$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \ldots\} \qquad + \quad \cdot$$

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

$$+ \quad \cdot$$

$$\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; m \in \mathbb{Z},\ n \in \mathbb{Z},\ n \neq 0 \right\}$$

"IT BELONGS TO"

$$-\frac{1}{2} \qquad \frac{3}{2} \qquad \frac{5}{7}$$

$$\frac{m}{n} + \frac{k}{q} = \frac{m \cdot q + k \cdot n}{n \cdot q} \qquad m, n, k, q \in \mathbb{Z}$$

$$n \neq 0 \qquad q \neq 0$$

$$\left(-\frac{1}{2}\right) + \frac{1}{5} = \frac{-5+2}{10} = \frac{3}{10}$$

$$\frac{m}{n} \cdot \frac{k}{q} = \frac{m \cdot k}{n q}$$

$$\frac{1}{\frac{k}{q}} = \frac{q}{k} = \left(\frac{k}{q}\right)^{-1}$$

$$\frac{1}{\frac{1}{3}} = 3 = \left(\frac{1}{3}\right)^{-1} = \frac{1}{3^{-1}}$$

$$\left(\frac{q}{n}\right)^k = \underbrace{\frac{q}{n} \cdot \ldots \cdot \frac{q}{n}}_{k\text{-TIMES}} = \frac{q^k}{n^k}$$

$$\left(\frac{7}{2}\right)^3 = \frac{7}{2} \cdot \frac{7}{2} \cdot \frac{7}{2} = \frac{7^3}{2^3}$$

$$\left(\frac{7}{2}\right)^{-3} = \left(\frac{2}{7}\right)^3 = \frac{2^3}{7^3}$$

$q \in \mathbb{Q} \qquad q \neq 0 \qquad k - k = 0$

$$q^0 = q^{k-k} = q^k \cdot q^{-k} = q^k \frac{1}{q^k} = 1$$

$$q^{(n+m)} = q^n \cdot q^m$$

$q \in \mathbb{Q} \qquad q \neq 0 \qquad \frac{1}{q} \qquad 0 = k - k$

$$q^0 = q^{k-k} = q^k \cdot q^{-k} = \cancel{q^k} \frac{1}{\cancel{q^k}} = 1$$

0^0 NOT DEFINED !

$\frac{3}{10} = 0.\underset{\substack{\uparrow \\ \text{1 DIGIT}}}{3}$ $\frac{5}{2} = 2.\underset{\substack{\uparrow \\ \text{1 DIGIT}}}{5}$

$\frac{1}{3} = 0.\underbrace{3333333}_{\substack{\text{INFINITE NUMBER OF DIGITS} \\ \text{BUT THEY ARE ALL THE SAME}}}\ldots\ldots$

$\frac{1}{22} = 0.4545454\ldots\ldots$

THM: IF $q \in \mathbb{Q}$ IS A RATIONAL NUMBER THERE ARE ONLY TWO POSSIBILITIES

1) q HAS A FINITE NUMBER OF DIGITS

2) q HAS AN INFINITE NUMBER OF DIGITS BUT THEY ARE PERIODIC

$\frac{9}{11} = 0,8181\ldots = 0,\overline{81}$

$\frac{1}{3} = 0,333\ldots = 0,\overline{3}$

$x^2 = 2$

$\nexists\, x \in \mathbb{Q} \;:\; x^2 = 2$

"IT DOES NOT EXIST" "SUCH THAT"

IT IS POSSIBLE TO EXTEND $\mathbb{Q}$ TO A SET OF NUMBERS $\mathbb{R}$ SUCH THAT THE PROBLEM $x^2=2$ HAS A SOLUTION IN $\mathbb{R}$

$\mathbb{N} = \{1,2,3,\ldots\} \subset \mathbb{N}_0 = \{0,1,2,\ldots\} \subset \mathbb{Z} = \{\ldots,-1,0,1,\ldots\} \subset$

"THE SET IS INCLUDED IN"

$\subset \mathbb{Q} \subset \mathbb{R}$

$\pi = 3{,}1459 \ldots\ldots -$

$e = 2{,}71 \ldots\ldots$

$\sqrt{2} \quad \sqrt{5} \quad \sqrt{7}$

$(\mathbb{R}, +, \cdot) \qquad a, b, c \in \mathbb{R}$

1) ASSOCIATIVE PROPERTY

$(a+b)+c = a+(b+c)$

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

2) COMMUTATIVE PROPERTY

$a+b = b+a$

$a \cdot b = b \cdot a$

3) DISTRIBUTIVE PROPERTY:

$a \cdot (b+c) = a \cdot b + a \cdot c$

4) EXISTENCE OF THE OPPOSITE.

