Real numbers

Mathematics I

University of Rome Tor Vergata

Week 1, AY 2022-2023

Natural numbers

We indicate with $\mathbb{N}$ the set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

We indicate with $\mathbb{N}_0$ the set of natural numbers with zero:

$$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$$

Of course, $\mathbb{N} \subset \mathbb{N}_0$.

On this sets the operation of sum is defined but subtraction cannot always be performed.

Integer numbers

We indicate with $\mathbb{Z}$ the set of integer numbers:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Thus, $\mathbb{Z}$ is the union of $\mathbb{N}_0$ and the set of negative numbers.
Note that $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z}$.

In this set we can sum and subtract numbers. We can also multiply numbers but division between number cannot always be performed

Rational numbers

We indicate with $\mathbb{Q}$ the set of rational numbers. Rational numbers are obtained by dividing an integer number by another integer number different from zero. In symbols:

$$\mathbb{Q} = \left\{ \frac{m}{n} \text{ such that } m \in \mathbb{Z}, n \in \mathbb{Z} \backslash \{0\} \right\}$$

Examples: $\frac{1}{2}$, $\frac{-1}{2}$, $\frac{25}{12}$, $\frac{2}{-3}$, etc.

Note that, if we set $n = 1$, we obtain the set $\mathbb{Z}$. Thus, $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}$.

The set $\mathbb{Q}$ is large enough to make sum, substraction, multiplication and division.

Operations on $\mathbb{Q}$

- Sum:
$$\frac{n}{m} + \frac{k}{q} = \frac{nq + km}{mq}, \quad \text{Ex: } \frac{1}{3} + \frac{7}{4} = \frac{1 \cdot 4 + 7 \cdot 3}{3 \cdot 4} = \frac{25}{12}$$
- Product:
$$\frac{n}{m} \cdot \frac{k}{q} = \frac{n \cdot k}{m \cdot q}, \quad \text{Ex: } \frac{1}{3} \cdot \frac{-7}{4} = -\frac{7}{12}$$
- Inverse:
$$\frac{1}{\frac{n}{m}} = \frac{m}{n}, \quad \text{Ex: } \frac{1}{\frac{2}{-3}} = -\frac{3}{2}$$

  Note that $\frac{1}{\frac{n}{m}}$ is also denoted by $\left(\frac{n}{m}\right)^{-1}$, and therefore $\left(\frac{n}{m}\right)^{-1} = \frac{1}{\frac{n}{m}} = \frac{m}{n}$.
- k-th power ($k \in \mathbb{N}$):
$$\left(\frac{m}{n}\right)^k = \underbrace{\frac{m}{n} \cdot \frac{m}{n} \cdots \frac{m}{n}}_{k \text{ times}} = \frac{m^k}{n^k}, \quad \text{Ex: } \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}$$

  and
$$\left(\frac{m}{n}\right)^{-k} = \left[\left(\frac{m}{n}\right)^{-1}\right]^k = \left[\frac{n}{m}\right]^k = \frac{n^k}{m^k}, \quad \text{Ex: } \left(\frac{1}{2}\right)^{-10} = 2^{10}$$

Operations on $\mathbb{Q}$, cont'd

Given $q \in \mathbb{Q}$, it is possible to show that:

$$q^m \cdot q^n = q^{m+n}$$

for $n, m \in \mathbb{Z}$.

Examples

- $\left(\frac{2}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^2 = \left(\frac{2}{5}\right)^{3+2} = \left(\frac{2}{5}\right)^5 = \frac{2^5}{5^5}$
- $\left(\frac{2}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^{-2} = \left(\frac{2}{5}\right)^{3-2} = \left(\frac{2}{5}\right)^1 = \frac{2}{5}$
- $\left(\frac{2}{5}\right)^2 \cdot \left(\frac{2}{5}\right)^{-3} = \left(\frac{2}{5}\right)^{2-3} = \left(\frac{2}{5}\right)^{-1} = \frac{5}{2}$
- $\left(-\frac{2}{5}\right)^2 \cdot \left(-\frac{2}{5}\right)^{-3} = \left(-\frac{2}{5}\right)^{2-3} = \left(-\frac{2}{5}\right)^{-1} = -\frac{5}{2}$

Given $q \in \mathbb{Q}$, $q \neq 0$, we have $q^0 = 1$. Indeed, for an arbitrary $k \in \mathbb{N}$, we can write:

$$q^0 = q^{k-k} = q^k \cdot q^{-k} = q^k \cdot \frac{1}{q^k} = 1$$

Decimal representation of $\mathbb{Q}$

So far we have expressed the elements of $\mathbb{Q}$ as fractions. They can also be expressed in decimal notation.

