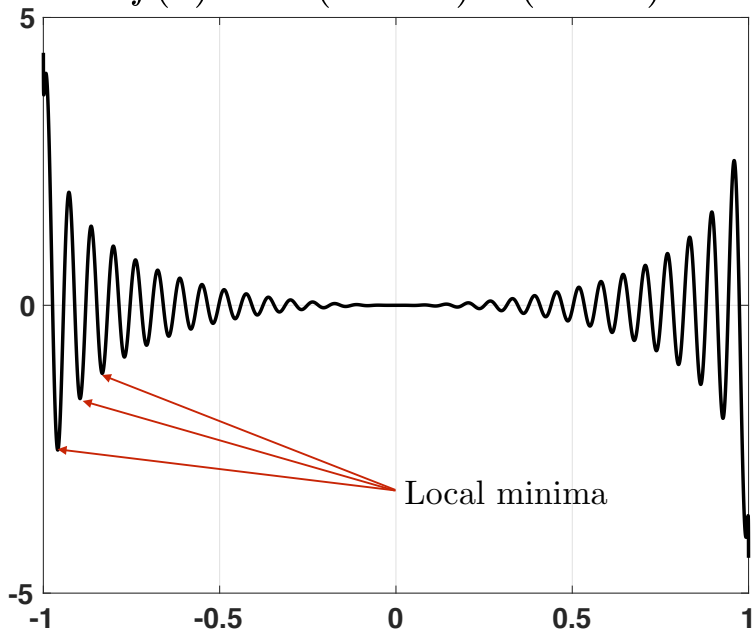
A point $x_0 \in (a, b)$ is a **local maximum** if there exists a sufficiently small $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subseteq (a, b) \text{ and } f(x) \leq f(x_0), \forall x \in (x_0 - \delta, x_0 + \delta).$$

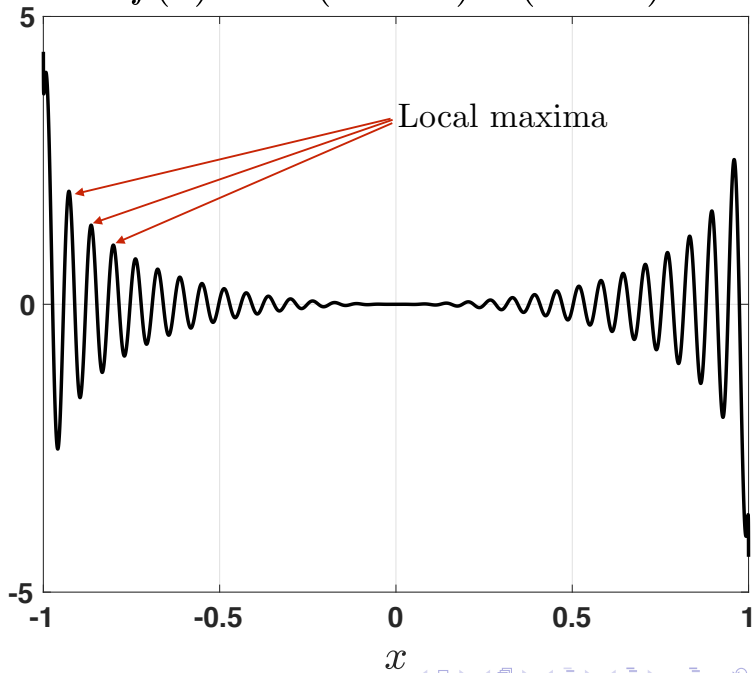
$$f(x) = \sin(100 * x) \ln(1 - x^2)$$



$$f(x) = \sin(100 * x) \ln(1 - x^2)$$



$$f(x) = \sin(100 * x) \ln(1 - x^2)$$



Fermat's Theorem on local extrema

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ and $x_0 \in (a, b)$. If f attains a local min or max in x_0 then either

$$f'(x_0) = 0$$

or

$$\nexists f'(x_0).$$

Proof. If x_0 is local min. then for suff. small h we have $f(x_0 + h) \geq f(x_0)$. If f is differentiable in x_0 then

$$\lim_{h \rightarrow 0^-} \frac{\overbrace{f(x_0 + h) - f(x_0)}^{\geq 0}}{\underbrace{h}_{<0}} = f'(x_0^-) \leq 0, \quad \lim_{h \rightarrow 0^+} \frac{\overbrace{f(x_0 + h) - f(x_0)}^{\geq 0}}{\underbrace{h}_{>0}} = f'(x_0^+) \geq 0$$

but since $\exists f'(x_0)$ then $f'(x_0) = 0$. The only option left is that $\nexists f'(x_0)$.

Fermat's Theorem on local extrema

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ and $x_0 \in (a, b)$. If f attains a local min or max in x_0 then either $f'(x_0) = 0$ or $\nexists f'(x_0)$.

Proof. If x_0 is local **max**. then for suff. small h we have $f(x_0 + h) \leq f(x_0)$. If f is differentiable in x_0 then

$$\lim_{h \rightarrow 0^-} \frac{\overbrace{f(x_0 + h) - f(x_0)}^{\leq 0}}{\underbrace{h}_{<0}} = f'(x_0^-) \geq 0, \quad \lim_{h \rightarrow 0^+} \frac{\overbrace{f(x_0 + h) - f(x_0)}^{\leq 0}}{\underbrace{h}_{>0}} = f'(x_0^+) \leq 0$$

but since $\exists f'(x_0)$ then $f'(x_0) = 0$. The only option left is that $\nexists f'(x_0)$.

Local minima and local maxima

Remark

The converse of the Fermat's Theorem on local extrema is not true!
 $f'(x_0) = 0$ is a **necessary but not sufficient** condition to have a local max/min in x_0 .

As always we have to find a counter-example ...

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f'(0) = 0.$$

Nevertheless for $x < 0$ trivially $x^3 < 0$ and for $x > 0$ trivially $x^3 > 0$.

Definition

Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable in x_0 . If $f'(x_0) = 0$ the point x_0 is called a **critical point**.

Local minima and local maxima

Exercise

Find all the critical points of $f(x) = x^x : (0, +\infty) \rightarrow (0, +\infty)$.

Solution. Define

$$g(x) = \ln(f(x)) = \ln(x^x) = x \ln(x).$$

Whence

$$g'(x) = (x \ln(x))' = (x)' \ln(x) + x (\ln(x))' = 1 \cdot \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1.$$

but also

$$g'(x) = (\ln(f(x)))' = \frac{1}{f(x)} f'(x).$$

So that

$$\frac{1}{f(x)} f'(x) = \ln(x) + 1 \Rightarrow f'(x) = f(x) (\ln(x) + 1) = x^x (\ln(x) + 1)$$

$$f'(x) = 0 \Leftrightarrow x^x (\ln(x) + 1) = 0 \Leftrightarrow \ln(x) + 1 = 0 \Leftrightarrow x = e^{-1},$$

which is thus the unique critical point.

Local minima and local maxima

Exercise

Establish if the critical points of $f(x) = x^x : (0, +\infty) \rightarrow (0, +\infty)$ are local min or local max or neither local min nor local max.

Solution. Since

$$f'(x) = (x^x)' = \underbrace{x^x}_{>0} (\ln(x) + 1)$$

then

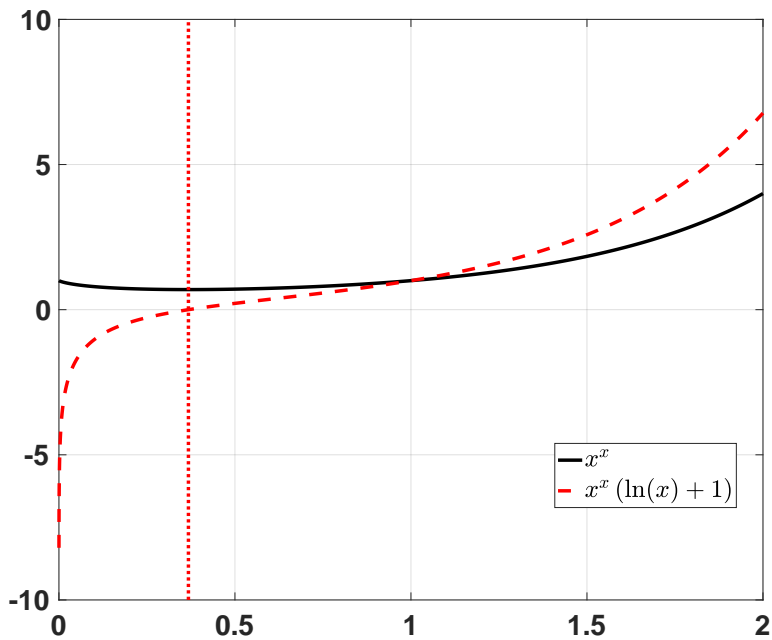
$$\text{sign}(f'(x)) = \text{sign}(\ln(x) + 1).$$

Whence

$$f'(x) > 0 \Leftrightarrow \ln(x) + 1 > 0 \Leftrightarrow \ln(x) > -1 \Leftrightarrow x > e^{-1} \text{ Increasing}$$

$$f'(x) < 0 \Leftrightarrow \ln(x) + 1 < 0 \Leftrightarrow \ln(x) < -1 \Leftrightarrow x < e^{-1} \text{ Decreasing}$$

$x = e^{-1}$ is a local minimum.



x

Local minima and local maxima

Exercise

Find a function f such that $f(x_0)$ is a local min. or local max., while $f'(x_0)$ does not exist.

Solution. Consider

$$f(x) = |x| \geq 0 = f(0) \quad \forall x \in \mathbb{R}.$$

Global minimum in $x = 0$, nevertheless:

$$\begin{aligned} f'(0^+) &= \lim_{h \rightarrow 0^+} \frac{|0 + h| - 0}{h} = 1 \\ f'(0^-) &= \lim_{h \rightarrow 0^+} \frac{|0 + h| - 0}{h} = -1, \end{aligned}$$

i.e. $f'(0)$ does not exist.

Local minima and local maxima

Exercise

Find all the critical points of $f(x) = x^{\frac{1}{x}} : (0, +\infty) \rightarrow (0, +\infty)$.

Solution. Define

$$g(x) = \ln(f(x)) = \ln\left(x^{\frac{1}{x}}\right) = \frac{1}{x} \ln(x).$$

Whence

$$g'(x) = \left(\frac{\ln(x)}{x}\right)' = \left(\frac{1}{x}\right)' \ln(x) + \frac{1}{x} (\ln(x))' = -\frac{\ln(x)}{x^2} + \frac{1}{x} \frac{1}{x} = \frac{1 - \ln(x)}{x^2}.$$

but also

$$g'(x) = (\ln(f(x)))' = \frac{1}{f(x)} f'(x).$$

So that

$$\frac{1}{f(x)} f'(x) = \frac{1 - \ln(x)}{x^2} \Rightarrow f'(x) = f(x) \frac{1 - \ln(x)}{x^2} = x^{\frac{1}{x}} \frac{1 - \ln(x)}{x^2}$$

$$f'(x) = 0 \Leftrightarrow x^{\frac{1}{x}} \frac{1 - \ln(x)}{x^2} = 0 \Leftrightarrow \ln(x) = 1 \Leftrightarrow x = e,$$

which is thus the unique critical point.

Local minima and local maxima

Exercise

Establish if the critical points of $f(x) = x^{\frac{1}{x}} : (0, +\infty) \rightarrow (0, +\infty)$ are local min or local max or neither local min nor local max.

Solution. Since

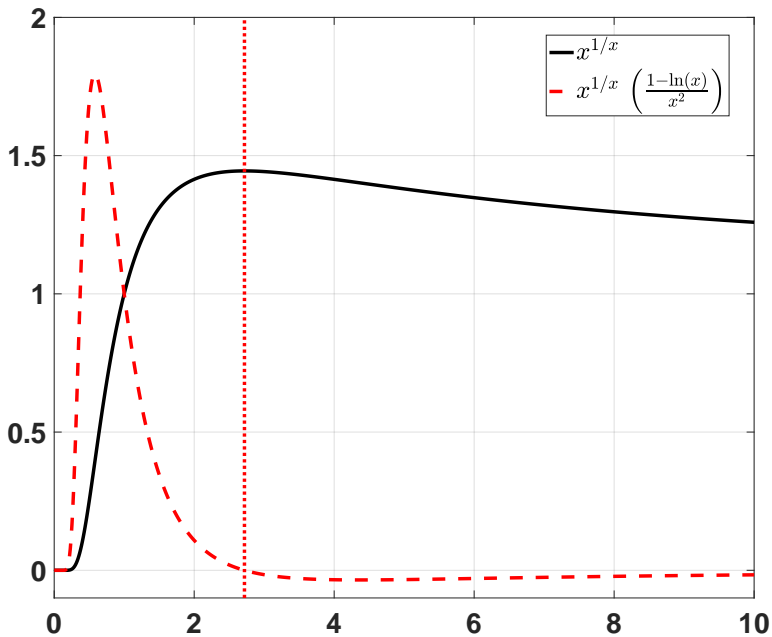
$$f'(x) = \left(x^{\frac{1}{x}}\right)' = \underbrace{x^{\frac{1}{x}}}_{>0} \left(\frac{1 - \ln(x)}{x^2}\right)$$

then $\text{sign}(f'(x)) = \text{sign}\left(\frac{1 - \ln(x)}{x^2}\right)$. Whence

$$f'(x) > 0 \Leftrightarrow \frac{1 - \ln(x)}{x^2} > 0 \Leftrightarrow \ln(x) < 1 \Leftrightarrow x < e \text{ Increasing}$$

$$f'(x) < 0 \Leftrightarrow \frac{1 - \ln(x)}{x^2} < 0 \Leftrightarrow \ln(x) > 1 \Leftrightarrow x > e \text{ Decreasing}$$

$x = e$ is a local maximum.



x

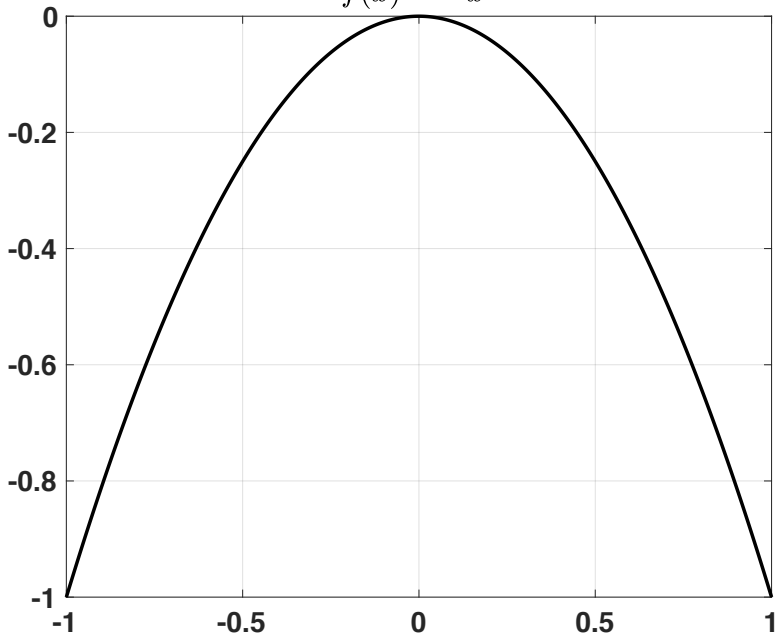
Definition

A function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be **concave** in D if for all x_1 and x_2 in D it holds that

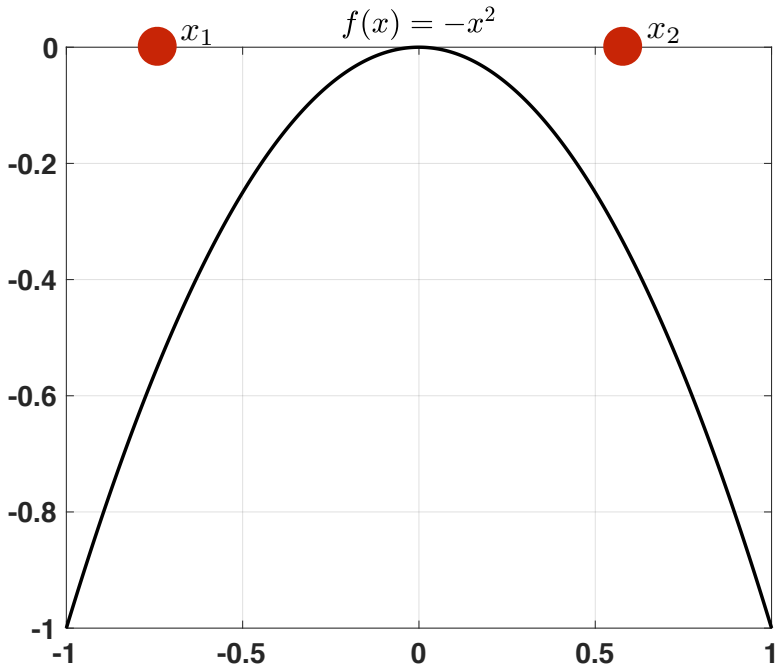
$$f((1 - \alpha)x_1 + \alpha x_2) \geq (1 - \alpha)f(x_1) + \alpha f(x_2), \forall \alpha \in [0, 1]$$

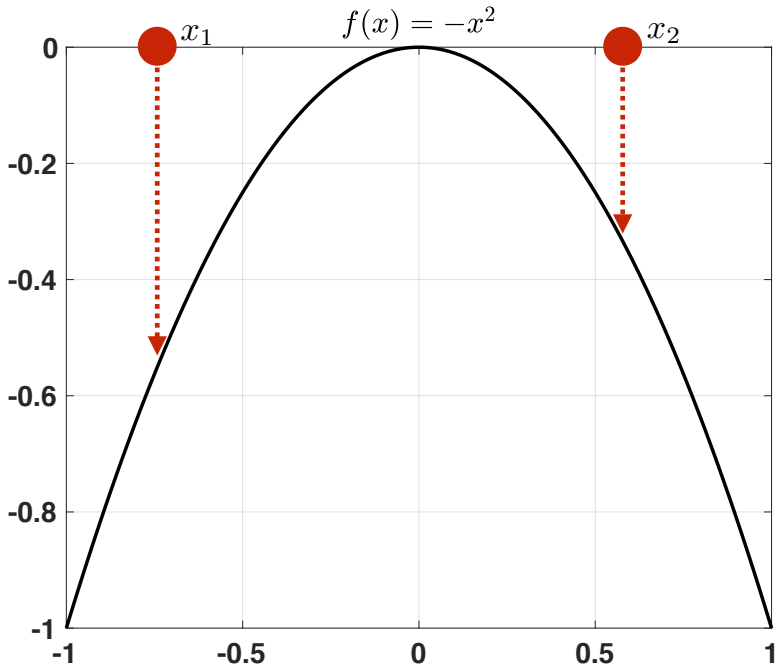
i.e. if the graph of the function is above the segment that joins $(x_1, f(x_1))$ with $(x_2, f(x_2))$.