$\boxed{+} \quad a \in \mathbb{R} \quad \exists (-a) : a + (-a) = 0$

OPPOSITE

5) EXISTENCE OF THE NEUTRAL ELEMENT

$\boxed{\cdot} \quad a \cdot 1 = a$

$\boxed{+} \quad a + 0 = a$

$\mathbb{Q}$

COMPLETENESS

$A \subseteq \mathbb{R}, \quad B \subseteq \mathbb{R} \qquad A \neq \emptyset \quad B \neq \emptyset$

EMPTY SET

SET WITHOUT ELEMENTS

A IS A SUBSET OF $\mathbb{R}$

AND IT MIGHT BE EQUAL TO $\mathbb{R}$

$A \subset \mathbb{R}$

ASSUME THAT

$\forall \; x \in A \implies x \le y \;\; \forall \; y \in B$

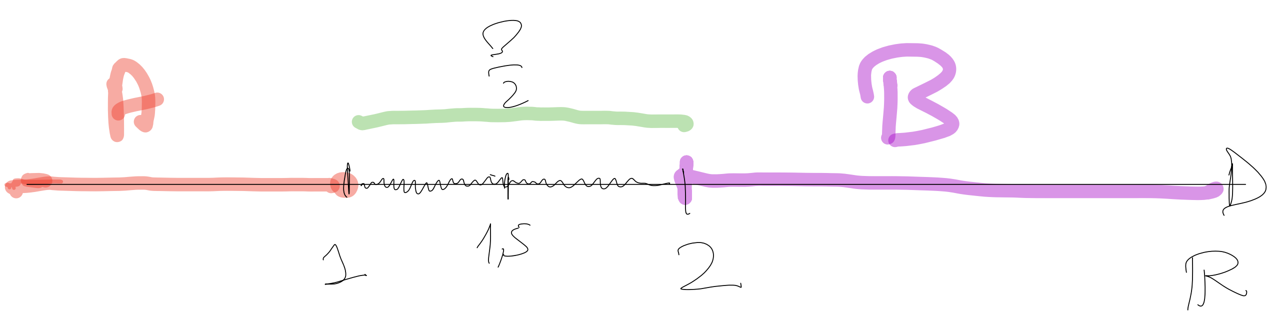
"FOR ALL" "IT HOLDS THAT"

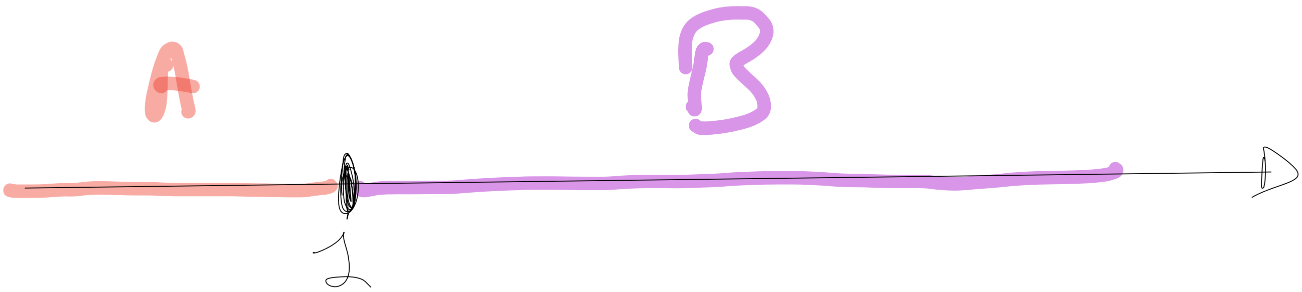
ALL THE ELEMENTS OF THE SET A ARE SMALLER OR EQUAL THAN ALL THE ELEMENTS OF THE SET B

$\exists \; c \in \mathbb{R} : \quad a \le c \le b \;\; \forall \; a \in A$
$\forall \; b \in B$

"IT EXISTS" "SUCH THAT"