Examples

- $\frac{3}{10} = 0.3$
- $-\frac{5}{2} = -2.5$
- $\frac{1}{3} = 0.33333\ldots$
- $\frac{1}{22} = 0454545\ldots$
- $\frac{7}{12} = 0.5833333\ldots$

The decimal representation of a rational number is either **finite**, as in $\frac{3}{10}$, $-\frac{5}{2}$, or **infinite with a period**, as in $\frac{1}{3}$, $\frac{1}{22}$, $\frac{7}{12}$.

Decimal representation of $\mathbb{Q}$, cont'd

Theorem

Let $q = \frac{n}{m}$ be a rational number. Then there are two ***mutually exclusive*** *possibilities:*

1. *The decimal representation of q is made by a* ***finite number of digits***
2. *The decimal representation of q is made by an* ***infinite number of digits but it is periodic.*** *In this case the period contains at most $m - 1$ digits*

Decimal representation of $\mathbb{Q}$, cont'd

Fraction	Decimal representation	Length of period
$\frac{9}{11}$	$0.818181\ldots = 0.\overline{81}$	2
$\frac{1}{7}$	$0.142857142857\ldots = 0.\overline{142857}$	6
$\frac{1}{81}$	$0.012345679012345679 = 0.\overline{012345679}$	9
$\frac{1}{29}$	$0.\overline{0344827586206896551724137931}$	28

Incompleteness of $\mathbb{Q}$

The set $\mathbb{Q}$ is insufficient for many purposes. For instance, assume we want to solve the following equation:

$$x^2 = 2$$

We know that the solutions are $x = \pm\sqrt{2}$. What about the decimal representation of $\sqrt{2}$?

$$\begin{aligned}\sqrt{2} = &1.4142135623730950488016887242 0\\ &969807856967187537694807317667 9\\ &737990732478462107038850387534 3\\ &27641572735013846230912297024924836 0...\end{aligned}$$

There is no period!

This means that there exist numbers, such as $\sqrt{2}$, which are not rational, meaning that they are NOT contained in $\mathbb{Q}$.

These numbers are called **irrational numbers**.

Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \middle| m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

Which are the numbers of $\mathbb{R}$ that are not in $\mathbb{Q}$? These numbers are called "irrational numbers" and are those whose decimal representation is not finite, nor periodic.

Examples of irrational numbers:

- $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ ... more generally all the square roots of numbers which are not perfect square (e.g. $4 = 2^2, 9 = 3^2, 16 = 4^2, 25 = 5^2, 36 = 6^2, \dots$)
- The Euler's number $e = 2.718281828459045235360287471352662497\dots$
- The Pi number $\pi = 3.141592653589793238462643383279502884\dots$

Real Numbers: Axioms

Operations: The operations of sum (+) and multiplication (·) between pairs of real numbers are defined and have the following properties

- Associative property: $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in \mathbb{R}$
- Commutative property: $a + b = b + a$, $a \cdot b = b \cdot a$, for all $a, b \in \mathbb{R}$
- Distributive property: $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{R}$
- Existence of the neutral element: for any $a \in \mathbb{R}$ there are two distinct numbers, namely 0 and 1 such that $a + 0 = a$ and $a \cdot 1 = a$. These numbers are called the neutral number of sum and multiplication, respectively.
- Existence of the opposite: for any $a \in \mathbb{R}$ there is a real number $-a$ such that $a + (-a) = 0$. $-a$ is called the opposite of a
- Existence of the inverse: for any $a \in \mathbb{R}$ there is a real number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$. $\frac{1}{a}$ is called the inverse of a

Real Numbers: Axioms

Order Relation There is a relation *minor or equal to*, $\leq$, between pairs of real numbers with the properties

- For any pair of real numbers a, b either $a \leq b$ or $b \leq a$.
- If $a \leq b$ **and** $b \leq a$ then necessarily $a = b$.
- If $a \leq b$, then also $a + c \leq b + c$, for any real number c.
- If $0 \leq a$ and $0 \leq b$ the $0 \leq a + b$ and $0 \leq a \cdot b$.

Real Numbers: Axioms

Completeness: Let A, B two non-empty subsets of real numbers:

$$A \subseteq \mathbb{R}, \quad B \subseteq \mathbb{R}, \quad A \neq \emptyset, \quad B \neq \emptyset.$$

Assume that any number in A is smaller than or equal to any other number in B:

$$\forall x \in A \Rightarrow x \leq y,\ \forall y \in B.$$

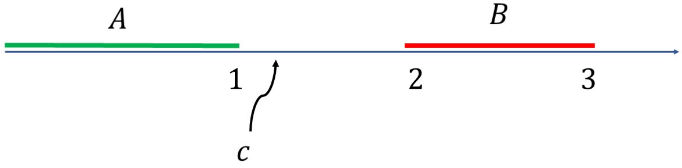
Then there exists a real number c such that c is larger than a and smaller than b, for any a in A and for any b in B.

$$\exists c \in \mathbb{R} : a \leq c \leq b, \forall a \in A, \forall b \in B.$$

The number c is called the separating point of A and B.