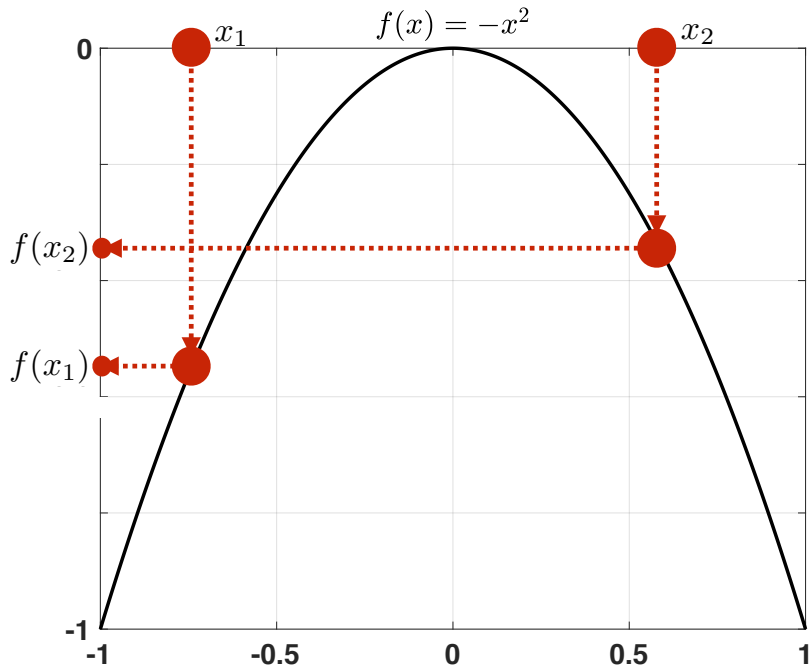
$$f(x) = -x^2$$

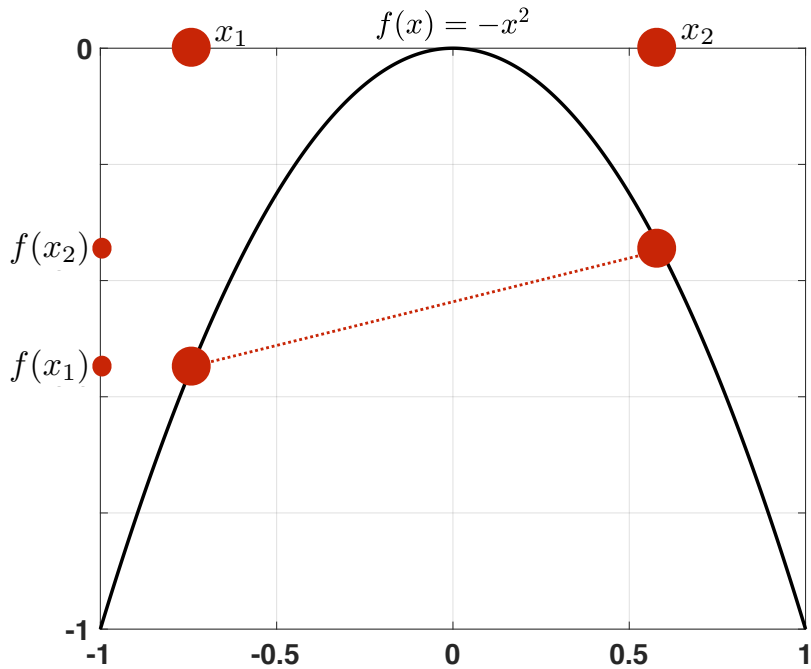


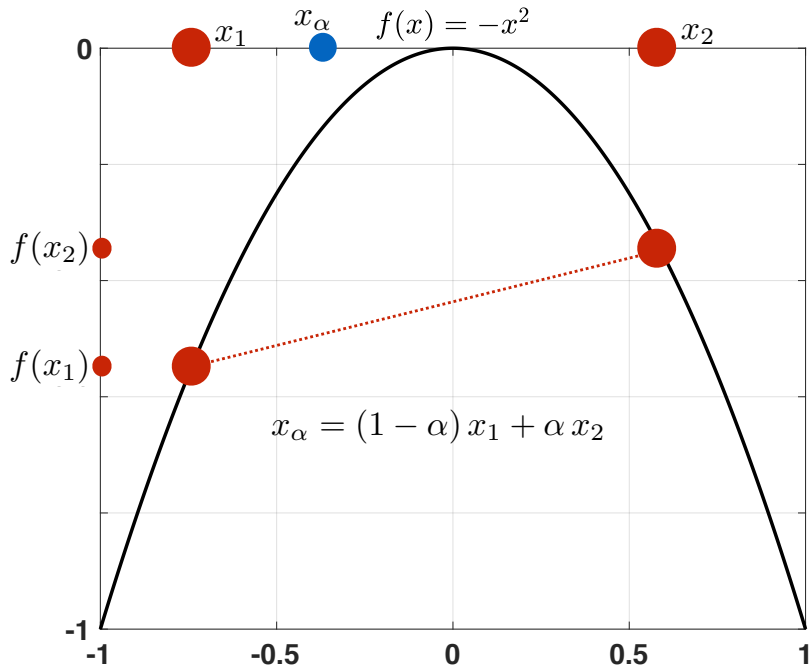
x

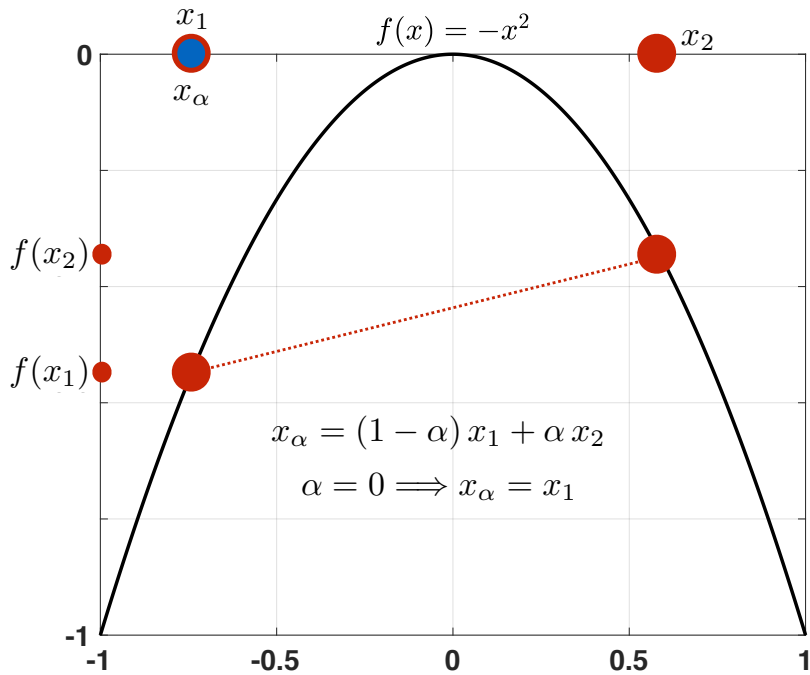


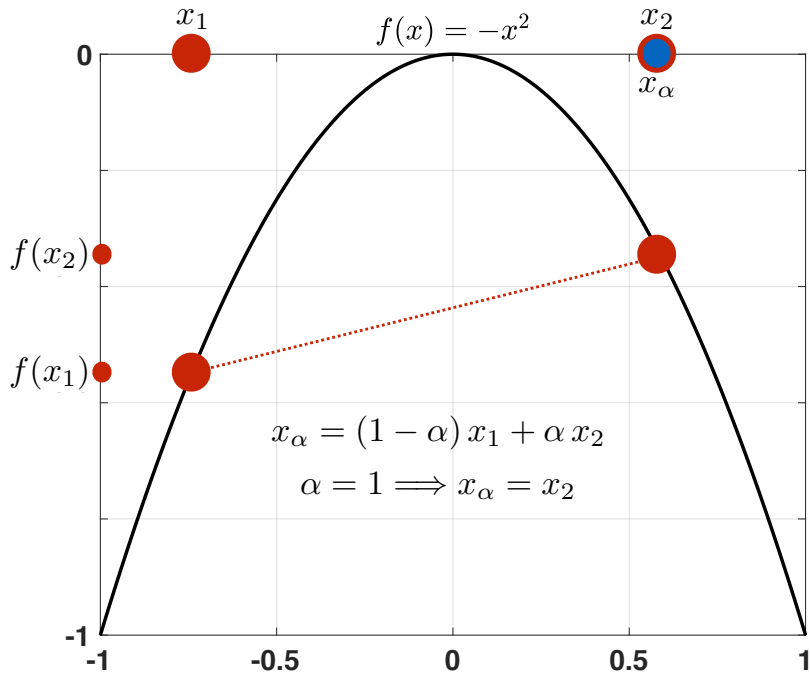


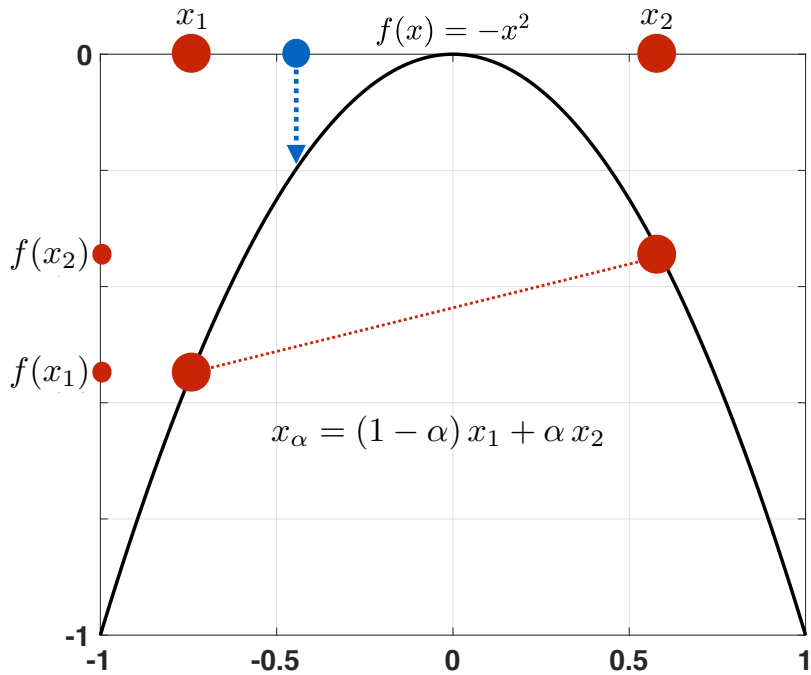


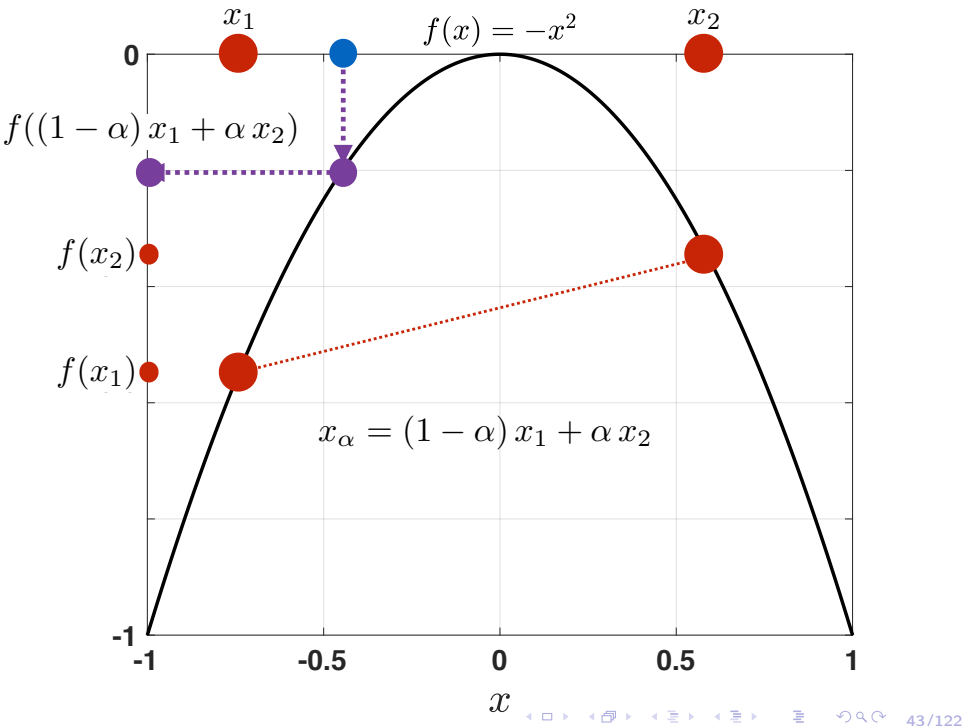


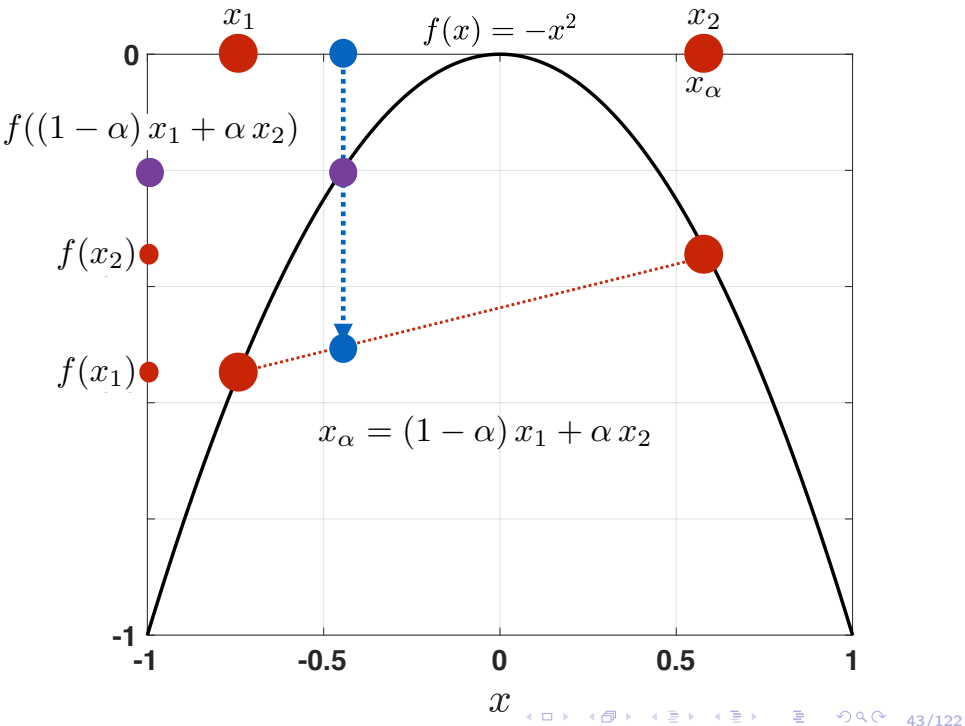


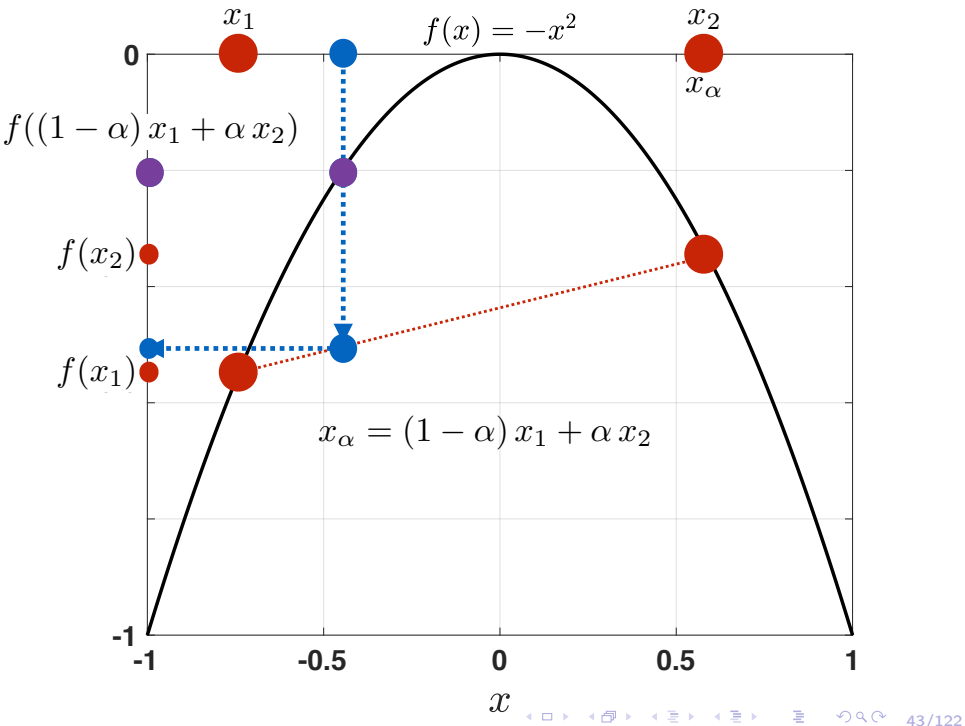


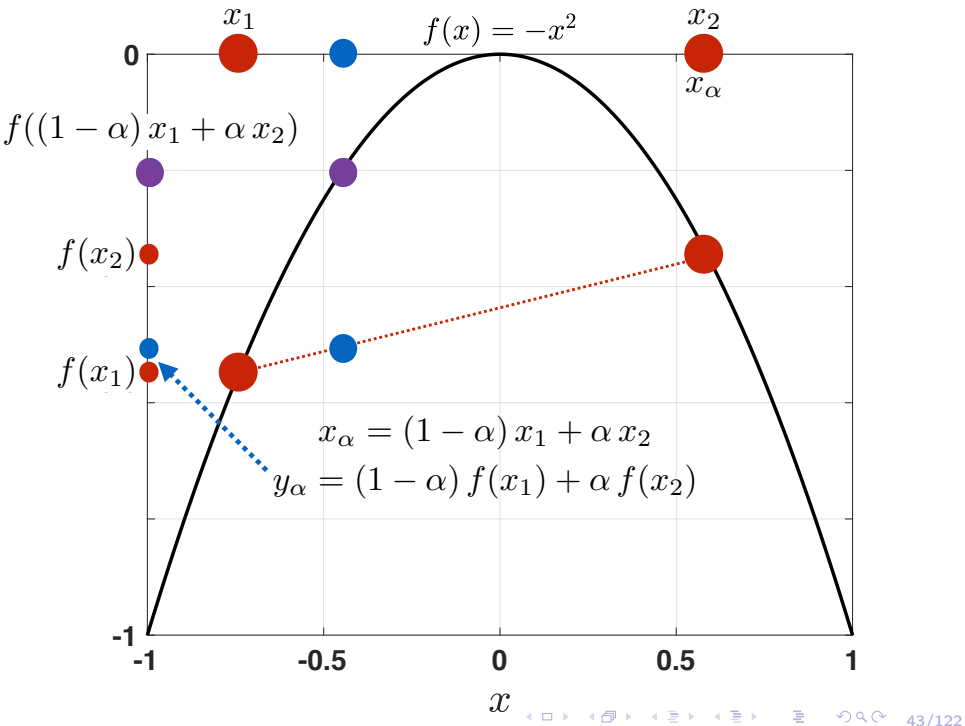


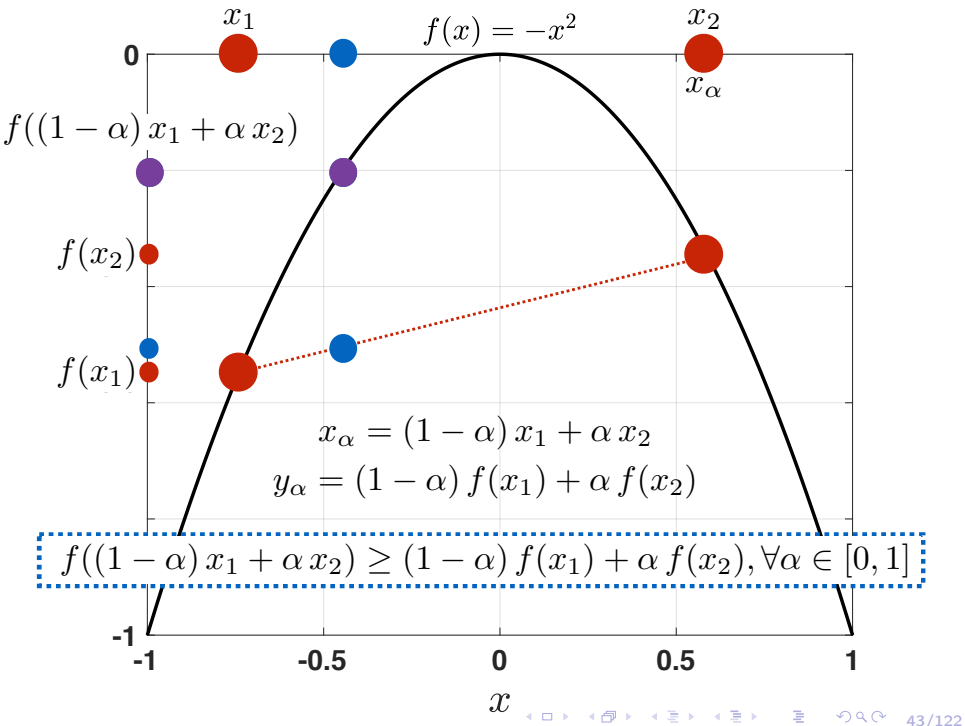


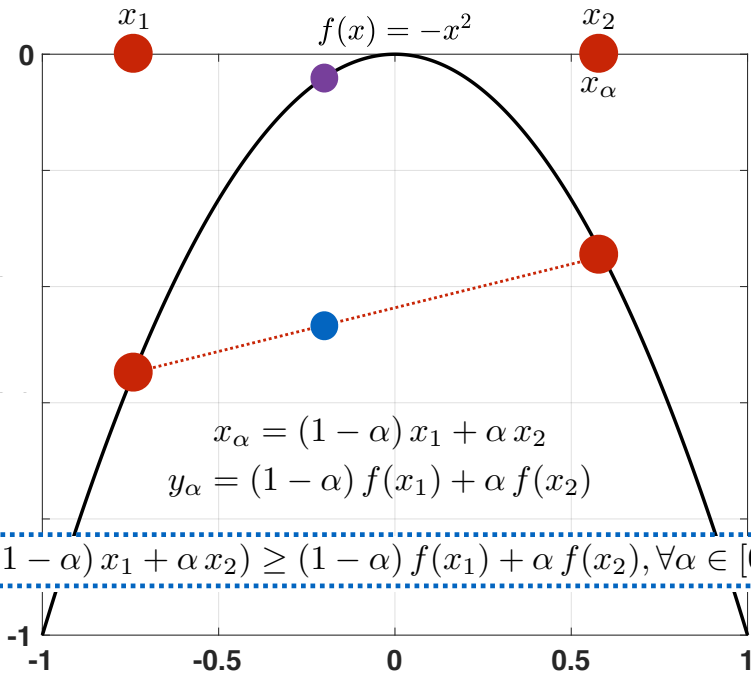


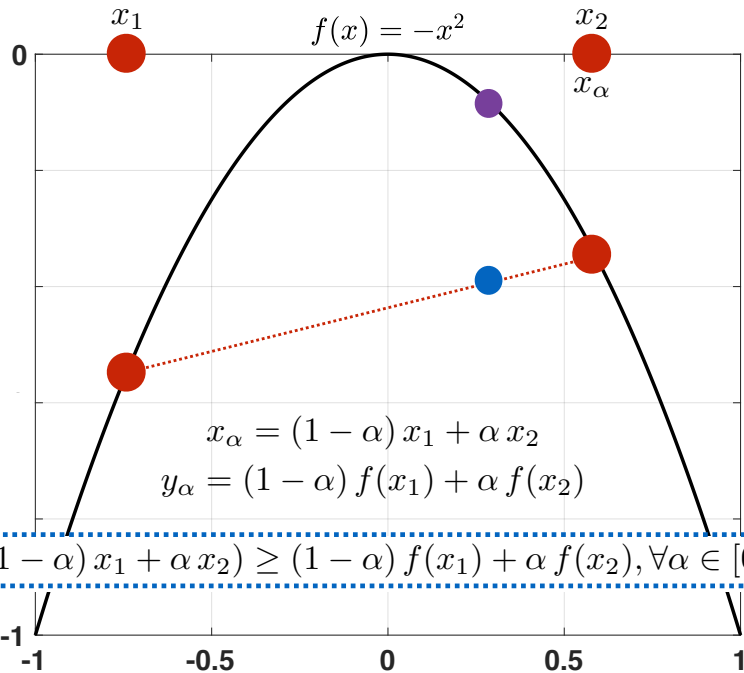


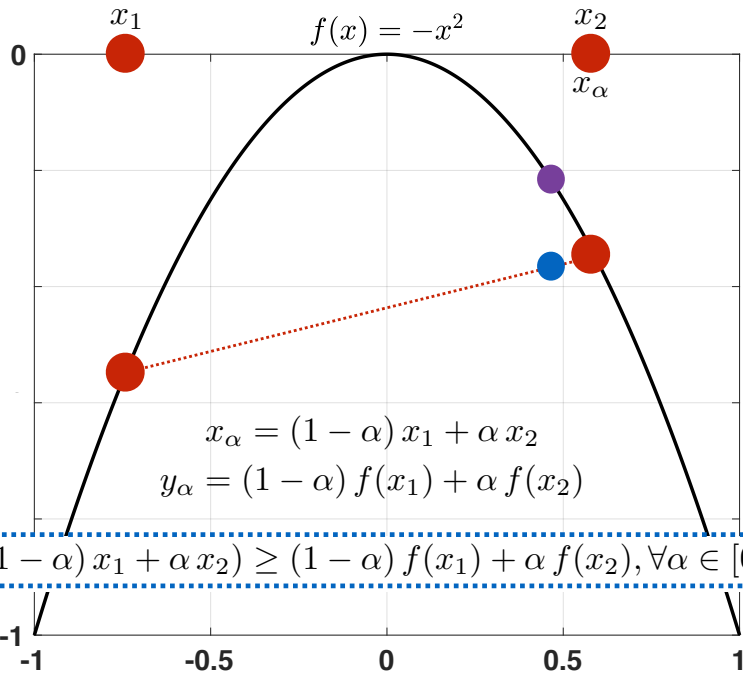












Definition

A function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be **convex** in D if for all x_1 and x_2 in D it holds that

$$f((1 - \alpha)x_1 + \alpha x_2) \leq (1 - \alpha)f(x_1) + \alpha f(x_2), \forall \alpha \in [0, 1]$$

i.e. if the graph of the function is above the segment that joins $(x_1, f(x_1))$ with $(x_2, f(x_2))$.

Concavity and Convexity

Theorem

A function f is convex on D if and only if

$$\forall x_1, x_2, x_3 \in D : x_1 < x_2 < x_3 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

or, equivalently, if and only if

$$\forall x_1, x_2, x_3 \in D : x_1 < x_2 < x_3 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}.$$

Concavity and convexity

Proof. By definition of convexity

$$\forall x, y \in D, \forall \alpha \in [0, 1] \Rightarrow f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \quad (\Delta).$$

Since $x_3 - x_2 < x_3 - x_1$ define $\alpha = \frac{x_3 - x_2}{x_3 - x_1} \in (0, 1)$. Then put α in the definition (Δ) with $x = x_1$ and $y = x_3$

$$\begin{aligned} f(\alpha x + (1 - \alpha)y) &= f\left(\frac{x_3 - x_2}{x_3 - x_1} x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3\right) \\ &= f\left(\frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3\right) \\ &= f\left(\frac{\cancel{x_3 - x_1} - x_2 x_1 + x_2 x_3 - \cancel{x_1 - x_3}}{x_3 - x_1}\right) = f\left(x_2 \frac{x_3 - x_1}{x_3 - x_1}\right) = f(x_2). \end{aligned}$$

Hence we can say that

$$f(x_2) = f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) = \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3)$$

Concavity and convexity

Summary:

$$f(x_2) = f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) = \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3)$$

which implies

$$(x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3)$$

$$\Leftrightarrow (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (+x_3 - x_3 + x_2 - x_1) f(x_3)$$

$$\Leftrightarrow (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_3 - x_1 - (x_3 - x_2)) f(x_3)$$

$$\Leftrightarrow (x_3 - x_2) f(x_3) + (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_3 - x_1) f(x_3)$$

$$\Leftrightarrow (x_3 - x_2) f(x_3) - (x_3 - x_2) f(x_1) \leq (x_3 - x_1) f(x_3) - (x_3 - x_1) f(x_2)$$

$$\Leftrightarrow \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

Concavity and convexity

Now we do similar computations as before

$$\begin{aligned}(x_3 - x_1) f(x_2) &\leq (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3) \\ \Leftrightarrow (x_3 - x_1) f(x_2) &\leq (\cancel{x_1} - \cancel{x_1} + x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3) \\ \Leftrightarrow (x_3 - x_1) f(x_2) &\leq (x_3 - x_1 - (x_2 - x_1)) f(x_1) + (x_2 - x_1) f(x_3) \\ \Leftrightarrow (x_3 - x_1) f(x_2) - (x_3 - x_1) f(x_1) &\leq -(x_2 - x_1) f(x_1) + (x_2 - x_1) f(x_3) \\ \Leftrightarrow (x_3 - x_1) (f(x_2) - f(x_1)) &\leq (x_2 - x_1) (f(x_3) - f(x_1)) \\ \Leftrightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} &\leq \frac{f(x_3) - f(x_1)}{x_3 - x_1},\end{aligned}$$

so summing up

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

Since we used only double implications \Leftrightarrow , the argument reverses throughout. □

Concavity and Convexity

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function which is differentiable on (a, b) .
Then f is convex on $(a, b) \Leftrightarrow f'$ is increasing on (a, b) .

Proof. \Rightarrow . Consider four points $a < x_1 < x_2 < x_3 < x_4 < b$. By the property of convex functions (used two times)

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2} \leq \frac{f(x_4) - f(x_3)}{x_4 - x_3}.$$

Now let $x_2 \rightarrow x_1^+$ and $x_3 \rightarrow x_4^-$ obtaining (since f is differentiable!)

$$f'(x_1) \leq f'(x_4),$$

the arbitrariness of x_1 and $x_4 \Rightarrow f'$ is increasing on (a, b) . □

Concavity and Convexity

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function which is differentiable on (a, b) .
Then f is convex on $(a, b) \Leftrightarrow f'$ is increasing on (a, b) .

Proof. \Leftarrow . Consider three points $a < x_1 < x_2 < x_3 < b$. By the mean value theorem

$$\exists \alpha \in (x_1, x_2) : \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\alpha)$$

and

$$\exists \beta \in (x_2, x_3) : \frac{f(x_3) - f(x_2)}{x_3 - x_2} = f'(\beta).$$

Since $\alpha < \beta$ then $f'(\alpha) \leq f'(\beta)$ and hence

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

which is the equivalent condition for convexity.

Concavity and Convexity

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function which is differentiable on (a, b) .
Then f is convex on $(a, b) \Leftrightarrow f'$ is increasing on (a, b) .

Corollary