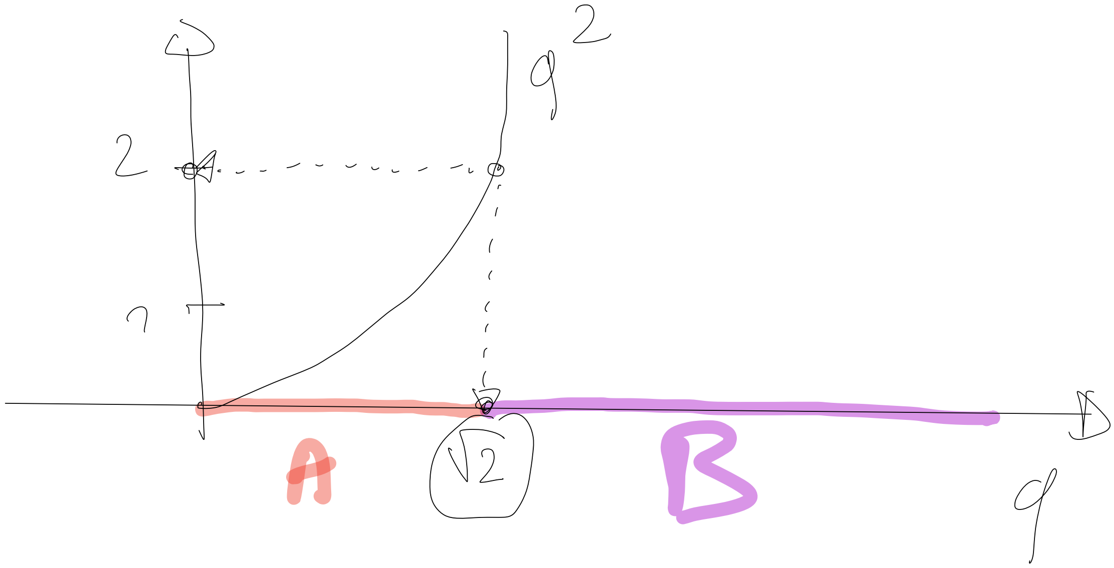
c IS CALLED A SEPARATION POINT





$$A = \{ q \in \mathbb{Q} \mid q \geq 0,\ q^2 < 2 \}$$

$$B = \{ q \in \mathbb{Q} \mid q \geq 0,\ q^2 \geq 2 \}$$



EXTENSIVE DECLARATION: TO DEFINE A SET

A SIMPLY LIST ALL THE ELEMENTS OF THE SET

$$A = \{\text{PARIS}, \text{LONDON}, \text{MILAN}\} = \{\text{PARIS}, \text{LONDON}, \text{MILAN}\}$$

$$A = \{-1, 0, 1\} \qquad A = \{5\}$$

INTENSIVE DECLARATION: THE ELEMENTS OF THE SET

ARE DEFINED ACCORDING TO A COMMON PROPERTY

$$A = \{\text{ALL CITIES OF EUROPE}\}$$

$$B = \{\text{ALL CITIES OF EUROPE} \mid \text{CAPITALS}\} \subset A$$

$$A = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\} =$$

$$= \{\text{THE SET OF ALL RATIONAL NUMBERS BETWEEN 0 AND 1}\}$$

$$\subset \mathbb{Q}$$

WE WRITE $a \in A$ TO SAY THAT THE ELEMENT a IS IN THE SET A

WE WRITE $a \notin A$ TO SAY THE OPPOSITE

THAT IS THE ELEMENT a DOES NOT BELONG TO A

$A = \{\text{ALL CITIES IN EUROPE}\}$

$B = \{x \in A \mid x \text{ IS A CAPITAL}\}$

WE WRITE $\forall a \in A$ TO STATE THAT A PROPERTY HOLDS FOR ALL THE ELEMENTS OF THE SET A

1) $A = \{0, \pi, 10, \sqrt{37}\}$

$\forall a \in A \Rightarrow a \geq 0$

2) $A = \{2, 4, 10, 20, 1000\}$

$\forall a \in A \Rightarrow \frac{a}{2} \in \mathbb{N}$

(a IS EVEN)

WE WRITE $\exists a \in A$ TO SAY THAT A CERTAIN PROPERTY HOLDS FOR AT LEAST ONE ELEMENT OF A

$A = \{0, \pi, 10, \sqrt{37}\}$

$\exists a \in A : a \in \mathbb{R}$

$A = \{-3, -1, 0, 1, 2\}$

$\exists a \in A : a \geq 0$

IF THE PROPERTY HOLDS FOR ONLY ONE ELEMENT WE WRITE $\exists! a \in A$

$A = \{0, \pi, 10\}$

$\exists! a \in A : a \in \mathbb{R}$ AND $a \notin \mathbb{Q}$

WE WRITE $\nexists a \in A$ TO SAY THAT NONE OF THE ELEMENTS OF THE SET A VERIFY THE PROPERTY

$A = \{0, \pi, 10\}$ $\nexists a \in A : a < 0$

$\in \quad \notin \quad \forall \quad \exists \quad \exists! \quad \nexists$

LET A AND B BE TWO GENERIC SETS

$A \cup B$ IS CALLED THE UNION SET BETWEEN A AND B AND IT IS DEFINED AS

$$A \cup B \equiv \{x \mid x \in A \text{ OR } x \in B\}$$

$$A = \{-1, 0, 1\} \qquad B = \{2\}$$

$$A \cup B = \{-1, 0, 1, 2\}$$

$$A \cap B = \{x \mid x \in A \text{ AND } x \in B\}$$

INTERSECTION SET

$\{\}$

$$A = \{-1, 0, 1\} \qquad B = \{2\}$$

$$A \cap B = \emptyset = \{\}$$

$\emptyset$ = IS THE SET WITH NO ELEMENTS

WE SAY $A \subseteq B$ IF ALL THE ELEMENTS OF A ARE ALSO ELEMENTS OF B

A IS CALLED A SUBSET OF B

$$\mathbb{Q} \subseteq \mathbb{R} \qquad A \subseteq B \quad \text{"} 1 \leq 2 \text{"} \qquad A \subset B \quad \text{"} 1 < 2 \text{"}$$