Illustrative example

Consider sets $A = \{x \in \mathbb{R} : x \leq 1\}$ and $B = \{x \in \mathbb{R} : 2 \leq x \leq 3\}$



In this example the separating point is not unique.

Illustrative example

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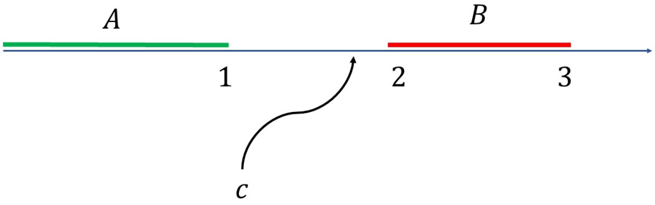
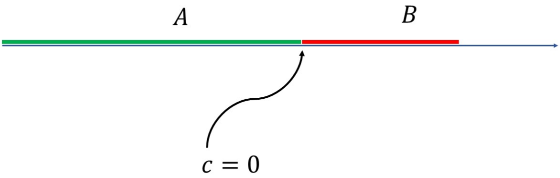


For instance this is another possible value for c.

Illustrative example

Consider sets $A = \{x \in \mathbb{R} : x \leq 0\}$ and $B = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$



In this case the separating point is unique.

$\mathbb{Q}$ is not $\mathbb{R}$

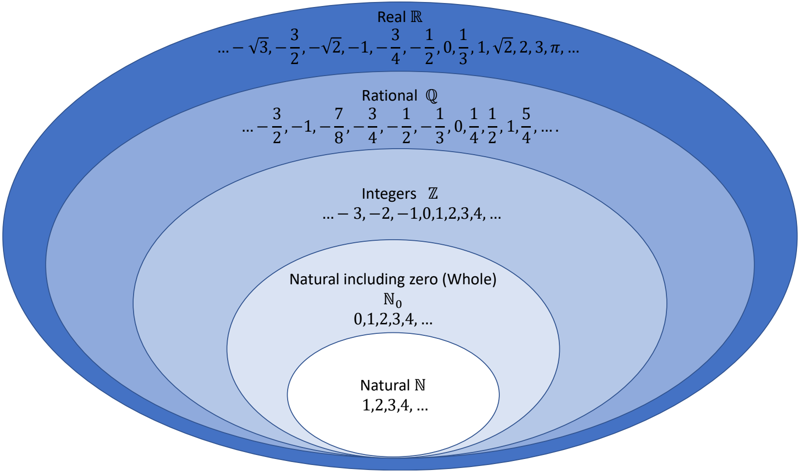
It is clear that the set $\mathbb{Q}$ satisfies all axioms of operations and ordering. Does it satisfy the axiom of completeness?

The answer is no!
Consider the sets:

$$A = \{q \in \mathbb{Q} : q \geq 0 \text{ and } q^2 < 2\}$$
$$B = \{q \in \mathbb{Q} : q \geq 0 \text{ and } q^2 \geq 2\}$$

These sets are disjoint, i.e. $A \cap B = \emptyset$ and the separation point is $c = \sqrt{2} \notin \mathbb{Q}$. Hence the axiom of completeness is not satisfied since **the separation point is not rational**.
The axiom of completeness is not valid for natural numbers $\mathbb{N}$, integer numbers $\mathbb{Z}$, rational numbers $\mathbb{Q}$.

In summary



Real $\mathbb{R}$
$\ldots -\sqrt{3}, -\frac{3}{2}, -\sqrt{2}, -1, -\frac{3}{4}, -\frac{1}{2}, 0, \frac{1}{3}, 1, \sqrt{2}, 2, 3, \pi, \ldots$
Rational $\mathbb{Q}$
$\ldots -\frac{3}{2}, -1, -\frac{7}{8}, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{3}, 0, \frac{1}{4}, \frac{1}{2}, 1, \frac{5}{4}, \ldots.$
Integers $\mathbb{Z}$
$\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots$
Natural including zero (Whole)
$\mathbb{N}_0$
$0, 1, 2, 3, 4, \ldots$
Natural $\mathbb{N}$
$1, 2, 3, 4, \ldots$

Extensional and intensional definitions

Observation

A set may contain infinite objects, so it may be complicated listing all of them. For this reason, sometimes we use *intensional* definitions instead of *extensional* definitions

Extensional definition

All objects are explicitly listed:

- $A = \{\triangle, \bigcirc, \square\}$
- $B = \{1, 7, 99, 1.23, -2\}$
- $C = \{\text{New York}, \text{London}, \text{Sydney}\}$

Intensional definition

We state the property which *unambiguously* defines the objects in the set:

- $A = \{\text{All cities in Europe}\}$
- $B = \{\text{All natural numbers larger than 100}\}$
- $C = \{\text{All people in this class whose surname has 'A' as a first letter}\}$

The $\in$, $\notin$ quantifiers

To say that the object a is contained in the set A, we write $a \in A$

To say that the object a is *not* contained in the set A, we write $a \notin A$

Examples

- If $A = \{\triangle, \bigcirc, \square\}$, then $\triangle \in A$, $\bigcirc \in A$, $\square \in A$ but $\diamond \notin A$
- If $A = \{1, 7, 99, 1.23, -2\}$, then $1 \in A$, $7 \in A$, but $5 \notin A$
- If $A = \{\text{All cities in Europe}\}$, then Paris $\in A$, but New York $\notin A$
- If $A = \{\text{All natural numbers larger than } 100\}$, then $101 \in A$, but $100 \notin A$

Remark: The $\in$ symbol is often used in intensional definitions:

$$A = \{\text{All cities in Europe}\}, \quad B = \{x \in A \mid x \text{ is a capital}\}$$

The set B includes the European cities that are capitals, so that **Milan** $\in A$, but **Milan** $\notin B$.

The ∀ quantifier

To say that a certain property holds for **ALL** objects in A we write $\forall a \in A$

Examples

- If $A = \{0, 0.5, 44.5, 2\}$, then we can write $\forall a \in A$, $a \geq 0$
- If $A = \{1, 77, 31, 5\}$, then we can write $\forall a \in A$, a is odd
- If $A = \{0.2, 0.5, 0.7\}$, then we can write $\forall a \in A$, $0 < a < 1$

The $\exists$, $\nexists$ quantifiers

To say that a certain property holds for **AT LEAST ONE** object in A we write $\exists a \in A$

To say that a certain property holds for **NO** objects in A we write $\nexists a \in A$

Examples

- If $A = \{0, 0.5, 44.5, 2\}$, then we can write $\exists a \in A$ such that $a \leq \frac{1}{2}$. Indeed $0 \leq \frac{1}{2}$ and $0.5 \leq \frac{1}{2}$
- If $A = \{0, 0.5, 44.5, 2\}$, then we can write $\exists a \in A$ such that $0 < a \leq \frac{1}{2}$. Indeed $0 < 0.5 \leq \frac{1}{2}$
- If $A = \{0, 0.5, 44.5, 2\}$, then we can write $\nexists a \in A$ such that $a < 0$. Indeed all objects in A are larger or equal to zero

Remark: In the second example, there exists **ONLY ONE** element in A such that $0 < a \leq \frac{1}{2}$. In this case we can write $\exists! a \in A$. Writing $\exists a \in A$ is correct as well, however $\exists! a \in A$ is more informative

Summary of quantifiers

- $a \in A$ means "the element a belongs to the set A"
- $\forall a \in A$ means "for all elements a in the set A"
- $\exists a \in A$ means "there exists at least one element a in the set A"
- $\nexists a \in A$ means "there exists no element in the set A"
- $\exists! a \in A$ means "there exists a unique element a in the set A"

Union of sets: the $\cup$ operator

Definition

Let A and B be two sets. We denote by $A \cup B$ the set containing all the elements of A and all the elements of B. In symbols:

$$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$$

Examples

- $A = \{\triangle, \bigcirc, \square\}$, $B = \{\triangle, \diamond, \square\} \Rightarrow A \cup B = \{\triangle, \bigcirc, \diamond, \square\}$
- $A = \{\text{Milan}, \text{Rome}\}$, $B = \{\text{Sydney}\} \Rightarrow A \cup B = \{\text{Milan}, \text{Rome}, \text{Sydney}\}$
- $A = \{1, 100, 4.4\}$, $B = \{4.4\} \Rightarrow A \cup B = \{1, 100, 4.4\}$
- $A = \{1, 3, -5\}$, $B = \{1, 2\}$, $C = \{1, 55\} \Rightarrow A \cup B \cup C = \{1, 3, -5, 2, 55\}$