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function which is **twice** differentiable on (a, b) .
Then f is convex on $(a, b) \Leftrightarrow f''(x) \geq 0$ for all $x \in (a, b)$.

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function which is differentiable on (a, b) .
Then f is concave on $(a, b) \Leftrightarrow f'$ is decreasing on (a, b) .

Corollary

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function which is **twice** differentiable on (a, b) .
Then f is concave on $(a, b) \Leftrightarrow f''(x) \leq 0$ for all $x \in (a, b)$.

Concavity and Convexity: The Second Derivative Test

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable on (a, b) and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$.

- If $f''(x_0) > 0$ then x_0 is a local minimum.
- If $f''(x_0) < 0$ then x_0 is a local maximum.

Proof. Assume $f''(x_0) > 0$. Since $f'(x_0) = 0$, we have

$$\lim_{h \rightarrow 0^+} \frac{f'(x_0 + h) - f'(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{f'(x_0 + h)}{h} = f''(x_0) > 0$$

Which means that I can find an ε such that, for $h > 0$ and sufficiently small

$$0 < f''(x_0) - \varepsilon < \frac{f'(x_0 + h)}{h} < f''(x_0) + \varepsilon \Rightarrow f'(x_0 + h) > h(f''(x_0) - \varepsilon) > 0$$

hence the function is increasing in a right neighborhood of x_0 . Similarly, for $h < 0$ and sufficiently small we get

$$f'(x_0 + h) < h(f''(x_0) - \varepsilon) < 0,$$

hence the function is decreasing in a left neighborhood of x_0 , and hence x_0 is a local minimum.

Concavity and Convexity: The Second Derivative Test

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable on (a, b) and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$.

- If $f''(x_0) > 0$ then x_0 is a local minimum.
- If $f''(x_0) < 0$ then x_0 is a local maximum.

Proof. Assume $f''(x_0) < 0$. Since $f'(x_0) = 0$, we have

$$\lim_{h \rightarrow 0^+} \frac{f'(x_0 + h) - f'(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{f'(x_0 + h)}{h} = f''(x_0) < 0$$

Which means that I can find an ε such that, for $h > 0$ and sufficiently small

$$f''(x_0) - \varepsilon < \frac{f'(x_0 + h)}{h} < f''(x_0) + \varepsilon < 0 \Rightarrow f'(x_0 + h) < h(f''(x_0) + \varepsilon) < 0$$

hence the function is decreasing in a right neighborhood of x_0 . Similarly, for $h < 0$ and sufficiently small we get

$$f'(x_0 + h) > h(f''(x_0) + \varepsilon) > 0,$$

hence the function is increasing in a left neighborhood of x_0 , and hence x_0 is a local maximum.

Concavity and Convexity: The Second Derivative Test

Exercise

Find minima/maxima of the following function

$$f(x) = \frac{\ln(x)}{x}$$

Solution. The function is defined in $D = \{x \in \mathbb{R} \mid x > 0\}$. The first derivative is

$$f'(x) = (\ln(x))' \frac{1}{x} + \ln(x) \left(\frac{1}{x}\right)' = \frac{1}{x^2} - \frac{\ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}.$$

Hence $f'(x) = 0$ if and only if $x = e$. Besides since

$$f''(x) = (1 - \ln(x))' \frac{1}{x^2} + (1 - \ln(x)) \left(\frac{1}{x^2}\right)' = -\frac{1}{x^3} - 2 \frac{1 - \ln(x)}{x^3} = -\frac{1 + 2 \ln(x)}{x^3},$$

we have that

$$f''(e) = -\frac{3}{e^3} < 0 \Rightarrow x_0 = e \text{ is a local maximum.}$$

Concavity and Convexity: The Second Derivative Test

Exercise

Find minima/maxima of the following function

$$f(x) = \ln(1 - \ln(x)) - \ln(x).$$

Solution. The domain of the function is determined by the two conditions

$$\begin{cases} x > 0 \\ 1 - \ln(x) > 0 \Rightarrow \ln(x) < 1 \Rightarrow x < e \end{cases},$$

whence $D = (0, e)$. The first derivative is

$$f'(x) = (\ln(1 - \ln(x)))' - (\ln(x))' = \frac{1}{1 - \ln(x)} \left(-\frac{1}{x}\right) - \frac{1}{x} = -\frac{2 - \ln(x)}{x(1 - \ln(x))},$$

hence $f'(x) = 0 \Leftrightarrow x = e^2$. Nevertheless $e^2 \notin D \Rightarrow$ the function has no minimum no maximum in $(0, e)$.

Concavity and Convexity: The Second Derivative Test

Exercise

Find minima/maxima of the following function

$$f(x) = \frac{\ln(x)}{x}$$

Solution. The domain of the function is $D = (0, +\infty)$. The first derivative is

$$f'(x) = (\ln(x))' \frac{1}{x} + \ln(x) \left(\frac{1}{x}\right)' = \frac{1}{x^2} - \frac{\ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2},$$

hence $f'(x) = 0 \Leftrightarrow x = e$.

$$\begin{aligned} f''(x) &= (1 - \ln(x))' \frac{1}{x^2} + (1 - \ln(x)) \left(\frac{1}{x^2}\right)' = -\frac{1}{x^3} - 2\frac{1 - \ln(x)}{x^3} \\ &= -\frac{3 - 2 \ln(x)}{x^3} \Rightarrow f''(e) = -\frac{3 - 2 \ln(e)}{e^3} = -\frac{1}{e^3} < 0, \quad (0.2) \end{aligned}$$

whence $x = e$ is a local maximum.

Concavity and Convexity: The Second Derivative Test

Exercise

Find minima/maxima of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. The domain of the function is determined by the two conditions

$$\begin{cases} x > 0 \\ 1 + \ln(x) > 0 \Rightarrow \ln(x) > -1 \Rightarrow x > e^{-1} = \frac{1}{e} \end{cases},$$

whence $D = (\frac{1}{e}, +\infty)$.

Concavity and Convexity: The Second Derivative Test

Exercise

Find minima/maxima of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. $D = (\frac{1}{e}, +\infty)$. The first derivative is

$$\begin{aligned} f'(x) &= (\ln(1 + \ln(x)))' - (\ln(x))' = \frac{1}{x(1 + \ln(x))} - \frac{1}{x} \\ &= \frac{1 - 1 - \ln(x)}{x(1 + \ln(x))} = -\frac{\ln(x)}{x(1 + \ln(x))}, \end{aligned}$$

hence $f'(x) = 0 \Leftrightarrow x = 1$.