WE SAY $A \subset B$ IF $A \subseteq B$ AND

$$\exists\, b \in B : b \notin A$$

A IS CALLED A PROPER SUBSET OF B

IF A AND B ARE SETS THEN

$$A \setminus B = \{x \in A \mid x \notin B\}$$

$$A = \{-1, 0, 1\} \qquad B = \{0\}$$

$$A \setminus B = \{-1, 1\}$$

$$A = \{1, 2, 3\} \qquad B = \{-1, -2, -3\}$$

$$A \setminus B = A$$

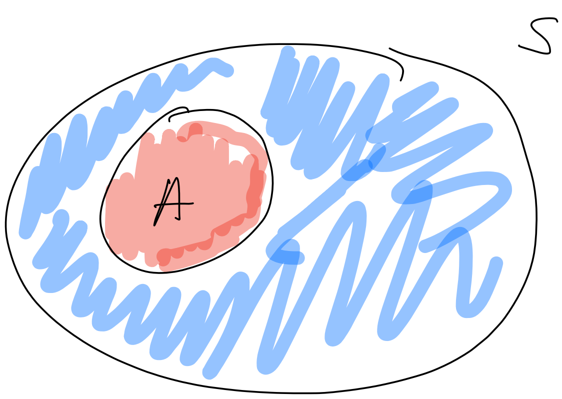
$$B \setminus A = B$$

$$C \cap \emptyset = \emptyset$$

$$C \setminus \emptyset = C$$

LET $A \subseteq S$. S IS SOMETIMES CALLED THE PARENT S OR THE UNIVERSE SET

$$A^c = S \setminus A$$



$$A \cup A^c = S$$

$\forall x \in \mathbb{R}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

$|-2| = 2 \qquad |3| = 3$

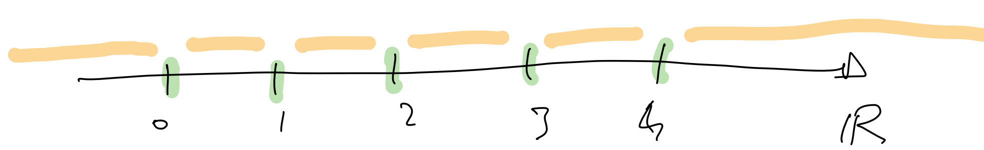
$$\sqrt{x^2} = |x|$$

$\mathbb{N}_0 \subseteq \mathbb{Z} \qquad \mathbb{N}_0^c = \{\ldots, -3, -2, -1\}$