Intersection of sets: the $\cap$ operator

Definition

Let A and B be two sets. We denote by $A \cap B$ the set containing the elements in common between A and B. In symbols:

$$A \cap B = \{x : x \in A \ \text{ and } \ x \in B\}$$

Examples

- $A = \{\triangle, \bigcirc, \square\}$, $B = \{\triangle, \diamond, \square\} \Rightarrow A \cap B = \{\triangle, \square\}$
- $A = \{\text{Milan}, \text{Rome}\}$, $B = \{\text{Sydney}\} \Rightarrow A \cap B = \emptyset$
- $A = \{1, 100, 4.4\}$, $B = \{4.4\} \Rightarrow A \cap B = \{4.4\}$
- $A = \{1, 3, -5\}$, $B = \{1, 2\}$, $C = \{1, 55\} \Rightarrow A \cap B \cap C = \{1\}$

The empty set

Definition

An empty set is a set with no elements, and is denoted by $\emptyset$

Remark

For any set A, $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$

Subsets

Let A and B be two sets. We write $A \subseteq B$ if all the elements of A are also elements of B.

Examples

- $A = \{1, 100\}$, $B = \{1, 100, 4.4\} \Rightarrow A \subseteq B$
- $A = \{\text{All cities in Europe}\}$, $B = \{\text{Paris}, \text{Milan}\} \Rightarrow B \subseteq A$
- $A = \{\triangle, \bigcirc, \square\}$, $B = \{\triangle, \bigcirc, \square\} \Rightarrow A \subseteq B$ and $B \subseteq A$

Let A and B be two sets. We write $A \subset B$ if all the elements of A are also elements of B **and we known that some elements of B are not in A**

Examples

- $A = \{1, 100\}$, $B = \{1, 100, 4.4\} \Rightarrow A \subset B$
- $A = \{\text{All cities in Europe}\}$, $B = \{\text{Paris}, \text{Milan}\} \Rightarrow B \subset A$
- $A = \{\triangle, \bigcirc, \square\}$, $B = \{\triangle, \bigcirc, \square\}$. This time we CANNOT write $A \subset B$ because B has no elements which are not in A. For the same reason, we CANNOT write $B \subset A$

Subsets, cont'd

Remark

- $A = \{1, 100\}$, $B = \{1, 100, 4.4\}$
- $A = \{\text{All cities in Europe}\}$, $B = \{\text{Paris}, \text{Milan}\}$

In the first example, it is correct writing either $A \subseteq B$ or $A \subset B$. However, $A \subset B$ is more informative, because it says us not only that all elements of A are in B, but also that B has some elements which are not in A. Similarly, in the second example, it is correct writing either $B \subseteq A$ or $B \subset A$, but the second is more informative.

The two relations $A \subseteq B$, $A \subset B$, can also be read from right to left:

- We write $B \supseteq A$ if $A \subseteq B$
- We write $B \supset A$ if $A \subset B$

Subtraction between sets

Definition

Let A and B be two sets. The “A minus B” set, denoted by $A \backslash B$, is the set containing the elements in A which are not in B. In symbols:

$$A \backslash B = \{x \in A : x \notin B\}$$

Examples

- $A = \{1, 100, 4\}$, $B = \{1\} \Rightarrow A \backslash B = \{100, 4\}$
- $A = \{1, 100, 4\}$, $B = \{2, 3\} \Rightarrow A \backslash B = \{1, 100, 4\}$
- $A = \{\triangle, \diamondsuit, \square\}$, $B = \{\triangle, \bigcirc\} \Rightarrow A \backslash B = \{\diamondsuit\, \square\}$
- $A = \{\text{All natural numbers}\}$, $B = \{\text{All natural numbers larger than } 10\}$

$$A \backslash B = \{\text{All natural numbers lower or equal to } 10\} = \{1, \ldots, 10\}$$

The complement set

Definition

Let S be the universal set and B a subset of S. The complement set of B is the "S minus B" set, namely the set of elements of S that are not contained in B. In symbols:

$$B^c = S \backslash B = \{x \in S : x \notin B\}$$

Example

- $S = \{1, 100, 4\}$, $B = \{1\} \Rightarrow B^c = \{100, 4\}$
- $S = \{\text{All cities in Europe}\}$, $B = \{x \in S | x \text{ is a capital city}\}$

$$B^c = \{x \in S | x \text{ is } \textit{not} \text{ a capital city}\}$$

Some subsets in $\mathbb{R}$: The intervals

Definition

A real interval with extremes $a, b \in \mathbb{R}$ such that $a \leq b$, is the set of all real numbers between a and b.
We say that a real interval is

- **open** if extremes a and b are not included and we denote it by (a, b)
- **closed** if extremes a and b are included and we denote it by $[a, b]$
- **not open nor closed** one of the extreme is included and the other is not, that is $[a, b)$ or $(a, b]$
- **bounded** if both a and b are finite numbers
- **unbounded** if either a, or b or both are infinite, e.g. $(-\infty, 1]$, $(-3, +\infty)$, $(-\infty, +\infty)$

Important: Numeric sets with just one element are denoted with the curly parentheses, for instance $\{2\}$ is the set that contains only the number 2.