Concavity and Convexity: The Second Derivative Test

Exercise

Find minima/maxima of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. $D = (\frac{1}{e}, +\infty)$. $f'(x) = -\frac{\ln(x)}{x(1+\ln(x))}$. The second derivative is

$$\begin{aligned} f''(x) &= -(\ln(x))' \frac{1}{x(1+\ln(x))} - \ln(x) \left(\frac{1}{x(1+\ln(x))} \right)' = \\ &= -\frac{1}{x^2(1+\ln(x))} + \frac{\ln(x)}{x^2(1+\ln(x))^2} (x(1+\ln(x)))' \\ &= -\frac{1}{x^2(1+\ln(x))} + \frac{\ln(x)}{x^2(1+\ln(x))^2} (1+\ln(x)+1) \\ &= \frac{-(1+\ln(x)) + \ln(x)(2+\ln(x))}{x^2(1+\ln(x))^2} = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2(1+\ln(x))^2}. \end{aligned}$$

Concavity and Convexity: The Second Derivative Test

Exercise

Find minima/maxima of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. Summary

$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x(1 + \ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2(1 + \ln(x))^2}.$$

Whence

$$f''(1) = \frac{-1}{1} = -1 < 0 \Rightarrow x = 1 \text{ is a local maximum.}$$

Concavity and Convexity: The Second Derivative Test

Exercise

Find concavity/convexity regions of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. Summary

$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x(1 + \ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2(1 + \ln(x))^2}.$$

Whence

$$f''(x) \geq 0 \Leftrightarrow (\ln(x))^2 + \ln(x) - 1 \geq 0.$$

Call $t = \ln(x)$. We have $f''(x) \geq 0 \Leftrightarrow t^2 + t - 1 \geq 0$. The roots of the polynomial $t^2 + t - 1$ are $t_{1,2} = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$. Whence

$$f''(x) \geq 0 \Leftrightarrow t^2 + t - 1 \geq 0 \Leftrightarrow t \leq -\frac{1 + \sqrt{5}}{2} \text{ or } t \geq \frac{\sqrt{5} - 1}{2}.$$

Concavity and Convexity: The Second Derivative Test

Exercise

Find concavity/convexity regions of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. Summary

$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x(1 + \ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2(1 + \ln(x))^2}.$$

$$f''(x) \geq 0 \Leftrightarrow t^2 + t - 1 \geq 0 \Leftrightarrow t \leq -\frac{1 + \sqrt{5}}{2} \text{ or } t \geq \frac{\sqrt{5} - 1}{2}.$$

Nevertheless $t = \ln(x)$ whence

$$t \leq -\frac{1 + \sqrt{5}}{2} \Rightarrow \ln(x) \leq -\frac{1 + \sqrt{5}}{2} \Rightarrow x \leq e^{-\frac{1 + \sqrt{5}}{2}},$$

Nevertheless $e^{-\frac{1 + \sqrt{5}}{2}} < e^{-1}$ hence $e^{-\frac{1 + \sqrt{5}}{2}} \notin D$, whence

$$f''(x) \geq 0 \Leftrightarrow x \geq e^{\frac{\sqrt{5} - 1}{2}}, \text{ and } f''(x) \leq 0 \Leftrightarrow \frac{1}{e} < x \leq e^{\frac{\sqrt{5} - 1}{2}}.$$

Concavity and Convexity: The Second Derivative Test

Exercise

Find concavity/convexity regions of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. Summary

$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x(1 + \ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2(1 + \ln(x))^2}.$$

$$f''(x) \geq 0 \Leftrightarrow x \geq e^{\frac{\sqrt{5}-1}{2}}, \text{ and } f''(x) \leq 0 \Leftrightarrow \frac{1}{e} < x \leq e^{\frac{\sqrt{5}-1}{2}}.$$

Which is the relative position of $e^{\frac{\sqrt{5}-1}{2}}$ with respect to the critical point $x = 1$? Consider that $\sqrt{5} \approx 2.2361$ then $\sqrt{5} - 1 > 0$ and hence

$$e^{\frac{-1+\sqrt{5}}{2}} > 1.$$

Concavity and Convexity: The Second Derivative Test

Exercise

Find concavity/convexity regions of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution.

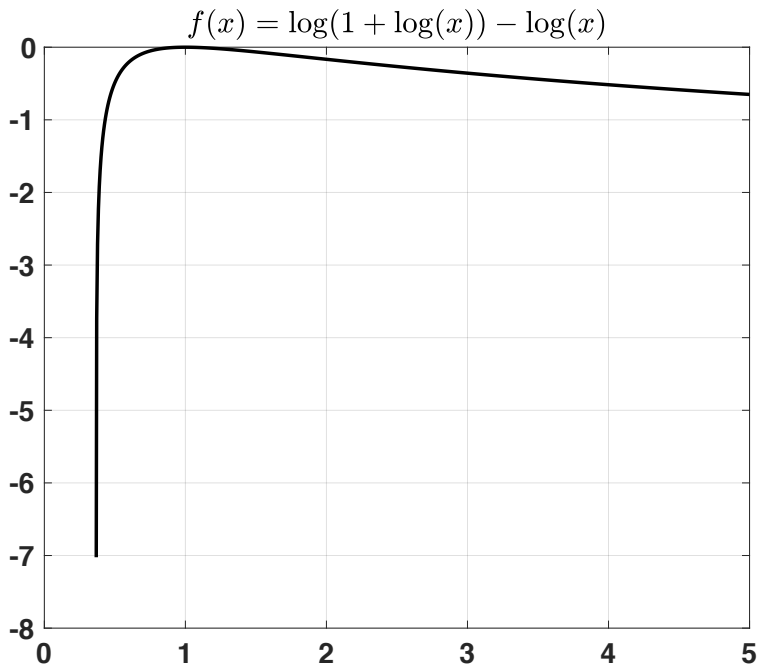
Final summary

$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x(1 + \ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2(1 + \ln(x))^2}.$$

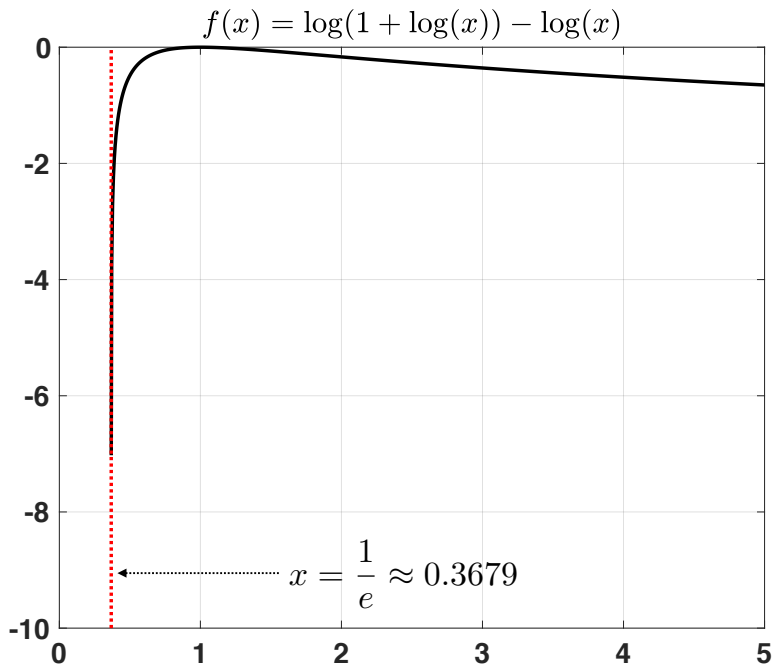
$x_0 = 1$ is a local maximum

$$f''(x) \geq 0 \Leftrightarrow x \geq e^{\frac{\sqrt{5}-1}{2}} \quad \text{and} \quad f''(x) \leq 0 \Leftrightarrow \frac{1}{e} < x \leq e^{\frac{\sqrt{5}-1}{2}}.$$

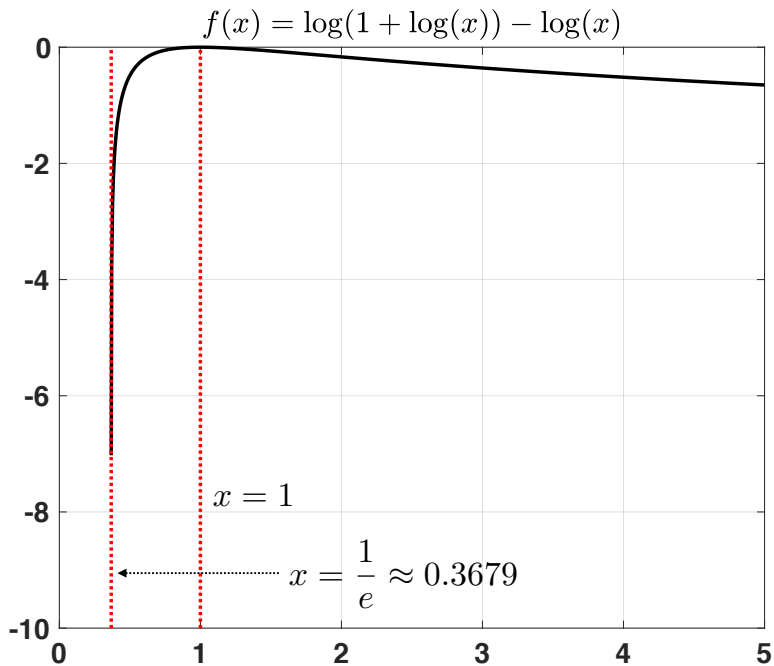
$$e^{\frac{-1+\sqrt{5}}{2}} > 1$$



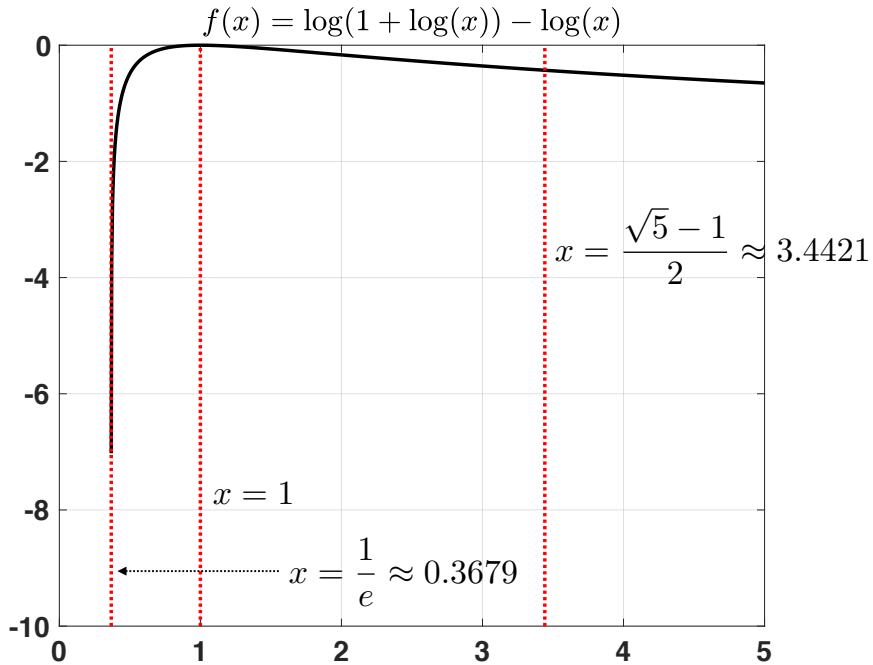
x

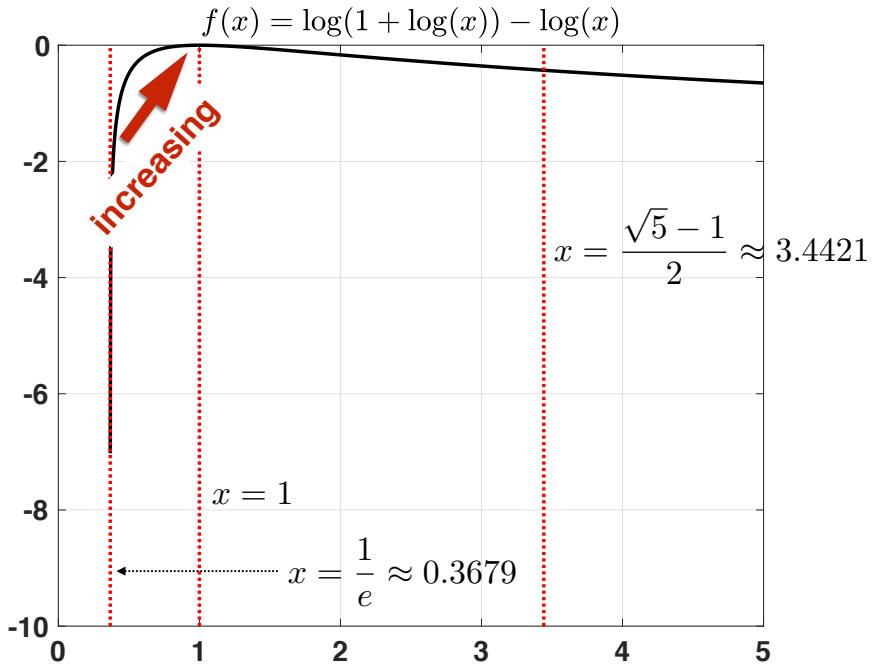


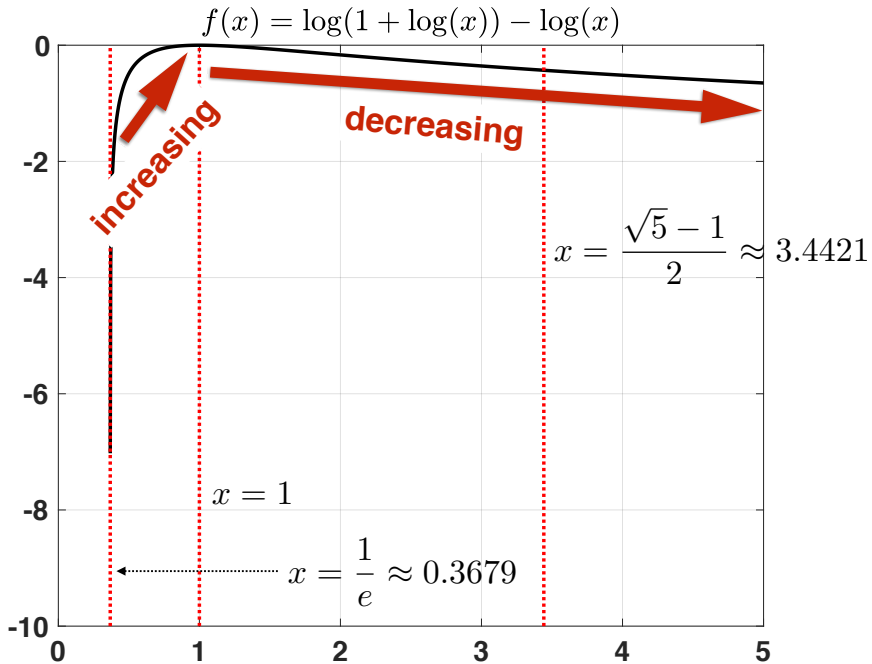
x

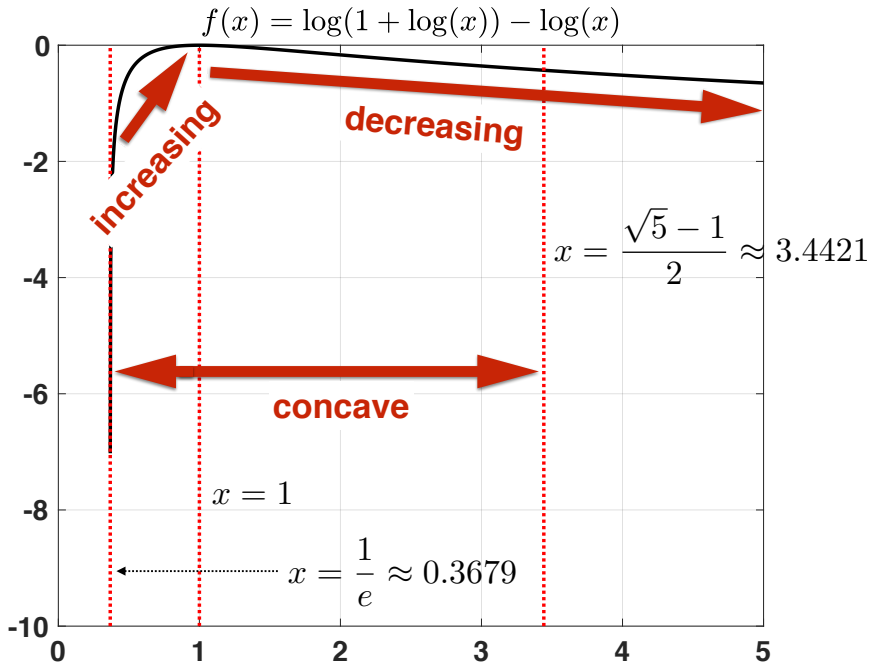


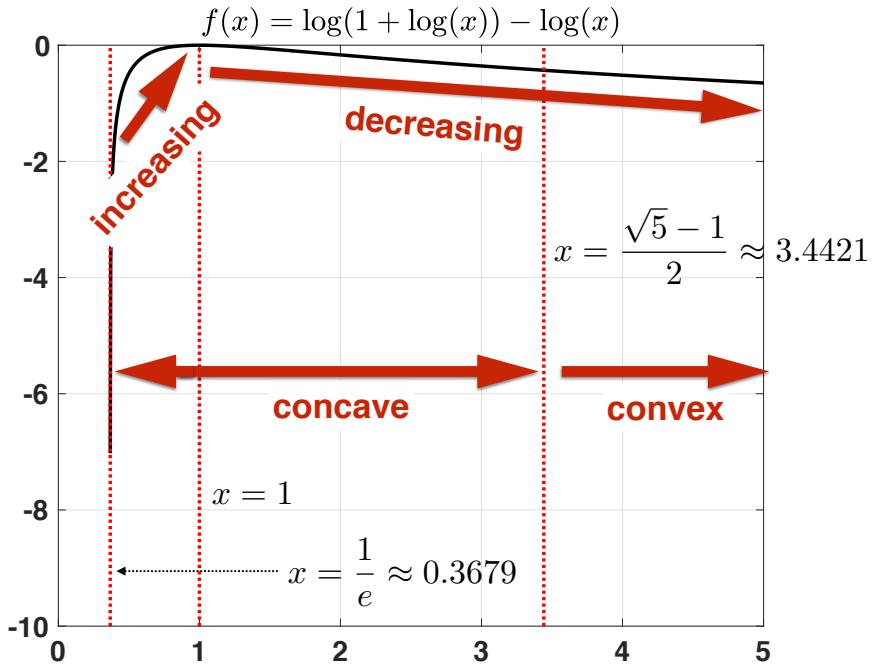
x











Optimization

Exercise

Monopolistic manufacturer.

$$p(x) = p_0 - x = \text{Price per unit to sell } x \text{ units}$$

$$c(x) = c_0 + \alpha x = \text{Cost to produce } x \text{ units}$$

$$R(x) = x p(x) = x p_0 - x^2 = \text{Total revenues}$$

$$\Pi(x) = R(x) - c(x) = x(p_0 - \alpha) - x^2 - c_0 = \text{Net profit.}$$

Remark. Only $0 \leq x \leq p_0$ are admissible.

- 1 Determine the x^* that maximizes profit. Does such an x^* exist for any p_0 and α ?
- 2 What is the maximum profit?
- 3 Which is the maximum value for c_0 that guarantees a positive maximum profit?
- 4 What price per unit must be charged in order to maximize the profit?