$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$

$\mathbb{Z} = \{\ldots, -2, -1, \cancel{0, 1, 2, \ldots}\}$

$\mathbb{N}_0 \subseteq \mathbb{R}$



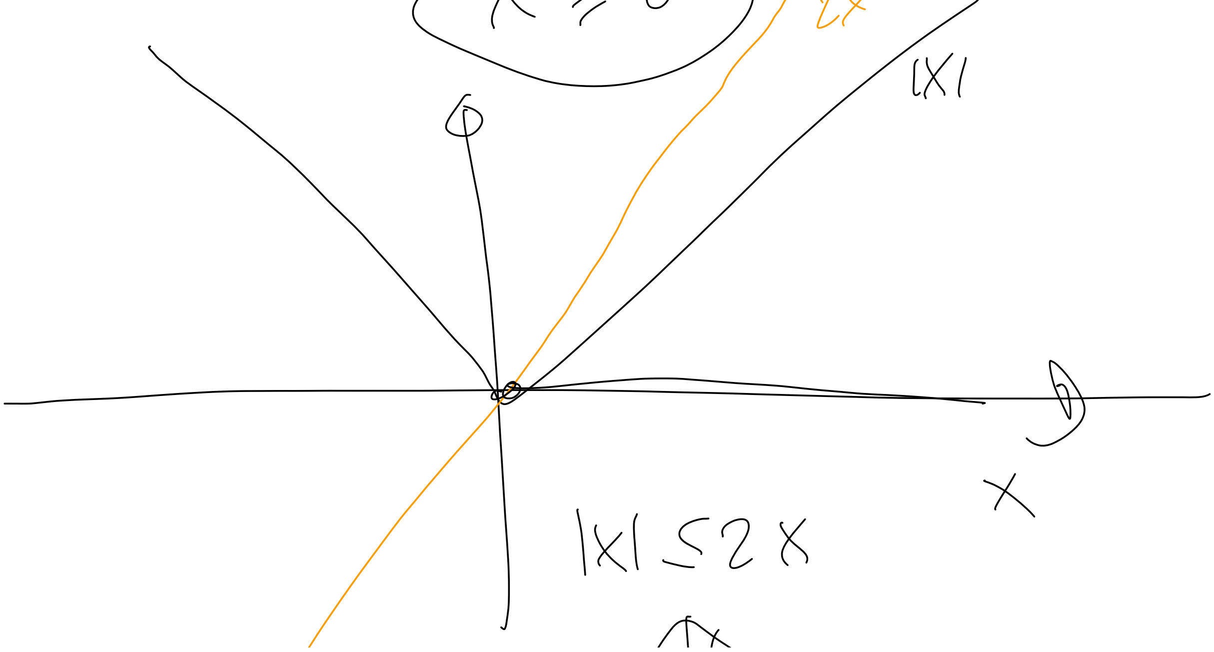
$$|x| - 2x \leq 0$$

$x \geq 0 \Rightarrow$

$$x - 2x \leq 0$$
$$-x \leq 0$$
$$x \geq 0$$

$x < 0$

$$-x - 2x \leq 0$$
$$-3x \leq 0$$
$$3x \geq 0$$
$$x \geq 0$$

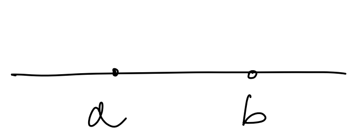


$$|x| \leq 2x$$

$x \geq 0$

INTERVALS ARE SPECIFIC SUBSETS OF THE REAL LINE $\mathbb{R}$

LET $a, b \in \mathbb{R}$: $a < b$



1) $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

CLOSED INTERVAL

2) $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

$a \notin (a, b)$
$b \notin (a, b)$

OPEN INTERVAL

3) $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

THIS SET IS NEITHER OPEN NOR CLOSED

4) $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

THIS SET IS NEITHER OPEN NOR CLOSED

IN CASE 1), 2), 3) AND 4) IF

a AND b ARE FINITE $\quad a \neq -\infty$
$b \neq +\infty$

THE INTERVAL IS CALLED

A BOUNDED INTERVAL

IN ANY OTHER CASE

$(a = -\infty$ OR $b = +\infty)$ IS

CALLED UNBOUNDED.

$$[-\infty, 1) \qquad (\sqrt{2}, +\infty)$$
$$(-\infty, 1) \qquad [\sqrt{2}, +\infty)$$
$$[\sqrt{2}, +\infty]$$

SUPER BRIEF INTRODUCTION TO

LOGIC

WE INDICATE STATEMENTS WITH
LATIN CAPITAL LETTERS

$$P, Q, R, \ldots\ldots$$

P = " THE NUMBER X IS EVEN "

GIVEN TWO STATEMENTS P AND Q

WE SAY THAT

$$P \Rightarrow Q$$

IF ASSUMING P TO BE TRUE IMPLIES

THAT Q IS TRUE

WE READ "P IMPLIES Q"

IN THIS CASE

1) P IS CALLED A SUFFICIENT CONDITION FOR Q

2) Q IS CALLED A NECESSARY CONDITION FOR P

$P \Rightarrow Q$ 1) P IMPLIES Q

2) IF P THEN Q

$P =$ " THE NUMBER n IS EVEN "

$Q =$ " THE NUMBER $n+1$ IS ODD "

$$P \Rightarrow Q$$

$P =$ " f IS DIFFERENTIABLE IN x_0 "

$Q =$ " f IS CONTINUOUS IN x_0 "

$P =$ " x IS A POSITIVE REAL NUMBER "

$Q =$ " THE $\sqrt{x}$ EXISTS "

$$P \Rightarrow Q$$

SUPPOSE THAT $P \Rightarrow Q$ AND $Q \Rightarrow P$

THEN P AND Q ARE CALLED EQUIVALENT AND P IS A SUFFICIENT AND -ESSARY CONDITION FOR Q AND VICE VERSA

$P \iff Q$ P IF AND ONLY IF Q

Q IF AND ONLY IF P

$P =$ " $x=0$ AND $y=0$ "

$Q =$ " $x^2+y^2=0$ "

$$\begin{cases} P \Rightarrow Q \\ Q \Rightarrow P \end{cases} \qquad P \iff Q$$

" $x=0$ AND $y=0$ " IF AND ONLY IF " $x^2+y^2=0$ "

FUNCTIONS

LET A AND B BE TWO GENERIC SETS.