Statements

An assertion that can be either *true* or *false* is a statement.

Consider two statements, say statements P and statement Q.

We say

$$P \Rightarrow Q$$

to mean If P holds true than also Q holds true or P implies Q.

Example:

- If it rains, then we open the umbrella
- If $x = 0$, then $x \cdot y = 0$

The arrow in $\Rightarrow$ gives the direction of the implication.

Statements

If

$$P \Rightarrow Q \text{ and } Q \Rightarrow P$$

then *assertion P and assertion Q are equivalent*. In this case we write

$$P \Leftrightarrow Q$$

to mean P if and only if Q, or P is equivalent to Q.

Example:

- If $x = 0$ and $y = 0$, then $x^2 + y^2 = 0$
- If $x^2 + y^2 = 0$, then $x = 0$ and $y = 0$

In this case assertion P ($x = 0$ and $y = 0$) and assertion Q ($x^2 + y^2 = 0$) are equivalent.

Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that ***P* is a sufficient condition for *Q***. Indeed it is sufficient that P is true to get that also Q is true.

We also say that ***Q* is a necessary condition for *P***. Indeed, if P is true then necessarily Q is true.

In summary the following are equivalent:

- $P \Rightarrow Q$
- P implies Q
- If P then Q
- P is a sufficient condition for Q
- P is a necessary condition for P

Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

We know that this expression means that both the implications $P \Rightarrow Q$ and $Q \Rightarrow P$ hold at the same time.

In this case P is a sufficient condition for Q (from the first implication) and P is also a necessary condition for Q (from the second implication). Hence we say that P is a **sufficient and necessary condition** for Q.

In summary the following are equivalent:

- $P \Leftrightarrow Q$
- P is equivalent to Q
- P if and only if Q
- P is a sufficient and necessary condition for Q

The structure of a theorem

A **theorem** is made of

- **Hypotheses**: these are the premises under which the conclusion holds
- Thesis: this is the conclusion which follows from the premises

When stating a theorem it must be clear what the hypotheses are and what the thesis is!

Example: The Pythagorean Theorem

Hypothesis: If a triangle is right-angled

Thesis: the square of the hypothenuse is equal to the sum of the squares of the other two sides

The absolute value

Definition

Let $a \in \mathbb{R}$. The absolute value of a, denoted by $|a|$, is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Examples

- $|1| = 1, \quad |-1| = 1, \quad \left|\frac{1}{2}\right| = \frac{1}{2}, \quad \left|-\frac{2}{127}\right| = \frac{2}{127}, \quad |-\pi| = \pi$

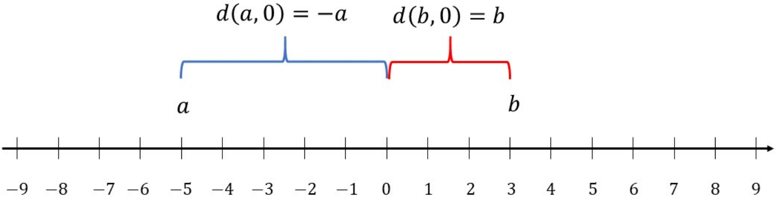
Distance between a point in $\mathbb{R}$ and the origin

Definition

Let $a \in \mathbb{R}$. The distance between a and the origin is the length of the segment line between the origin and point a, hence it corresponds to a if $a > 0$ and $-a$ if $a < 0$. Put in other words

$$d(a, 0) = |a|$$

Notice that the length must be positive (or equal to zero).



Distance between two points in $\mathbb{R}$

Definition

Let $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. The distance between x_1 and x_2 is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

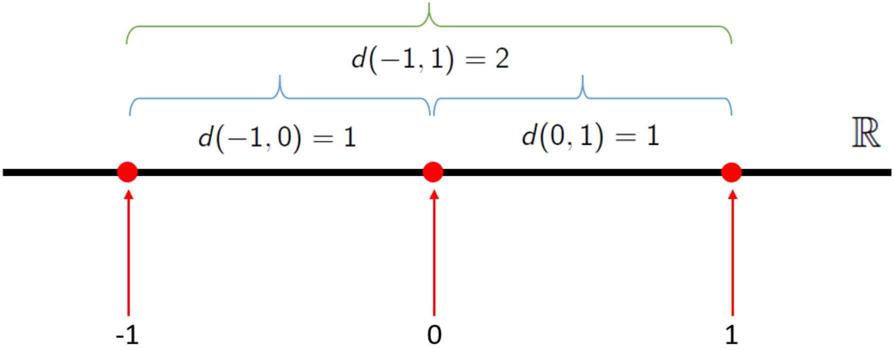
Note that, by definition:

- $d(x_1, x_2) \geq 0, \quad \forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$ if and only if x_1 and x_2 are the same point
- $d(x_1, x_2) = d(x_2, x_1)$, since $|x_1 - x_2| = |x_2 - x_1|$ (that is the order does not matter when we measure distances).