Optimization

$$p(x) = p_0 - x = \text{Price per unit to sell } x \text{ units}$$

$$c(x) = c_0 + \alpha x = \text{Cost to produce } x \text{ units}$$

$$R(x) = x p(x) = x p_0 - x^2 = \text{Total revenues}$$

$$\Pi(x) = R(x) - c(x) = x(p_0 - \alpha) - x^2 - c_0 = \text{Net profit.}$$

Determine x^* that maximizes profit. Does such an $x^* \exists \forall p_0$ and α ?

$$\Pi'(x) = p_0 - \alpha - 2x \Rightarrow \Pi'(x^*) = 0 \Leftrightarrow p_0 - \alpha - 2x^* = 0 \Leftrightarrow x^* = \frac{p_0 - \alpha}{2}$$

Besides

$$\Pi''(x) = -2 \Rightarrow x^* \text{ is a maximum.}$$

Finally

$$x^* \in [0, p_0] \Leftrightarrow p_0 \geq \alpha.$$

Optimization

$$p(x) = p_0 - x = \text{Price per unit to sell } x \text{ units}$$

$$c(x) = c_0 + \alpha x = \text{Cost to produce } x \text{ units}$$

$$R(x) = x p(x) = x p_0 - x^2 = \text{Total revenues}$$

$$\Pi(x) = R(x) - c(x) = x(p_0 - \alpha) - x^2 - c_0 = \text{Net profit.}$$

What is the maximum profit?

$$x^* = \frac{p_0 - \alpha}{2} \Rightarrow$$

$$\Pi(x^*) = \frac{p_0 - \alpha}{2} (p_0 - \alpha) - \frac{(p_0 - \alpha)^2}{4} - c_0 = \frac{(p_0 - \alpha)^2}{4} - c_0.$$

Which is the max. c_0 that guarantees a positive maximum profit?

$$\Pi(x^*) > 0 \Leftrightarrow c_0 < \frac{(p_0 - \alpha)^2}{4}.$$

If $c_0 = \text{cost of the production plant} \geq \frac{(p_0 - \alpha)^2}{4}$ no production.

Optimization

$$p(x) = p_0 - x = \text{Price per unit to sell } x \text{ units}$$

$$c(x) = c_0 + \alpha x = \text{Cost to produce } x \text{ units}$$

$$R(x) = x p(x) = x p_0 - x^2 = \text{Total revenues}$$

$$\Pi(x) = R(x) - c(x) = x(p_0 - \alpha) - x^2 - c_0 = \text{Net profit.}$$

What price per unit must be charged in order to maximize the profit?

$$x^* = \frac{p_0 - \alpha}{2} \Rightarrow p(x^*) = p_0 - \frac{p_0 - \alpha}{2} = \frac{1}{2}(p_0 + \alpha).$$

Exercise

Let $\alpha_n > 0$ be a positive sequence.

$$c_n(x) = \ln(\alpha_n + x^2) = \text{cost of producing } x \text{ units at time } t = n$$

$$r(x) = \ln(x) = \text{revenues}$$

$$\pi_n(x) = r(x) - c_n(x) = \ln(x) - \ln(\alpha_n + x^2) = \text{Net profit.}$$

- Determine, at each time n , the optimal amount x_n of units that must be produced.
- Assume $\alpha_n = \frac{1}{4^n}$.

Which is the first date (i.e. the first n) in which the producer faces a strictly positive optimal profit?

Which is the total amount produced from the initial time ($n = 0$) to infinity ($n = \infty$)?

Optimization

$$c_n(x) = \ln(\alpha_n + x^2) = \text{cost of producing } x \text{ units at time } t = n$$

$$r(x) = \ln(x) = \text{revenues}$$

$$\pi_n(x) = r(x) - c_n(x) = \ln(x) - \ln(\alpha_n + x^2) = \text{Net profit.}$$

Determine, at each time n , the optimal amount x_n .

$$\pi'_n(x) = \frac{1}{x} - \frac{2x}{\alpha_n + x^2} = \frac{\alpha_n + x^2 - 2x^2}{x(\alpha_n + x^2)} = \frac{\alpha_n - x^2}{x(\alpha_n + x^2)} = 0 \Leftrightarrow x_n = \pm\sqrt{\alpha_n},$$

only $x_n = +\sqrt{\alpha_n}$ is admissible. Since

$$\pi''_n(x) = \frac{-2x}{x(\alpha_n + x^2)} - \frac{\alpha_n - x^2}{x^2(\alpha_n + x^2)^2} (\alpha_n + x^2 + 2x^2),$$

we get

$$\pi''_n(x_n) = -\frac{2\sqrt{\alpha_n}}{2\alpha_n\sqrt{\alpha_n}} < 0,$$

whence $x_n = \sqrt{\alpha_n}$ is a maximum.

Optimization

Assume $\alpha_n = \frac{1}{4^n}$

$$c_n(x) = \ln\left(\frac{1}{4^n} + x^2\right) = \text{cost of producing } x \text{ units at time } t = n$$

$$r(x) = \ln(x) = \text{revenues}$$

$$\pi_n(x) = r(x) - c_n(x) = \ln(x) - \ln\left(\frac{1}{4^n} + x^2\right) = \text{Net profit.}$$

Which is the first date in which the producer faces a strictly positive optimal profit?

$$\pi_n(x_n) = \ln\left(\frac{2^n}{2}\right) \Rightarrow \begin{cases} \pi_0(x_0) = \ln(1/2) < 0 \\ \pi_1(x_1) = \pi_n(x_1) = \ln(1) = 0 \\ \pi_2(x_1) = \ln(2) > 0. \end{cases}$$

so the answer is $n = 2$.

Optimization

Assume $\alpha_n = \frac{1}{4^n}$

$$c_n(x) = \ln\left(\frac{1}{4^n} + x^2\right) = \text{cost of producing } x \text{ units at time } t = n$$

$$r(x) = \ln(x) = \text{revenues}$$

$$\pi_n(x) = r(x) - c_n(x) = \ln(x) - \ln\left(\frac{1}{4^n} + x^2\right) = \text{Net profit.}$$

Which is the total amount produced from the initial time ($n = 0$) to infinity ($n = \infty$)?

Remember that $x_n = \sqrt{\alpha_n} = \frac{1}{2^n}$.

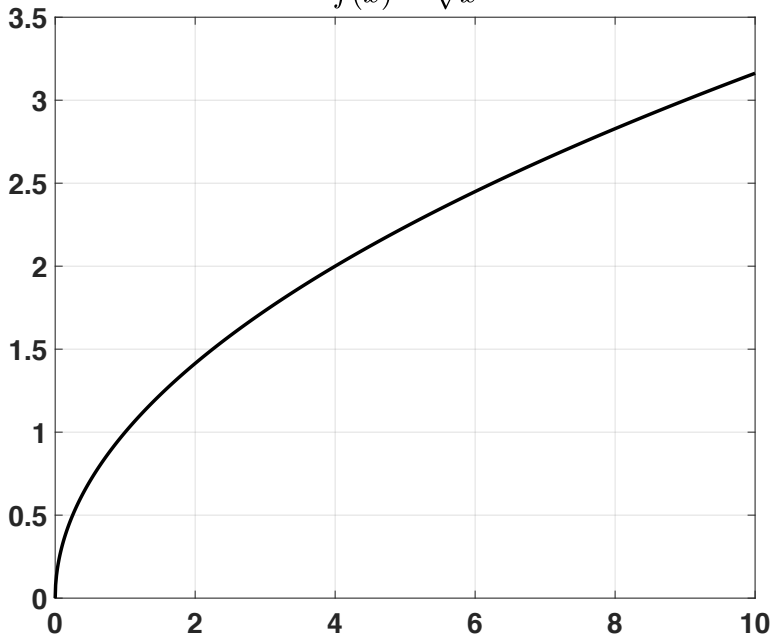
The total amount produced is

$$\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2.$$

Exercise

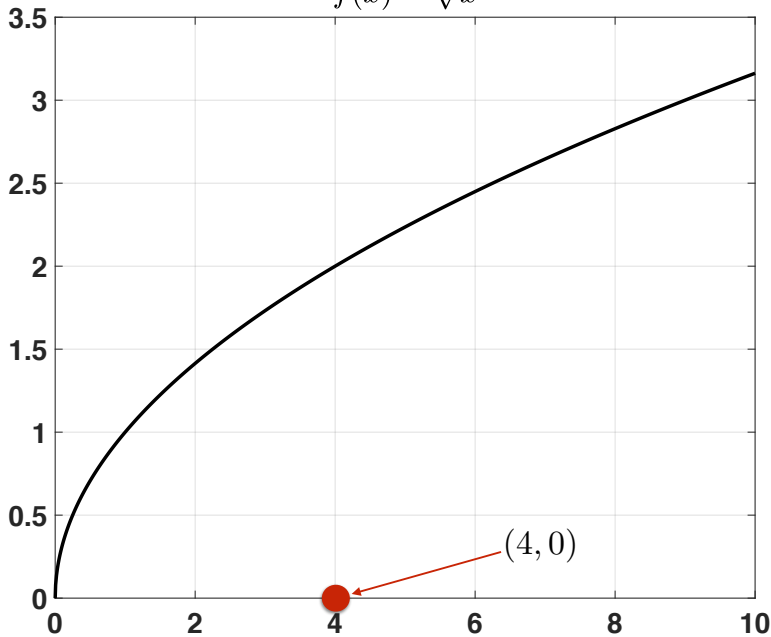
Find the point on the graph of $y = \sqrt{x}$ nearest to the point $(4, 0)$.

$$f(x) = \sqrt{x}$$



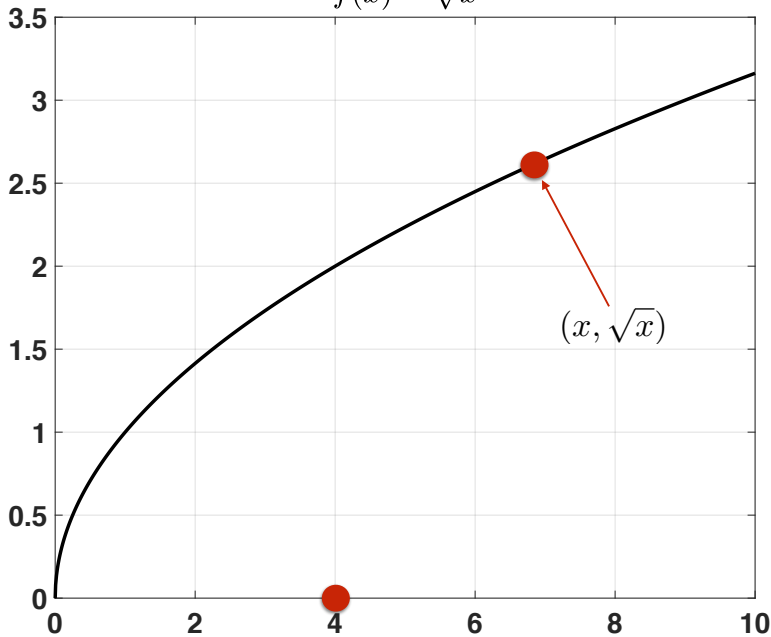
x

$$f(x) = \sqrt{x}$$



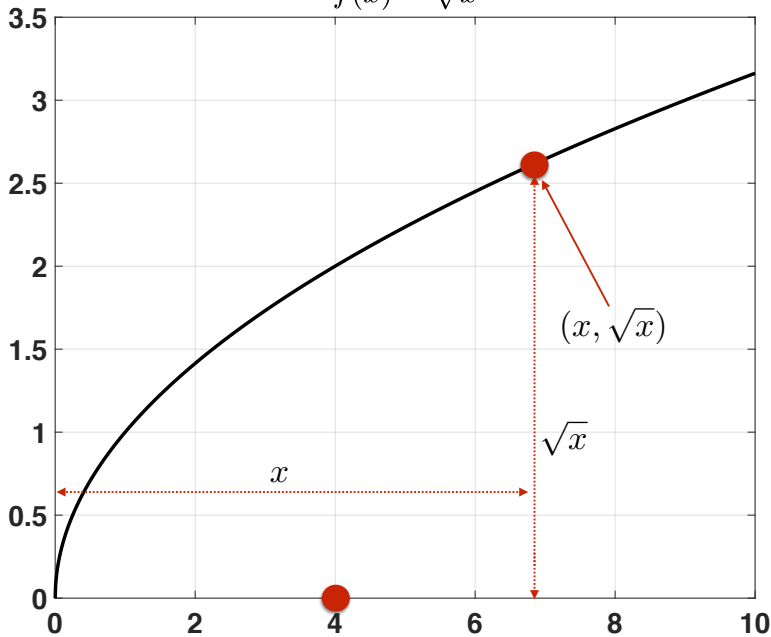
x

$$f(x) = \sqrt{x}$$



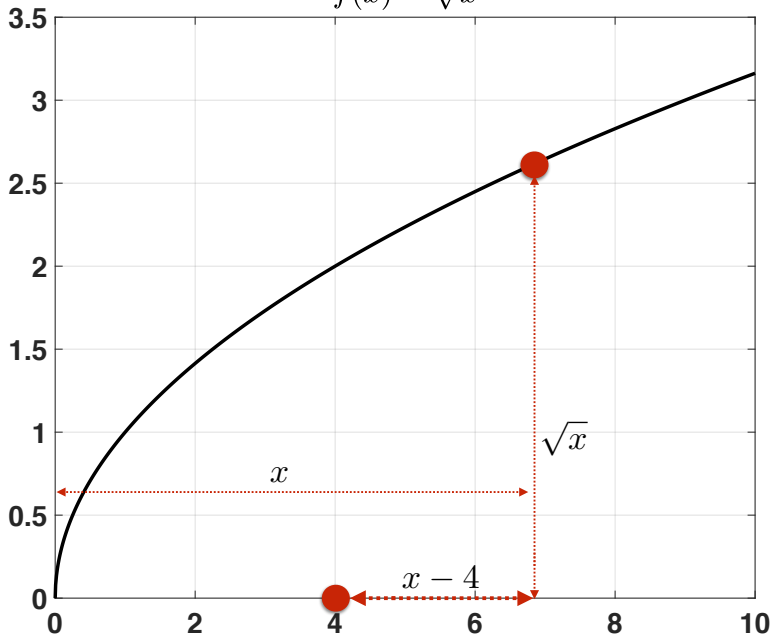
x

$$f(x) = \sqrt{x}$$



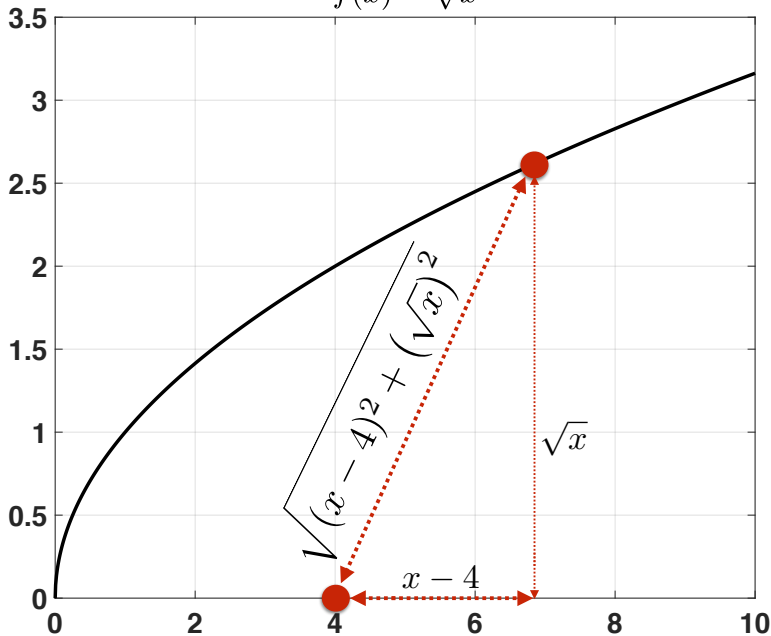
x

$$f(x) = \sqrt{x}$$



x

$$f(x) = \sqrt{x}$$



Exercise

Find the point on the graph of $y = \sqrt{x}$ nearest to the point $(4, 0)$.

Solution. We have to find, if it exists, the minimum of

$$f(x) = \sqrt{(x-4)^2 + (\sqrt{x})^2} = \sqrt{(x-4)^2 + x} = \left((x-4)^2 + x\right)^{\frac{1}{2}}.$$

By the rule of derivation of composite functions:

$$(g(x)^\alpha)' = \alpha (g(x))^{\alpha-1} g'(x) \Rightarrow \left(g(x)^{\frac{1}{2}}\right)' = \frac{1}{2} (g(x))^{-\frac{1}{2}} g'(x).$$

Hence

$$f'(x) = \frac{\left((x-4)^2 + x\right)'}{2 \sqrt{(x-4)^2 + x}} = \frac{(2(x-4) + 1)}{2 \sqrt{(x-4)^2 + x}} = \frac{(2x-7)}{2 \sqrt{(x-4)^2 + x}}$$

Exercise

Find the point on the graph of $y = \sqrt{x}$ nearest to the point $(4, 0)$.