A FUNCTION FROM A TO B IS ANY LAW/CORRESPONDENCE THAT ASSOCIATE TO ANY $a \in A$ A

UNIQUE

ELEMENT OF B

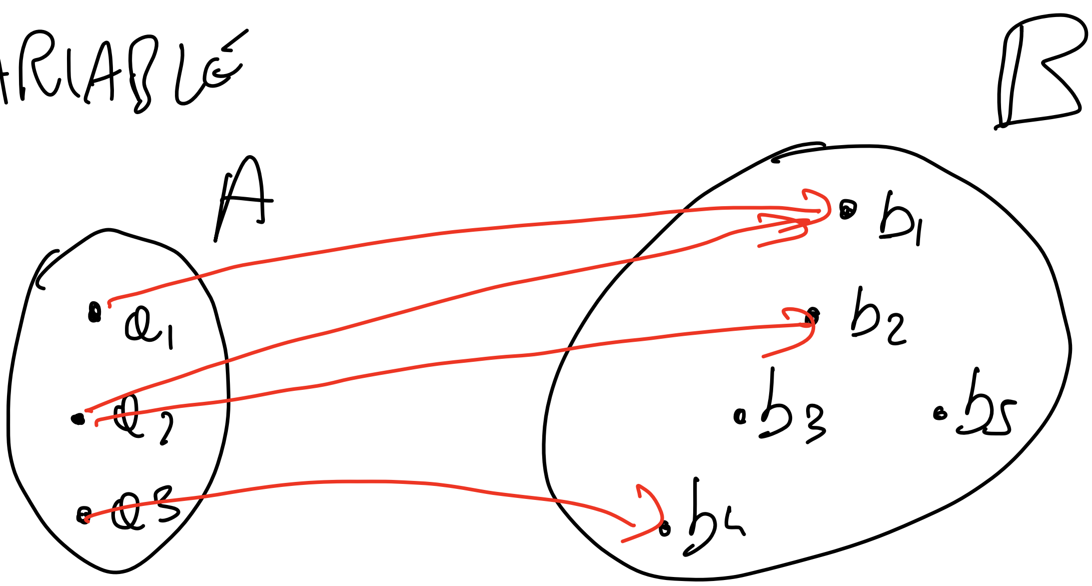
$$f : A \longrightarrow B$$

$$a \in A \longrightarrow f(a) \in B$$

$f(a)$ IS CALLED THE IMAGE OF a THROUGH THE FUNCTION f

IF $A \subseteq \mathbb{R}$ AND $B \subseteq \mathbb{R}$

THE FUNCTION IS CALLED A REAL FUNCTION OF A REAL VARIABLE



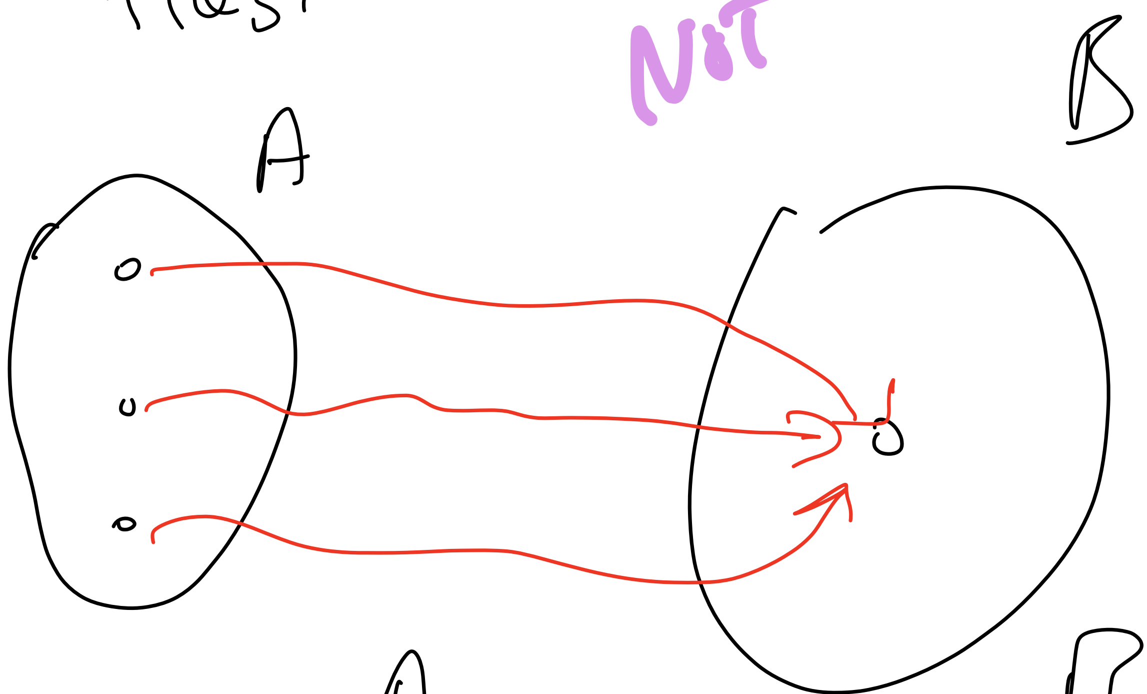
TION!