Examples

- $d(1,2) = |1-2| = 1, \quad d(2,1) = |2-1| = 1$
- $d(1,-1) = |1-(-1)| = 2, \quad d(-1,1) = |-1-1| = 2$
- $d\left(0,-\frac{1}{2}\right) = \left|0-\left(-\frac{1}{2}\right)\right| = \frac{1}{2}, \quad d\left(-\frac{1}{2},0\right) = \left|-\frac{1}{2}-0\right| = \frac{1}{2}$

Distance between two points in $\mathbb{R}$: example



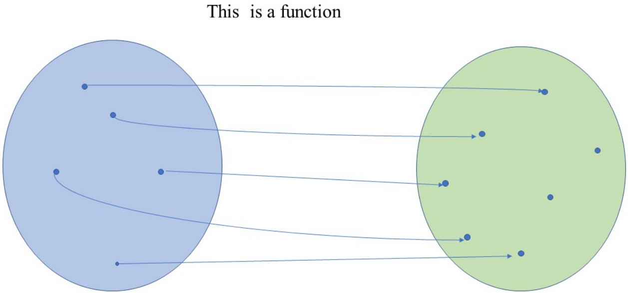
Functions

Definition (General definition)

Let A and B be two sets. A function f defined on A and with values in B is a law that associates to any element $x \in A$ **one and only one** element $y \in B$

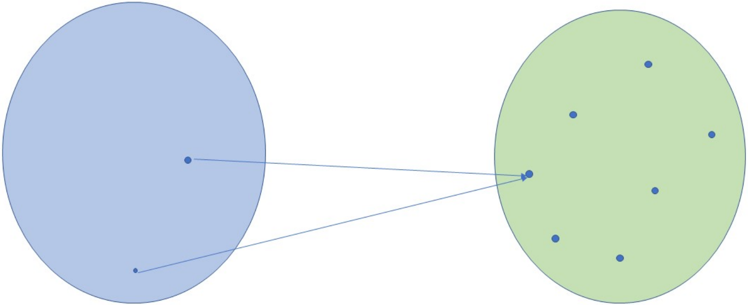
To indicate a function we use the notation

$$f : A \to B \quad \text{or} \quad y = f(x).$$

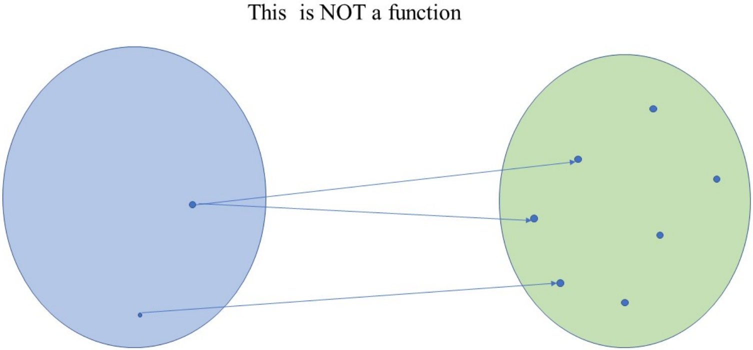


Functions

This is a function



Functions



Functions

Definition (Real function of a Real variable)

Let $D \subseteq \mathbb{R}$ (eventually $D = \mathbb{R}$). A real function of a real variable is a law that associates to any real number $x \in D$ **one and only one** real number $y \in \mathbb{R}$ such that $y = f(x)$.

The variable x is called the **independent variable**, the variable y is called the **dependent variable**. In Economics x is also called the exogenous variable and y the endogenous variable.

y is called **the image of x through the function f**

Functions

Definition (Domain and Range)

The set D of all real values for which the law f makes sense is called the *domain* of the function f:

$$D = \{x \in \mathbb{R} : \ f(x) \text{ is well defined } \}$$

The set of all images $f(x)$, for all $x \in D$ is called the *range* of the function

$$R = \{y \in \mathbb{R} : \ y = f(x), \ \forall x \in D\}$$

Definition (Graph)

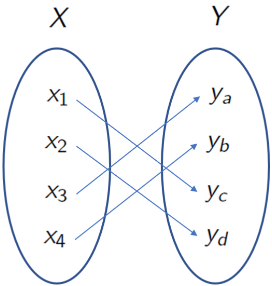
Let $f : D \to \mathbb{R}$ be a real function of a real variable. The graph of f is the set

$$G = \{(x, f(x)) : \ x \in D\}$$

The plot of the function f is the representation of the graph on a Cartesian plane.