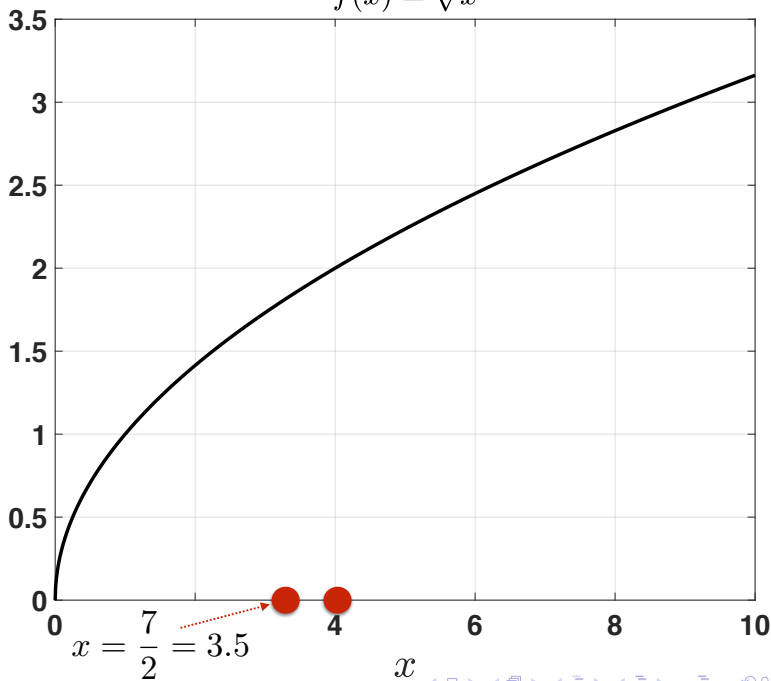
Solution. Summary

$$f(x) = \left((x-4)^2 + x \right)^{\frac{1}{2}}, \quad f'(x) = \frac{(2x-7)}{2\sqrt{(x-4)^2 + x}}.$$

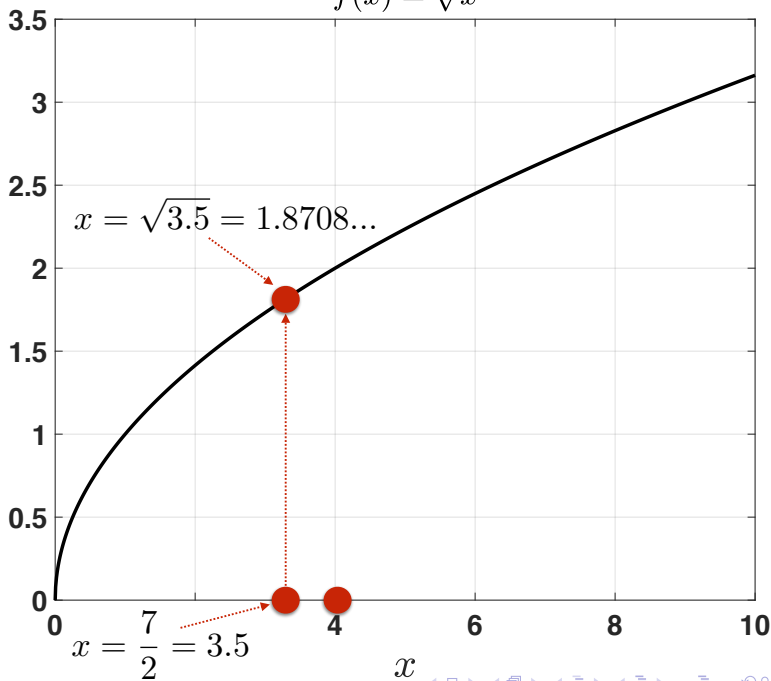
Whence $f'(x) = 0 \Leftrightarrow x = \frac{7}{2}$. Is it a minimum?

$$\begin{aligned} f''(x) &= (2x-7)' \frac{1}{2\sqrt{(x-4)^2 + x}} + (2x-7) \left(\frac{1}{2\sqrt{(x-4)^2 + x}} \right)' \\ &= \frac{2}{2\sqrt{(x-4)^2 + x}} - \frac{1}{2} \frac{(2x-7)}{\left((x-4)^2 + x \right)^{3/2}} \Rightarrow f''\left(\frac{7}{2}\right) > 0 \end{aligned}$$

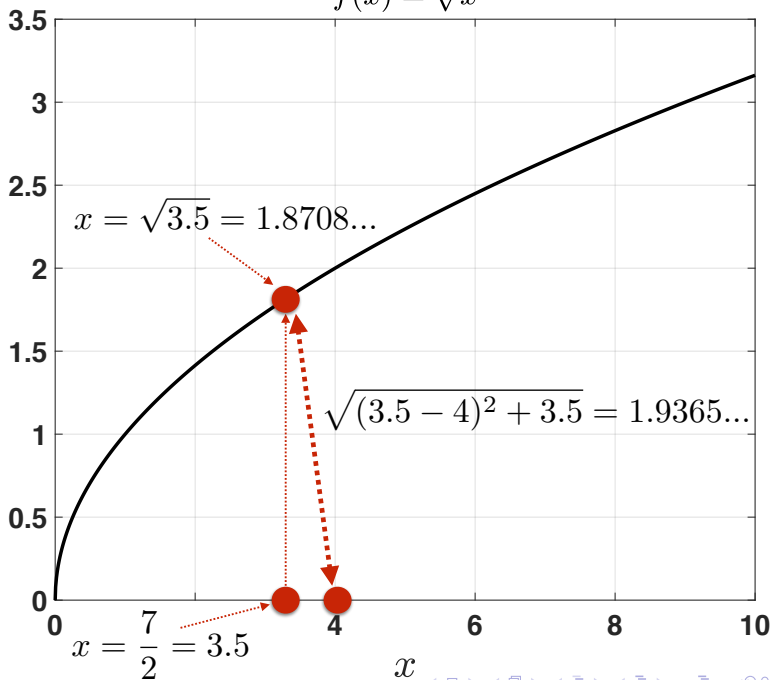
$$f(x) = \sqrt{x}$$



$$f(x) = \sqrt{x}$$



$$f(x) = \sqrt{x}$$



Optimization

Exercise

Find the minimum distance between the point $(0,0)$ and the graph of the function $g(x) = \frac{1}{\sqrt{x}}$.

Solution. The (squared) distance between $(0,0)$ and $(x, \frac{1}{\sqrt{x}})$ is

$$f(x) = (x - 0)^2 + \left(\frac{1}{\sqrt{x}} - 0\right)^2 = x^2 + \frac{1}{x}.$$

whence

$$f'(x) = 2x - \frac{1}{x^2} \Rightarrow f'(x) = 0 \Leftrightarrow 2x^3 - 1 = 0 \Leftrightarrow x^3 = \frac{1}{2} \Leftrightarrow x = \frac{1}{2^{1/3}}$$

The second derivative is

$$f''(x) = 2 + \frac{2}{x^3} \Rightarrow f''\left(\frac{1}{2^{1/3}}\right) > 0 \Rightarrow x = \frac{1}{2^{1/3}} \text{ is a minimum.}$$

$$\text{The minimum distance is } \sqrt{f\left(\frac{1}{2^{1/3}}\right)} = \sqrt{\frac{1}{2^{2/3}} + 2^{1/3}}.$$

Optimization

Utility function

From **WIKIPEDIA**: Consider a set of alternatives facing an individual, and over which the individual has a preference ordering.

A utility function is able to represent those preferences if it is possible to assign a real number to each alternative, in such a way that alternative a is assigned a number greater than alternative b if, and only if, the individual prefers alternative a to alternative b .

In a rational choice framework every consumer decides to consume the amount of good x that maximizes the utility $U(x)$.

$$a \text{ is preferred to } b \Leftrightarrow U(a) > U(b)$$

Optimization

Exercise

Let $u_0 > 0$ and $u_1 > 0$:

$$U(x) = u_0 \ln(x^2) - u_1 x = \text{utility of buying } x \text{ units of good.}$$

For which value of u_0 and u_1 will the consumer buy an amount of good larger than 1?

For which value of u_0 and u_1 the optimal utility is positive?

Solution. The consumer has to maximize the utility $U(x)$:

$$U'(x) = \frac{2u_0}{x} - u_1 = 0 \Leftrightarrow x = \frac{2u_0}{u_1} = x_0.$$

Note that

$$U''(x) = -\frac{2u_0}{x^2} < 0, \forall x \in \mathbb{R}$$

so the function is concave everywhere, whence x_0 is a maximum. Finally

$$U(x_0) = 2u_0 \ln\left(\frac{2u_0}{u_1}\right) - 2u_0 = 2u_0 \left(\ln\left(\frac{2u_0}{u_1}\right) - 1 \right) > 0 \Leftrightarrow \ln\left(\frac{2u_0}{u_1}\right) > 1 \Leftrightarrow \frac{2u_0}{u_1} > e.$$

$x_0 > 1 \Leftrightarrow u_1 < 2u_0.$

Optimization

Exercise

Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is maximal.

Solution. We have to find $x \geq 0$ and $y \geq 0$ such that $x + y = 9$ and such that

$$F(x, y) = x y^2$$

is maximal. Since $y = 9 - x$ we have to find the maximum of

$$f(x) = x(9-x)^2 \Rightarrow f'(x) = (9-x)^2 - 2x(9-x) = (9-x)(9-x-2x) = (9-x)(9-3x) = 0 \Leftrightarrow x = 9 \text{ or } x = 3.$$
$$f''(x) = -2(9-x) + 2x - 2(9-x) = -4(9-x) + 2x$$

$f''(3) = -18 < 0$ and $f''(9) = 18 > 0$ so $x = 9$ is the minimum and $x = 3$ the maximum. The final answer is thus

$$x = 3, \quad y = 9 - 3 = 6.$$

The Derivative of the Inverse Function

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable in (a, b) . Assume f is invertible and call $f^{(-1)} : I_f \subseteq \mathbb{R} \rightarrow \mathbb{R}$ the inverse function, where I_f denotes the image of f . Then $f^{(-1)}$ is differentiable in I_f and

$$\left[f^{(-1)} \right]' (y) = \frac{1}{f' (f^{-1} (y))}.$$

for all $y \in I_f$.

Proof. Take a point $x_0 \in (a, b)$ and call $y_0 = f(x_0)$, that is $x_0 = f^{-1}(y_0)$.

Hence

$$\lim_{y \rightarrow y_0} \frac{f^{(-1)}(y) - f^{(-1)}(y_0)}{y - y_0} = \lim_{x \rightarrow x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)},$$

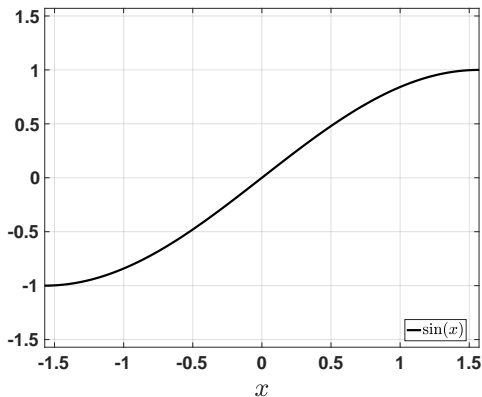
whence

$$\left[f^{(-1)} \right]' (y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}. \quad \square$$

The Derivative of the Inverse Function

Definition

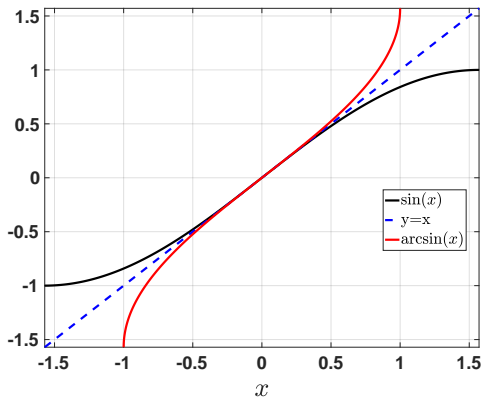
The function $\sin(x)$ is strictly monotonic and increasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so it can be inverted and the inverse is called $\arcsin(x)$ and it is defined in $[-1, 1]$ with values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



The Derivative of the Inverse Function

Definition

The function $\sin(x)$ is strictly monotonic and increasing in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so it can be inverted and the inverse is called $\arcsin(x)$ and it is defined in $[-1, 1]$ with values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



The arcsin function: some important values.

Remember that, by definition,

$$x_0 \xrightarrow{f} y_0 = f(x_0) \Leftrightarrow y_0 \xrightarrow{f^{(-1)}} f^{(-1)}(y_0) = x_0.$$

$$\sin(0) = 0 \Rightarrow \arcsin(0) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 1 \Rightarrow \arcsin(1) = \frac{\pi}{2}$$

$$\sin\left(-\frac{\pi}{2}\right) = -1 \Rightarrow \arcsin(-1) = -\frac{\pi}{2}$$

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \Rightarrow \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \Rightarrow \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \Rightarrow \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \Rightarrow \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \Rightarrow \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \Rightarrow \arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

The Derivative of the Inverse Function

Exercise

Compute the derivative of the $\arcsin(x)$.

Solution. Recall the formula for the derivative of the inverse

$$\left[f^{(-1)}\right]'(y) = \frac{1}{f'(f^{-1}(y))}.$$

In our case, for $x \in [-1, 1]$, it means that

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}.$$

Since $\arcsin(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have

$$\cos(\arcsin(x)) = +\sqrt{1 - \sin^2(\arcsin(x))} = \sqrt{1 - x^2},$$

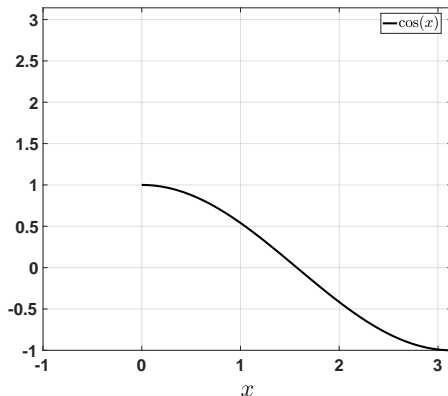
whence

$$(\arcsin(x))' = \frac{1}{\sqrt{1 - x^2}}.$$

The Derivative of the Inverse Function

Definition

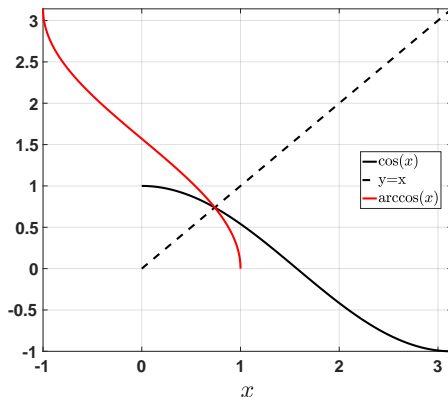
The function $\cos(x)$ is strictly monotonic and decreasing in $[0, \pi]$ so it can be inverted and the inverse is called $\arccos(x)$ and it is defined in $[-1, 1]$ with values in $[0, \pi]$.



The Derivative of the Inverse Function

Definition

The function $\cos(x)$ is strictly monotonic and decreasing in $[0, \pi]$ so it can be inverted and the inverse is called $\arccos(x)$ and it is defined in $[-1, 1]$ with values in $[0, \pi]$.



The arccos function: some important values.

Remember that, by definition,

$$x_0 \xrightarrow{f} y_0 = f(x_0) \Leftrightarrow y_0 \xrightarrow{f^{(-1)}} f^{(-1)}(y_0) = x_0.$$

$$\cos(0) = 1 \Rightarrow \arccos(1) = 0$$

$$\cos\left(\frac{\pi}{2}\right) = 0 \Rightarrow \arccos(0) = \frac{\pi}{2}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \Rightarrow \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \Rightarrow \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \Rightarrow \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

The Derivative of the Inverse Function

Exercise

Compute the derivative of the $\arccos(x)$.

Solution. Recall the formula for the derivative of the inverse

$$\left[f^{(-1)}\right]'(y) = \frac{1}{f'(f^{-1}(y))}.$$

In our case, for $x \in [-1, 1]$, it means that

$$(\arccos(x))' = \frac{1}{-\sin(\arccos(x))}.$$

Since $\arccos(x) \in [0, \pi]$, we have

$$\sin(\arccos(x)) = +\sqrt{1 - \cos^2(\arccos(x))} = \sqrt{1 - x^2},$$

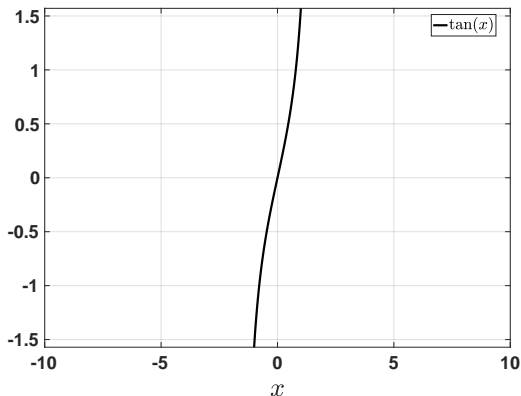
whence

$$(\arccos(x))' = -\frac{1}{\sqrt{1 - x^2}}.$$

The Derivative of the Inverse Function

Definition

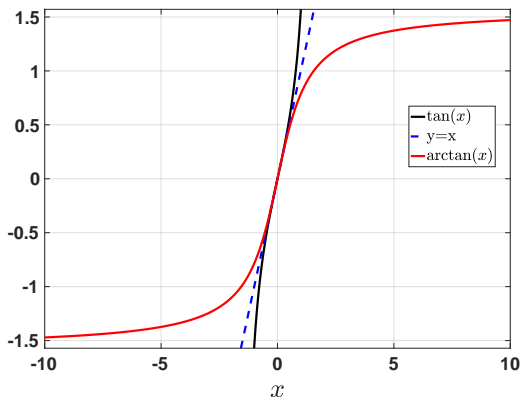
The function $\tan(x)$ is strictly monotonic and increasing in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so it can be inverted and the inverse is called $\arctan(x)$ and it is defined in \mathbb{R} with values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



The Derivative of the Inverse Function

Definition

The function $\tan(x)$ is strictly monotonic and increasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so it can be inverted and the inverse is called $\arctan(x)$ and it is defined in \mathbb{R} with values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



The arctan function: some important values.

Remember that, by definition,

$$x_0 \xrightarrow{f} y_0 = f(x_0) \Leftrightarrow y_0 \xrightarrow{f^{(-1)}} f^{(-1)}(y_0) = x_0.$$

$$\tan(0) = 0 \Rightarrow \arctan(0) = 0$$

$$\lim_{x \rightarrow (\frac{\pi}{2})^-} \tan(x) = +\infty \Rightarrow \lim_{x \rightarrow +\infty} \arctan(x) = (\frac{\pi}{2})^-$$

$$\lim_{x \rightarrow (\frac{\pi}{2})^+} \tan(x) = -\infty \Rightarrow \lim_{x \rightarrow -\infty} \arctan(x) = (\frac{\pi}{2})^+$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \Rightarrow \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

$$\tan\left(\frac{\pi}{4}\right) = 1 \Rightarrow \arctan(1) = \frac{\pi}{4}$$

$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3} \Rightarrow \arctan(\sqrt{3}) = \frac{\pi}{3}.$$

The Derivative of the Inverse Function

Exercise

Compute the derivative of the $\arctan(x)$.