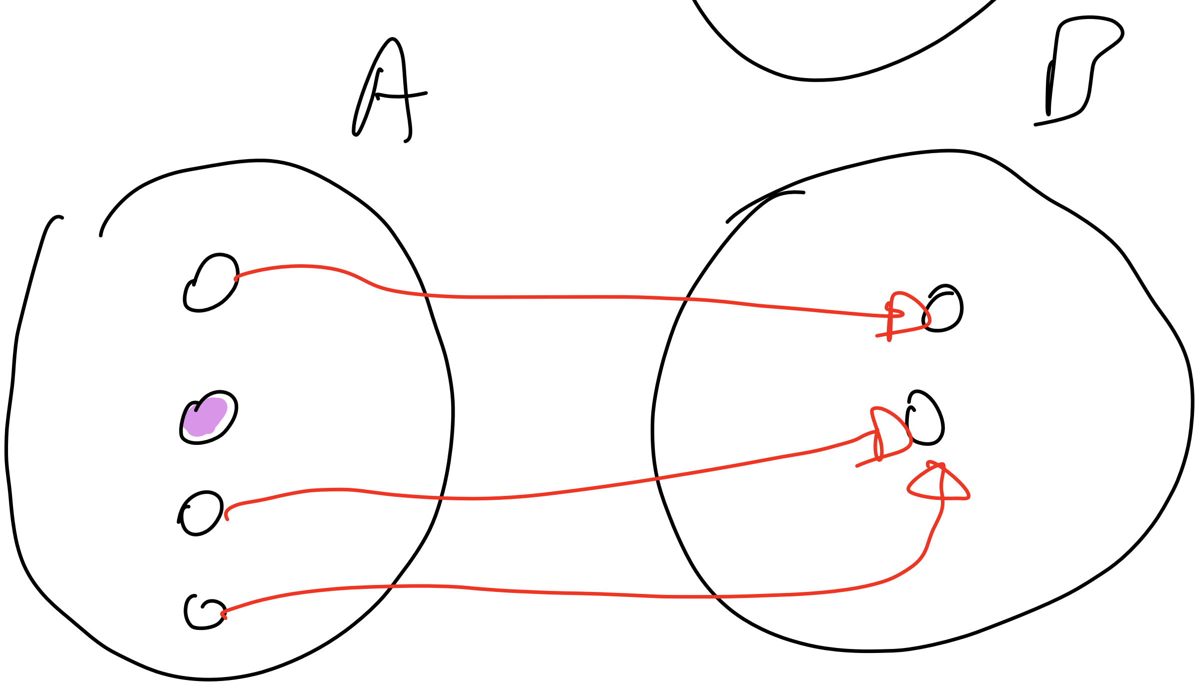
$f(a_1) = b_1$

$f(a_2) = b_2$ OR b_1

$f(a_3) = b_4$

NOT A FUNC





IF f IS A FUNCTON DEFINED ON $\mathbb{R}$ WE CALL

$$D = \{ x \in \mathbb{R} \mid f(x) \text{ IS WELL-DEFINED} \}$$

D = DOMAIN OF THE FUNCTION

$f(x) = \sqrt{x}$ $\quad D = \{ x \in \mathbb{R} \mid x \geq 0 \}$

$$f : D \longrightarrow \mathbb{R}$$

1) THE FUNCTION f CONTAINS A DIVISION

2) THE FUNCTION f CONTAINS

THE EVEN ROOTS

$\left(\sqrt[2]{}, \sqrt[4]{}, \sqrt[6]{}, \sqrt[8]{} \right)$

3) THE FUNCTION f CONTAINS

A LOGARITHM

$$f(x) = \frac{P(x)}{Q(x)} \qquad Q(x) \neq 0$$

$$f(x) = \frac{1}{3x-1} \qquad x \longrightarrow \frac{1}{3x-1}$$

$$f(0) = \frac{1}{3 \cdot 0 - 1} = \frac{1}{-1} = -1$$

-1 IS THE IMAGE OF 0 THROUGH f

$$D = \{ x \in \mathbb{R} \mid 3x - 1 \neq 0 \}$$

$$= \{x \in \mathbb{R} \mid x \neq \frac{1}{3}\}$$

$$= \mathbb{R} \setminus \{\frac{1}{3}\}$$

$$= [-\infty, \frac{1}{3}) \cup (\frac{1}{3}, +\infty]$$

$$f(x) = 2^{\left(\frac{1}{3x-1}\right)}$$

$$x \to \frac{1}{3x-1} \to 2^{\boxed{\frac{1}{3x-1}}}$$

$$D = \{x \in \mathbb{R} \mid 3x-1 \neq 0\} = \{x \in \mathbb{R} \mid x \neq \frac{1}{3}\}$$

$$f(x) = \frac{1}{7+x^4} \qquad D = \mathbb{R}$$

$$7 + x^4 \neq 0$$

$$7 > 0 \quad x^4 \geq 0 \Rightarrow 7 + x^4 > 0$$

$$\sqrt[2]{f(x)} \quad \sqrt[4]{f(x)} \quad \sqrt[6]{f(x)} \quad \cdots \sqrt[2n]{f(x)}$$

$$n \in \mathbb{N}$$

$$f(x) = (x^2 - 1)^{\frac{1}{4}} = \sqrt[4]{x^2 - 1}$$

$$x^2 - 1 \geq 0 \Rightarrow x \leq -1 \text{ OR } x \geq +1$$

$$D = \{x \in \mathbb{R} \mid x \leq -1 \text{ OR } x \geq +1\}$$

$$= [-\infty, -1] \cup [+1, +\infty]$$

$$f(x) = (x^2 - 1)^{\frac{1}{3}} \quad D = \mathbb{R}$$

$$\mathbb{R} \rightsquigarrow \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$$