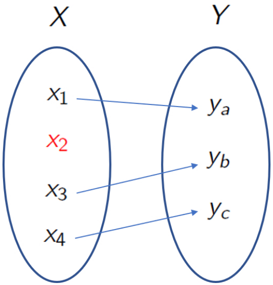
Functions: the intuition

Intuitively, a function is a rule that associates to **each** element of a set X, **only and only one** element in another set Y.

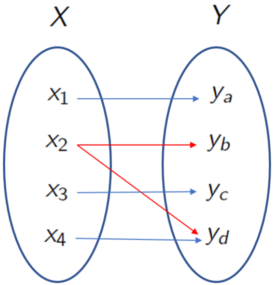


This is a function between X and Y

Functions: the intuition, cont'd



This is **not** a function between X and Y because $x_2 \in X$ is not mapped into any element in Y



This is **not** a function between X and Y because x_2 is mapped into more than one element in Y

Functions: the definition

Definition

Let $D \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. A function is a rule that associates to each element of D one and only one element of $\mathbb{R}$. In symbols we write:

$$f : D \to \mathbb{R}$$

meaning that

$$\forall x \in D \ \Rightarrow \exists!\ y \in \mathbb{R} : \ y = f(x)$$

The set D is called the **domain** of the function.

- The variable x is called "independent variable", it can take values in D
- The variable y is called the "dependent variable", it can take values in $\mathbb{R}$.
- In economics x is called the exogenous variable and y is called the endogenous variable

The domain of a function

Definition

Let $f : D \to \mathbb{R}$ be a function. The domain of the function, $D \subseteq \mathbb{R}$, is the set of all values $x \in \mathbb{R}$ for which the expression $f(x)$ makes sense.

Three cases require computations:

1. $f(x)$ contains a division
2. $f(x)$ contains a root with even power
3. $f(x)$ contains a logarithm

or any combinations of the above conditions.

Domain of rational functions

Let $f(x) = \dfrac{P(x)}{Q(x)}$, and assume that $Q(x)$ makes sense for all $x \in \mathbb{R}$.
Then the function $f(x)$ is well defined if and only if $Q(x) \neq 0$. This means that

$$D = \{x \in \mathbb{R} : \; Q(x) \neq 0\}$$

Examples

- $f(x) = \dfrac{x+3}{x^2-1}$

$$D = \{x \in \mathbb{R} : \; x \neq \pm 1\}$$

- $f(x) = e^{\frac{x+5}{x-3}}$

$$D = \{x \in \mathbb{R} : \; x \neq 3\}$$

Domain of irrational functions

Let $f(x) = \sqrt[n]{G(x)}$, and assume that $G(x)$ makes sense for all $x \in \mathbb{R}$. There are two possibilities:

- if n is even, then the function $f(x)$ is well defined if and only if $G(x) \geq 0$. This means that

$$D = \{x \in \mathbb{R} : \; G(x) \geq 0\}$$

- if n is odd, then the function $f(x)$ is well defined for all $x \in \mathbb{R}$

Examples

- $f(x) = \sqrt{x^2 - 5}$
  This is an irrational function with even index ($n = 2$). Then we have

$$D = \{x \in \mathbb{R} : \; x \leq -\sqrt{5} \text{ or } x \geq \sqrt{5}\}$$

- $f(x) = \sqrt[3]{x + 2}$
  This is an irrational function with odd index ($n = 3$). Then we have

$$D = \mathbb{R}$$

Domain of logarithmic functions

Let $f(x) = \log H(x)$, and assume that $H(x)$ makes sense for all $x \in \mathbb{R}$.
Then the function $f(x)$ is well defined if and only if $H(x) > 0$. This means that

$$D = \{x \in \mathbb{R} : \ H(x) \geq 0\}$$

Examples

- $f(x) = \log(1 - x^2)$

$$D = \{x \in \mathbb{R} : \ -1 < x < 1\}$$

- $f(x) = \log(x^2 + 2)$
  Since $x^2 + 2 > 0$ for all $x \in \mathbb{R}$ we get that

$$D = \mathbb{R}$$

Example

These three condition must be combined together if a function contains fractions, roots and logarithms.

Example

$$f(x) = \frac{x}{\log(x+2)}$$

We have that:

- $\log(x+2) \neq 0$ for the existence of the fraction
- $x + 2 > 0$ for the existence of the logarithm

Hence we have

$$\begin{cases} \log(x+2) & \neq 0 \\ x+2 & > 0 \end{cases}$$

which implies that

$$D = \{x \in \mathbb{R} : \ x > -2 \text{ and } x \neq -1\}$$