Solution. Recall the formula for the derivative of the inverse

$$\left[f^{(-1)}\right]'(y) = \frac{1}{f'(f^{-1}(y))}.$$

In our case, for $x \in \mathbb{R}$, it means that

$$(\arctan(x))' = \frac{1}{\frac{1}{\cos^2(\arctan(x))}} = \cos^2(\arctan(x)).$$

Now use

$$\cos^2(x) = \frac{1}{1 + \tan^2(x)},$$

to have

$$(\arctan(x))' = \frac{1}{1 + \tan^2(\arctan(x))} = \frac{1}{1 + x^2}$$

De L'Hôpital rule

Theorem

Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $g : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be continuous on D and let x_0 be a limit point of D . Assume that f and g are both differentiable in $D \setminus \{x_0\}$ and $g'(x_0) \neq 0$. If:

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0 \quad \text{AND} \quad \exists \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = L,$$

or if

$$\lim_{x \rightarrow x_0} f(x) = \pm\infty, \quad \lim_{x \rightarrow x_0} g(x) = \pm\infty \quad \text{AND} \quad \exists \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = L,$$

then

$$\exists \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = L.$$

De L'Hôpital rule

Proof. Assume

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0 \quad \text{AND} \quad \exists \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = L,$$

The red condition implies

$$\forall \epsilon > 0 \exists \delta_\epsilon > 0 : \forall x \in D : |x - x_0| < \delta_\epsilon \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon$$

Take $x_1 < x_2$ in $(x_0 - \delta, x_0)$. Cauchy applied to f and g in $[x_1, x_2]$ gives

$$\exists \xi \in (x_1, x_2) : \frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \frac{f'(\xi)}{g'(\xi)}.$$

Note that $x_0 - \delta_\epsilon < x_1 < \xi < x_2 < x_0$ hence ξ is such that $|\xi - x_0| < \delta_\epsilon$

$$\left| \frac{f'(\xi)}{g'(\xi)} - L \right| = \left| \frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} - L \right| < \epsilon.$$

Now take the limit for $x_2 \rightarrow x_0^-$ and use the blue condition

$$\left| \frac{f(x_1)}{g(x_1)} - L \right| < \epsilon.$$

Now for $\epsilon \rightarrow 0$ we have that (remember that $x_1 \in (x_0 - \delta_\epsilon, x_0)$) $x_1 \rightarrow x_0^-$. Hence:

$$\exists \lim_{x \rightarrow x_0^-} \frac{f(x)}{g(x)} = L.$$

with an identical argument we arrive at: $\exists \lim_{x \rightarrow x_0^+} \frac{f(x)}{g(x)} = L.$

De L'Hôpital rule

Theorem

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and differentiable on \mathbb{R} . If:

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} g(x) = 0 \quad \text{AND} \quad \exists \lim_{x \rightarrow \pm\infty} \frac{f'(x)}{g'(x)} = L,$$

or if

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty, \quad \lim_{x \rightarrow \pm\infty} g(x) = \pm\infty \quad \text{AND} \quad \exists \lim_{x \rightarrow \pm\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\exists \lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = L.$$

De L'Hôpital rule

Exercise

Compute the limit

$$\lim_{x \rightarrow 0^+} x \ln(x)$$

$$\begin{aligned}\lim_{x \rightarrow 0^+} x \ln(x) &= 0 \times (-\infty) \\&= \lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{x}} \\&= \frac{-\infty}{+\infty} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{(\ln(x))'}{(\frac{1}{x})'} \\&= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -\frac{x^2}{x} = \lim_{x \rightarrow 0^+} -x = 0.\end{aligned}$$

Exercise

Compute the limit

$$\lim_{x \rightarrow +\infty} \frac{\ln(x)}{\sqrt{x}}$$

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{\ln(x)}{\sqrt{x}} &= \frac{+\infty}{+\infty} \stackrel{H}{=} \lim_{x \rightarrow +\infty} \frac{(\ln(x))'}{(\sqrt{x})'} = \lim_{x \rightarrow +\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow +\infty} \frac{2\sqrt{x}}{x} \\ &= \lim_{x \rightarrow +\infty} \frac{2}{\sqrt{x}} = 0. \end{aligned}$$

De L'Hôpital rule

Exercise

Compute the limit

$$\lim_{x \rightarrow \infty} \left(\arctan x - \frac{\pi}{2} \right) e^x.$$

Since the arc whose tangent is $+\infty$ is $\frac{\pi}{2}$

$$\lim_{x \rightarrow +\infty} \arctan(x) = +\frac{\pi}{2}$$

whence

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\arctan x - \frac{\pi}{2} \right) e^x &= 0 \times (+\infty) = \lim_{x \rightarrow \infty} \left(\frac{\arctan x - \frac{\pi}{2}}{e^{-x}} \right) \\ &= \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{-e^{-x}} = - \lim_{x \rightarrow \infty} \frac{e^x}{1+x^2} \\ &= \frac{\infty}{\infty} \stackrel{H}{=} - \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \frac{\infty}{\infty} \stackrel{H}{=} - \lim_{x \rightarrow \infty} \frac{e^x}{2} = -\infty. \end{aligned}$$

De L'Hôpital rule

WARNING!

The hypothesis

$$\exists \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental!

Example

$$\lim_{x \rightarrow +\infty} \frac{x + \sin(x)}{x} = \frac{+\infty}{+\infty} \stackrel{H}{=} \lim_{x \rightarrow +\infty} \frac{1 + \cos(x)}{1} = \nexists.$$

However

$$\lim_{x \rightarrow +\infty} \frac{x + \sin(x)}{x} = \lim_{x \rightarrow +\infty} \left(1 + \underbrace{\frac{\sin(x)}{x}}_{\rightarrow 0} \right) = 1.$$

De L'Hôpital rule

Exercise

$$\lim_{x \rightarrow 0^+} \left(\ln \left(1 + e^{-1/x} \right) \right)^x = 0^0 = ???$$

Solution. Use the identity

$$\left(\ln \left(1 + e^{-1/x} \right) \right)^x = e^{x \ln(\ln(1 + e^{-1/x}))},$$

and compute

$$\begin{aligned} & \lim_{x \rightarrow 0^+} x \ln \left(\ln \left(1 + e^{-1/x} \right) \right) \\ &= \lim_{x \rightarrow 0^+} \frac{\ln \left(\ln \left(1 + e^{-1/x} \right) \right)}{\frac{1}{x}} = \frac{-\infty}{+\infty} \stackrel{H}{=} \lim_{x \rightarrow 0^+} - \frac{\left(\frac{1}{\ln(1 + e^{-1/x})} \right) \left(\frac{1}{1 + e^{-1/x}} \right) \frac{e^{-1/x}}{x^2}}{\frac{1}{x^2}} \\ &= \lim_{x \rightarrow 0^+} - \left(\frac{1}{\ln(1 + e^{-1/x})} \right) \left(\frac{1}{1 + e^{-1/x}} \right) = \lim_{y \rightarrow 0^+} - \left(\frac{1}{\ln(1 + y)} \right) \left(\frac{1}{1 + 1/y} \right) \\ &= \lim_{y \rightarrow 0^+} - \frac{1}{\ln(1 + y) + \ln(1 + y)^{1/y}} = -1 \Rightarrow \lim_{x \rightarrow 0^+} \left(\ln \left(1 + e^{-1/x} \right) \right)^x = e^{-1} = \frac{1}{e}. \end{aligned}$$

De L'Hôpital rule

Exercise

Compute the limit

$$\lim_{x \rightarrow \infty} \frac{5x + \sin(x) + \ln(\sqrt{x})}{3x}$$

Solution. Blindly applying De L'Hôpital gives

$$\lim_{x \rightarrow \infty} \frac{5x + \sin(x) + \ln(\sqrt{x})}{3x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{5 + \cos(x) + \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}{3},$$

which does not exist! Nevertheless

$$\lim_{x \rightarrow \infty} \frac{5x}{3x} = \frac{5}{3}, \quad \lim_{x \rightarrow \infty} \frac{\sin(x)}{3x} = 0.$$

$$\lim_{x \rightarrow \infty} \frac{\ln(\sqrt{x})}{3x} = \frac{+\infty}{+\infty} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}{3} = 0,$$

whence

$$\lim_{x \rightarrow \infty} \frac{5x + \sin(x) + \ln(\sqrt{x})}{3x} = \frac{5}{3}.$$

De L'Hôpital rule

Exercise

Compute the limit

$$\lim_{x \rightarrow 1^+} \left(\frac{1}{x^2 - 1} - \frac{1}{\ln x} \right).$$

Solution.

$$\begin{aligned} \lim_{x \rightarrow 1^+} \left(\frac{1}{x^2 - 1} - \frac{1}{\ln x} \right) &= +\infty - \infty \\ &= \lim_{x \rightarrow 1^+} \frac{\ln x - (x^2 - 1)}{(x^2 - 1) \ln x} = \frac{0}{0} \\ &\stackrel{H}{=} \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 2x}{2x \ln x + \frac{x^2 - 1}{x}} = \frac{1 - 2}{0} = -\infty. \end{aligned}$$

De L'Hôpital rule

Exercise

Establish for which values of $\alpha > 0$ the following limit exists and it is finite:

$$\lim_{x \rightarrow 1^-} \frac{\arcsin(x) - \frac{\pi}{2}}{(1 - x^2)^\alpha}.$$

Solution. Recall that $\arcsin(1) = \frac{\pi}{2}$. Whence

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{\arcsin(x) - \frac{\pi}{2}}{(1 - x^2)^\alpha} &= \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow 1^-} \frac{(\arcsin(x) - \frac{\pi}{2})'}{((1 - x^2)^\alpha)'} \\ &= \lim_{x \rightarrow 1^-} \frac{1}{\sqrt{1 - x^2}} \frac{1}{\alpha (1 - x^2)^{\alpha-1} (-2x)} \\ &= \lim_{x \rightarrow 1^-} \frac{1}{\alpha (1 - x^2)^{\alpha-\frac{1}{2}} (-2x)}. \end{aligned}$$

so if $\alpha > 0$ it must be $\alpha \leq \frac{1}{2}$ to have a finite limit.

De L'Hôpital rule

Exercise

Establish for which values of $\alpha > 0$ the following limit exists and it is finite:

$$\lim_{x \rightarrow +\infty} \frac{\arctan(x) - \frac{\pi}{2}}{\ln\left(1 + \frac{1}{x^\alpha}\right)}.$$

Solution. Recall that $\lim_{x \rightarrow +\infty} \arctan(x) = \frac{\pi}{2}$. Whence

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{\arctan(x) - \frac{\pi}{2}}{\ln\left(1 + \frac{1}{x^\alpha}\right)} &= \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow +\infty} \frac{(\arctan(x) - \frac{\pi}{2})'}{(\ln(1 + \frac{1}{x^\alpha}))'} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{1+x^2} \left(1 + \frac{1}{x^\alpha}\right) \frac{1}{-\alpha x^{-\alpha-1}} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{1+x^2} \left(1 + \frac{1}{x^\alpha}\right) \frac{x^{\alpha+1}}{-\alpha}. \end{aligned}$$

so, since $\alpha > 0$, the answer is $\alpha + 1 \leq 2 \Rightarrow \alpha \leq 1$.

Exercise

Compute the limit

$$\lim_{x \rightarrow 0} \frac{\arctan(x)}{\ln(1+x)}.$$

Solution.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\arctan(x)}{\ln(1+x)} &= \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{(\arctan(x))'}{(\ln(1+x))'} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{1+x^2}}{\frac{1}{1+x}} \\ &= \lim_{x \rightarrow 0} \frac{1+x}{1+x^2} = 1. \end{aligned}$$

Exercise

Compute the limit

$$\lim_{x \rightarrow 0} \frac{\arctan(x^2)}{\ln(1+x)}.$$

Solution.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\arctan(x^2)}{\ln(1+x)} &= \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{(\arctan(x^2))'}{(\ln(1+x))'} \\ &= \lim_{x \rightarrow 0} \frac{\frac{2x}{1+x^2}}{\frac{1}{1+x}} \\ &= \lim_{x \rightarrow 0} \frac{2x(1+x)}{1+x^2} = 0. \end{aligned}$$

De L'Hôpital rule

Exercise

Compute, as a function of α , the following limit

$$\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^\alpha}.$$

Solution.

$$\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^\alpha} = \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{(\sin(x^2))'}{(x^\alpha)'} = \lim_{x \rightarrow 0} \frac{\cos(x^2) 2x}{\alpha x^{\alpha-1}} = \lim_{x \rightarrow 0} 2 \frac{\cos(x^2)}{\alpha x^{\alpha-2}}.$$

In summary

$$\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^\alpha} = \begin{cases} 0 & \text{if } \alpha < 2 \\ 1 & \text{if } \alpha = 2 \\ +\infty & \text{if } \alpha > 2 \end{cases}$$

De L'Hôpital rule

Exercise

Compute, as a function of α , the following limit

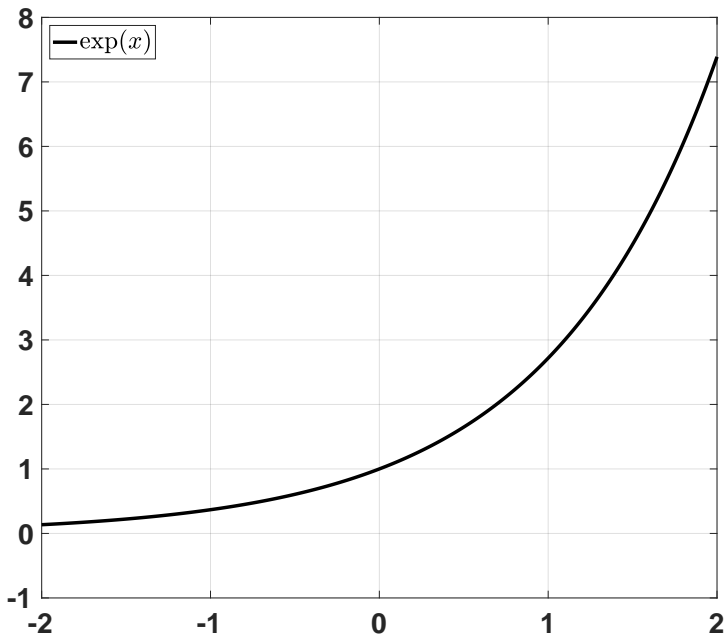
$$\lim_{x \rightarrow 0} \frac{\ln(1 + x^\alpha)}{\arcsin(x)}.$$

Solution.

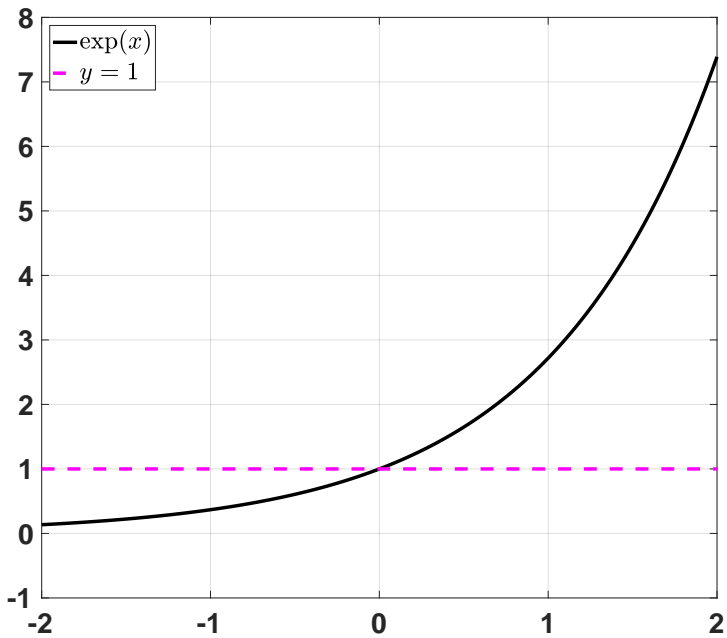
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln(1 + x^\alpha)}{\arcsin(x)} &= \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{(\ln(1 + x^\alpha))'}{(\arcsin(x))'} = \lim_{x \rightarrow 0} \frac{\frac{\alpha x^{\alpha-1}}{1+x^\alpha}}{\frac{1}{(1+x^2)^{1/2}}} \\ &= \lim_{x \rightarrow 0} \frac{\alpha x^{\alpha-1} \sqrt{1+x^2}}{1+x^\alpha} = \lim_{x \rightarrow 0} \frac{\alpha \sqrt{1+x^2}}{x^{1-\alpha} + x} \end{aligned}$$

In summary

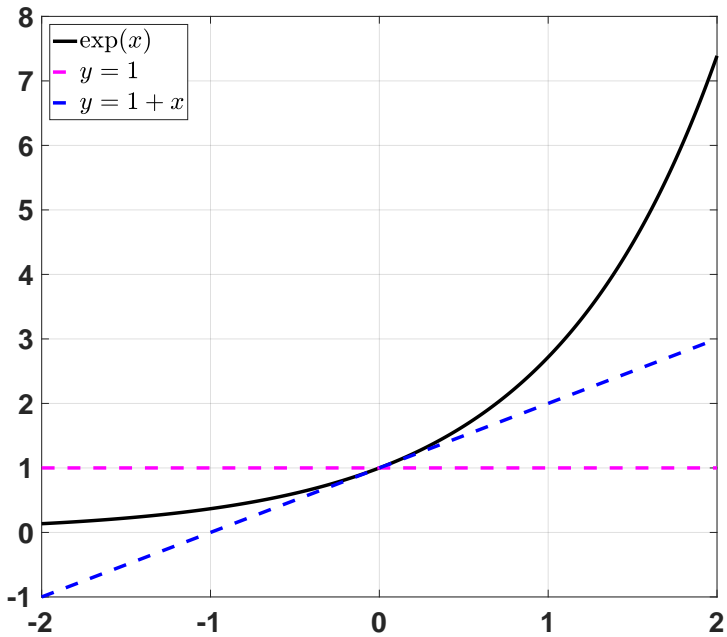
$$\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^\alpha} = \begin{cases} +\infty & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases}$$



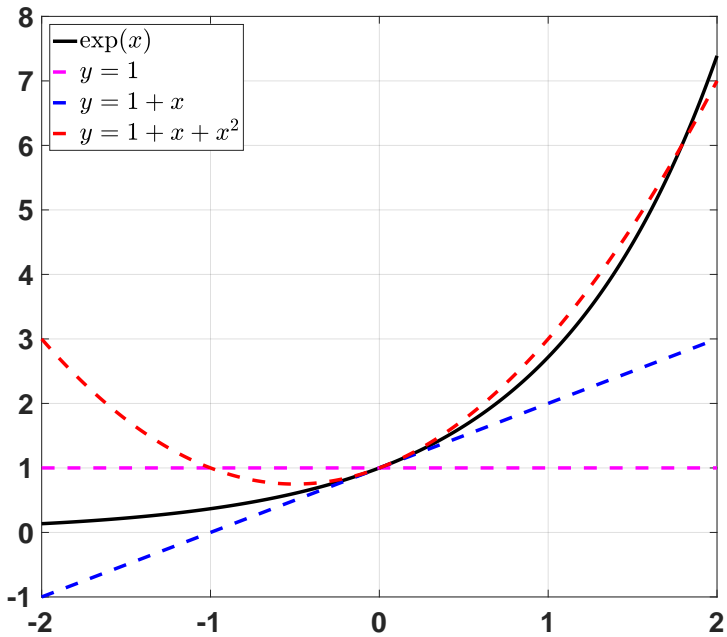
x

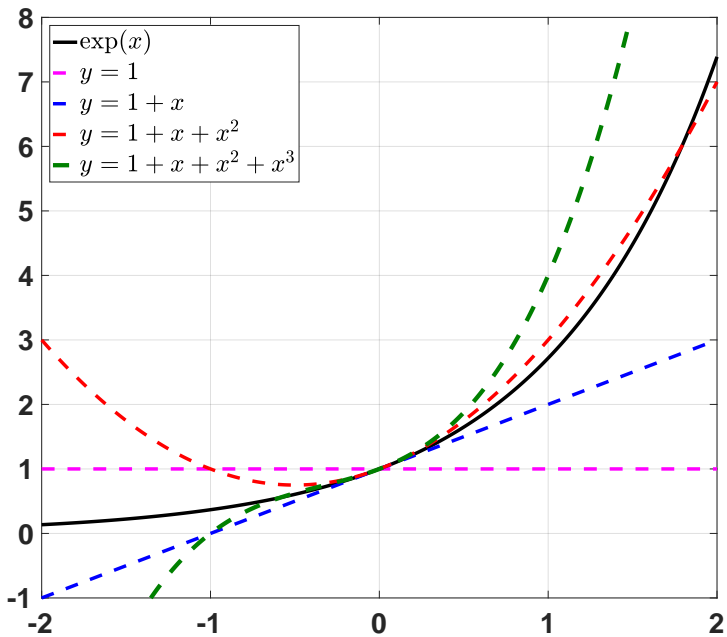


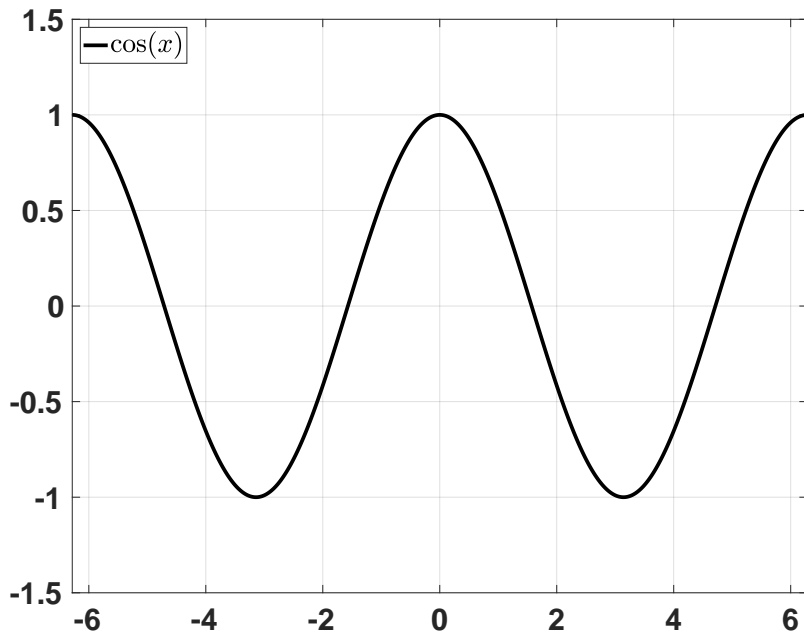
x



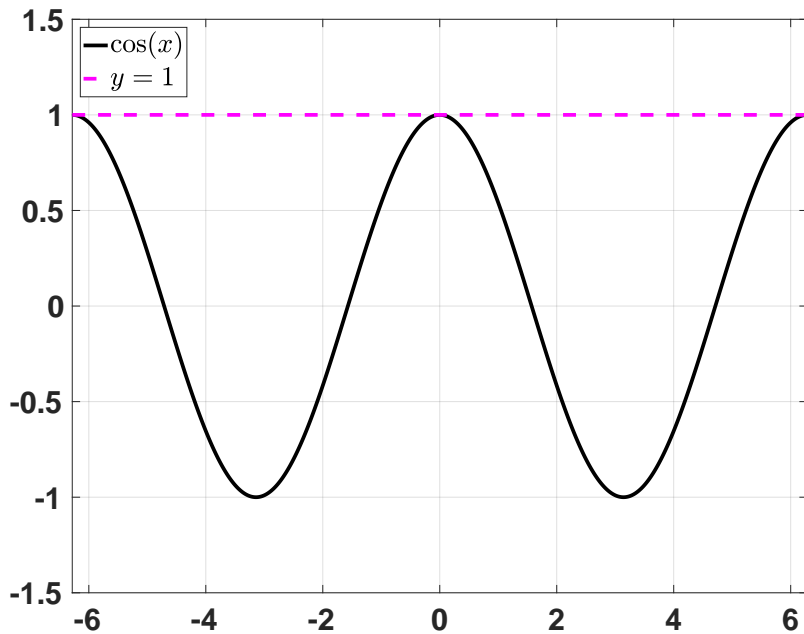
x



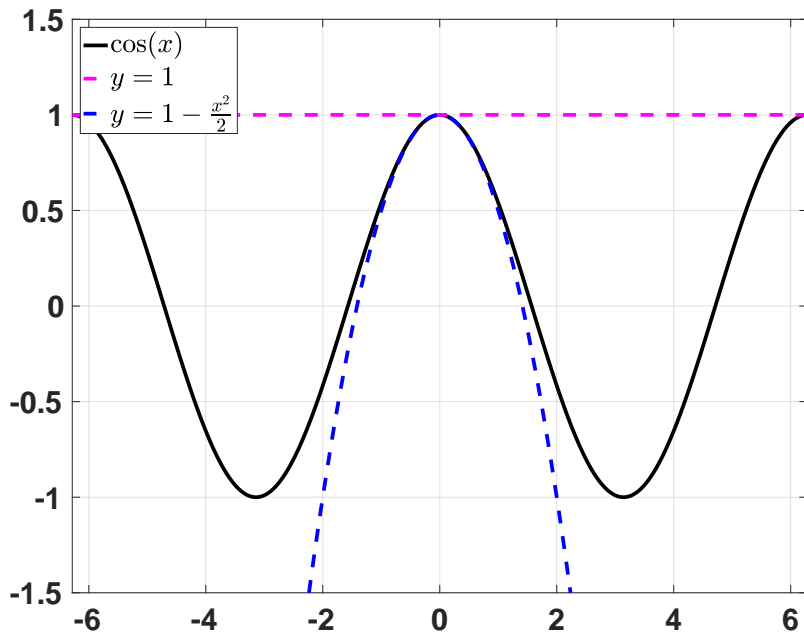




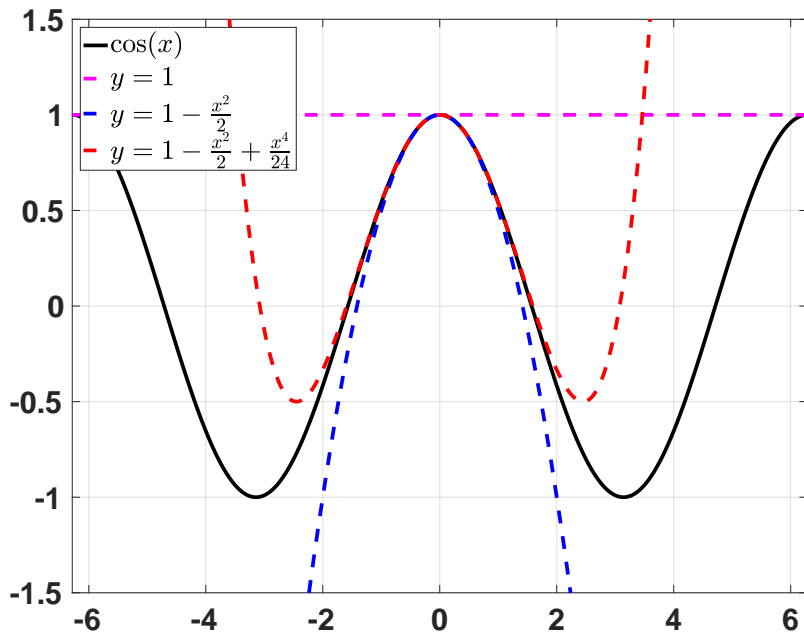
x



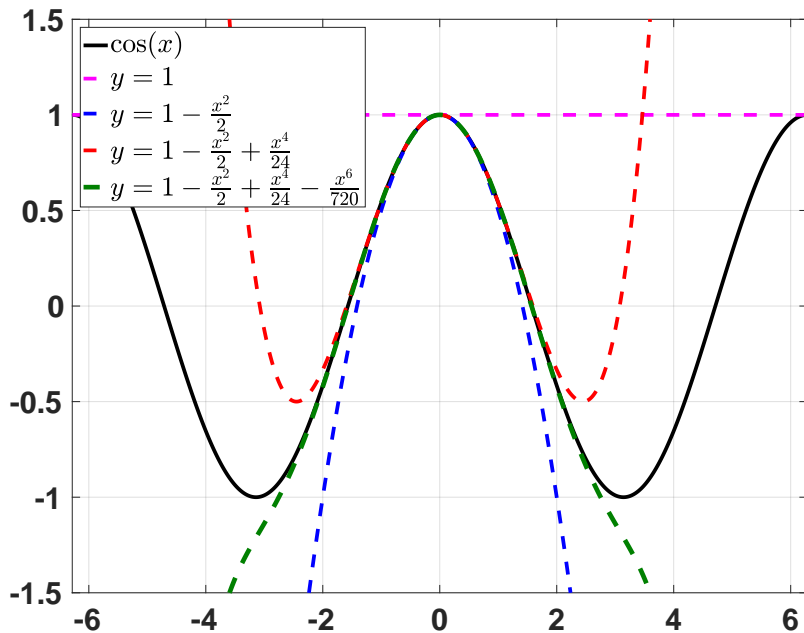
x



x



x



x

The Taylor's formula

Before proceeding...let's talk about "little-o" functions.

Definition

Let $\alpha > 0$. We indicate with $o((x - x_0)^\alpha)$ (we say "little oh of $(x - x_0)^\alpha$ ") every function that, as $x \rightarrow x_0$, goes to zero faster than $(x - x_0)^\alpha$. In formula

$$\lim_{x \rightarrow x_0} \frac{o((x - x_0)^\alpha)}{(x - x_0)^\alpha} = 0.$$

Example

$$\lim_{x \rightarrow 0} \frac{\ln(1 + x^2)}{x} = \lim_{x \rightarrow 0} x \underbrace{\frac{\ln(1 + x^2)}{x^2}}_{\rightarrow 1} = 0 \Rightarrow \ln(1 + x^2) = o(x).$$

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = 0 \Rightarrow x^2 = o(x).$$

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{\sqrt{x}} = \lim_{x \rightarrow 0} \sqrt{x} \underbrace{\frac{\sin(x)}{x}}_{\rightarrow 1} = 0 \Rightarrow \sin(x) = o(\sqrt{x}).$$

Algebra of little-oh

Strange things happen with little-oh:

$$o(x) + o(x^2) = o(x).$$

This is because

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{o(x) + o(x^2)}{x} &= \lim_{x \rightarrow 0} \left(\frac{o(x)}{x} + \frac{o(x^2)}{x} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{o(x)}{x} + \frac{o(x^2)}{x^2} x \right) = 0.\end{aligned}$$

More generally

$$o(x^m) + o(x^n) = o(x^{\min(m,n)}).$$

Besides, trivially:

$$o(x^m) o(x^n) = o(x^{m+n}).$$

The Taylor's formula

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function n -times differentiable in a neighborhood of $x_0 \in (a, b)$ and assume that the n -th derivative is continuous in x_0 . Then $f(x)$ can be written as:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \dots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + o((x - x_0)^n).$$

In other words

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \dots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n}_{\text{Approximating polynomial}} + \underbrace{o((x - x_0)^n)}_{\text{Reminder}}$$

we can approximate the function as polynomial plus a reminder that goes to zero faster than $(x - x_0)^n$.

The Taylor's formula: proof

Case $n = 1 \Rightarrow \exists f'(x)$ and it is continuous in x_0 . Let $g(x)$ be defined by

$$g(x - x_0) = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

that is

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{Approximating polynomial}} + \underbrace{g(x - x_0)}_{\text{Reminder}}$$

f is differentiable $x_0 \Rightarrow f$ is continuous in $x_0 \Rightarrow \lim_{x \rightarrow x_0} g(x - x_0) = 0$.

Moreover $g(x - x_0)$ is differentiable as f . Now compute:

$$\lim_{x \rightarrow x_0} \frac{g(x - x_0)}{(x - x_0)} = \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow x_0} \frac{g'(x - x_0)}{1} = \lim_{x \rightarrow x_0} [f'(x) - f'(x_0)] = 0,$$

therefore we have proved that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o((x - x_0)). \quad \square$$

The Taylor's formula: proof

Case $n = 2 \Rightarrow \exists f''(x)$ and it is continuous in x_0 . Let $g(x)$ be defined by

$$g(x - x_0) = f(x) - f(x_0) - f'(x_0)(x - x_0) - \frac{1}{2!} f''(x_0)(x - x_0)^2$$

that is

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2}_{\text{Approximating polynomial}} + \underbrace{g(x - x_0)}_{\text{Reminder}}$$

f is differentiable $x_0 \Rightarrow f$ is continuous in $x_0 \Rightarrow \lim_{x \rightarrow x_0} g(x - x_0) = 0$. Moreover $g(x - x_0)$ is differentiable as f with

$$g'(x - x_0) = f'(x) - f'(x_0) - f''(x_0)(x - x_0) \Rightarrow \lim_{x \rightarrow x_0} g'(x - x_0) = 0$$

$$g''(x - x_0) = f''(x) - f''(x_0) \Rightarrow \lim_{x \rightarrow x_0} g''(x - x_0) = 0$$

Now compute:

$$\lim_{x \rightarrow x_0} \frac{g(x - x_0)}{(x - x_0)^2} = \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow x_0} \frac{g'(x - x_0)}{2(x - x_0)} = \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow x_0} \frac{g''(x - x_0)}{2} = 0$$

therefore we have proved that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + o((x - x_0)^2). \quad \square$$

The Taylor's formula of $\cos(x)$ around $x_0 = 0$.

Compute the Taylor polynomial expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \dots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + o((x - x_0)^n).$$

of $f(x) = \cos(x)$ around $x_0 = 0$.

$$f(x) = \cos(x) \Rightarrow f(0) = 1$$

$$f'(x) = -\sin(x) \Rightarrow f'(0) = 0$$

$$f''(x) = -\cos(x) \Rightarrow f''(0) = -1$$

$$f'''(x) = \sin(x) \Rightarrow f'''(0) = 0$$

$$f^{(4)}(x) = \cos(x) \Rightarrow f^{(4)}(0) = 1$$

$$\vdots \quad \vdots$$

whence

$$\cos(x) = 1 - \frac{x}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n}).$$

The Taylor's formula of $\sin(x)$ around $x_0 = 0$.

Compute the Taylor polynomial expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \dots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + o((x - x_0)^n).$$

of $f(x) = \sin(x)$ around $x_0 = 0$.

$$f(x) = \sin(x) \Rightarrow f(0) = 0$$

$$f'(x) = \cos(x) \Rightarrow f'(0) = 1$$

$$f''(x) = -\sin(x) \Rightarrow f''(0) = 0$$

$$f'''(x) = -\cos(x) \Rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = \sin(x) \Rightarrow f^{(4)}(0) = 0$$

$$\vdots$$

whence

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1}).$$

The Taylor's formula of $\ln 1 + x$ around $x_0 = 0$.

Compute the Taylor polynomial expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \dots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + o((x - x_0)^n).$$

of $f(x) = \ln(1 + x)$ around $x_0 = 0$.

$$f(x) = \ln(1 + x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = 2\frac{1}{(1+x)^3} \Rightarrow f'''(0) = 2$$

$$f^{(4)}(x) = -6\frac{1}{(1+x)^4} \Rightarrow f^{(4)}(0) = -6.$$

whence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{2}{3 \cdot 2} x^3 - \frac{6}{4 \cdot 3 \cdot 2} x^4 + \dots = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^n \frac{x^n}{n} + o(x^n).$$

The Taylor's formula of e^x around $x_0 = 0$.

Compute the Taylor polynomial expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \dots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + o((x - x_0)^n).$$

of $f(x) = e^x$ around $x_0 = 0$.

$$f(x) = e^x \Rightarrow f(0) = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

$$f''(x) = e^x \Rightarrow f''(0) = 1$$

$$f'''(x) = e^x \Rightarrow f'''(0) = 1$$

$$f^{(4)}(x) = e^x \Rightarrow f^{(4)}(0) = 1.$$

whence

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + o(x^n).$$

The Taylor's formula.

Exercise

Write the Taylor's polynomial of **degree 3** of $\sqrt{1+x}$ in $x_0 = 0$ and use it to find an approximation of $\sqrt{2}$.

How to proceed

- Write the generic formula with the requested **degree**

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \frac{1}{3!} f^{(3)}(x_0)(x - x_0)^3 + o((x - x_0)^3).$$

- Plug in the formula the value for x_0

$$f(x) = f(0) + f'(0)x + \frac{1}{2!} f^{(2)}(0)x^2 + \frac{1}{3!} f^{(3)}(0)x^3 + o(x^3).$$

- From the last formula isolate the **quantities that must be computed**

$$f(0), \quad f'(0), \quad f^{(2)}(0), \quad f^{(3)}(0).$$

The Taylor's formula.

Exercise

Write the Taylor's polynomial of **degree 3** of $\sqrt{1+x}$ in $x_0 = 0$ and use it to find an approximation of $\sqrt{2}$.

How to proceed

- Write the generic formula with the requested **degree**

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \frac{1}{3!} f^{(3)}(x_0)(x - x_0)^3 + o((x - x_0)^3).$$

- Plug in the formula the value for x_0

$$f(x) = f(0) + f'(0)x + \frac{1}{2!} f^{(2)}(0)x^2 + \frac{1}{3!} f^{(3)}(0)x^3 + o(x^3).$$

- From the last formula isolate the **quantities that must be computed**

$$f(0), f'(0), f^{(2)}(0), f^{(3)}(0).$$

- Compute the quantities and conclude with some algebra.

The Taylor's formula for $\sqrt{2} = 1.41421\dots$

Exercise

Write the Taylor's polynomial of degree 3 of $\sqrt{1+x}$ in $x_0 = 0$ and use it to find an approximation of $\sqrt{2}$.

$$f(x) = \sqrt{1+x} \Rightarrow f(0) = 1$$

$$f'(x) = \frac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4(1+x)^{3/2}} \Rightarrow f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8(1+x)^{5/2}} \Rightarrow f'''(0) = \frac{3}{8}$$

By plugging the above quantities into the formula, we get

$$\sqrt{1+x} = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + o(x^3) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + o(x^3).$$

Write the **approximated formula**:

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \Rightarrow \sqrt{2} = \sqrt{1+1} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} = \frac{23}{16} = 1.4375$$

The Taylor's formula for $\ln(2) = 0.69314\dots$

Exercise

Write the Taylor's polynomial of degree 3 of $\ln(1+x)$ in $x_0 = 0$ and use it to find an approximation of $\ln(2)$.

$$f(x) = \ln(1+x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = 2\frac{1}{(1+x)^3} \Rightarrow f'''(0) = 2.$$

By plugging the above quantities into the formula, we get

$$\ln(1+x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + o(x^3) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3).$$

Write the **approximated formula**:

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \Rightarrow \ln(2) = \ln(1+1) \approx 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 0.8333$$

The Taylor's formula for $\sin(1) = 0.84147\dots$

Exercise

Write the Taylor's polynomial of degree 5 of $\sin(x)$ in $x_0 = 0$ and use it to find an approximation of $\sin(1)$.

$$f(x) = \sin(x) \Rightarrow f(0) = 0$$

$$f^{(1)}(x) = \cos(x) \Rightarrow f'(0) = 1$$

$$f^{(2)}(x) = -\sin(x) \Rightarrow f''(0) = 0$$

$$f^{(3)}(x) = -\cos(x) \Rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = \sin(x) \Rightarrow f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos(x) \Rightarrow f^{(5)}(0) = 1.$$

By plugging the above quantities into the formula, we get

$$\begin{aligned}\sin(x) &= f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + o(x^5) \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5).\end{aligned}\tag{0.3}$$

$$\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120} \Rightarrow \sin(1) \approx 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} = 0.841666\dots$$