$+$

$\circ$

$\log(f(x))$ IS WELL-DEFINED

IF AND ONLY IF $f(x) > 0$

$$g(x) = \log(1 + x^2)$$

$1 + x^2 > 0 \qquad D = \mathbb{R}$

$$g(x) = \log\left(\frac{1}{1+x}\right)$$

$1 + x \neq 0 \Rightarrow x \neq -1$

$$\frac{1}{1+x} > 0 \iff x > -1$$

$$D = (1, +\infty) = \{x \in \mathbb{R} \mid x > -1\}$$

$$\frac{1}{x^2-1}$$

$$x^2 - 1 \neq 0 \iff x^2 \neq 1$$

$$\iff x \neq \pm 1$$

$$f(x) = \frac{\sqrt{x^2-1}}{\log(x-2)}$$

$x^2 - 1 \geq 0 \Rightarrow x \leq -1$ OR $x \geq +1$

$x - 2 > 0 \Rightarrow x > 2$

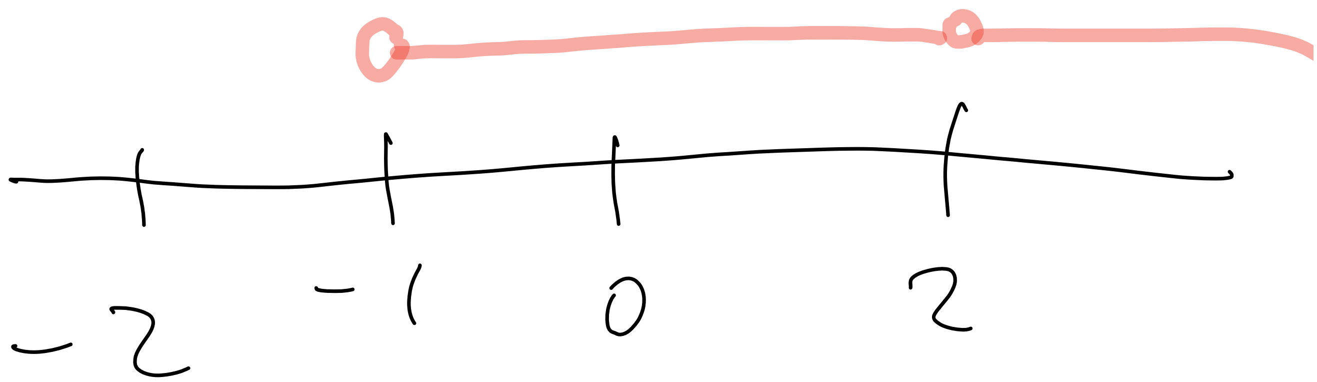
$$x - 2 = 1 \Rightarrow \log(1) = 0$$

$$x - 2 > 1 \Rightarrow x > 3$$

$$f(x) = \frac{\log(x+1)}{x^2 - 4}$$

$$x + 1 > 0 \Rightarrow x > -1$$

$$x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$$



- $D \subseteq \mathbb{R}$
- $x \in D \longrightarrow f(x) \qquad (f, D)$

RANGE OF A FUNCTION
IMAGE OF A FUNCTION

$$R_f = \{ y \in \mathbb{R} \mid \exists\, x \in D : y = f(x) \}$$

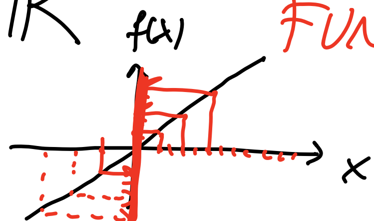
THE COLLECTION (THE SUBSET) OF ALL REAL NUMBERS THAT ARE IMAGES OF ELEMENTS IN THE DOMAIN OF THE FUNCTION

$$R_f = \{ f(x) \mid x \in D \}$$

1) $\boxed{f(x) = x}$ $\longrightarrow$ IDENTITY FUNCTION

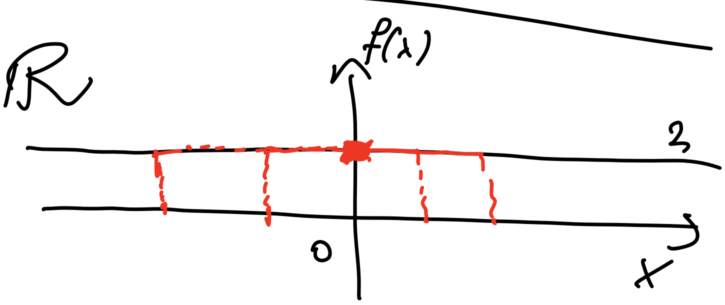
$D = \mathbb{R}$

$R_f = \mathbb{R}$



2) $f(x) = 2 \quad \forall\, x \in \mathbb{R}$

$R_f = \{2\}$



3) $f(x) = \frac{1}{x} = x^{-1}$ $\quad D = \mathbb{R} \setminus \{0\}$

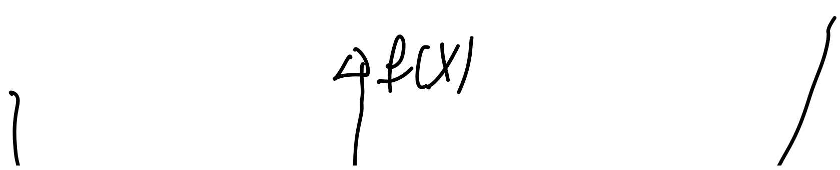
$R_f = \mathbb{R} \setminus \{0\}$ $\quad \forall y \in \mathbb{R} \setminus \{0\} \; \exists \, x : y = \frac{1}{x}$

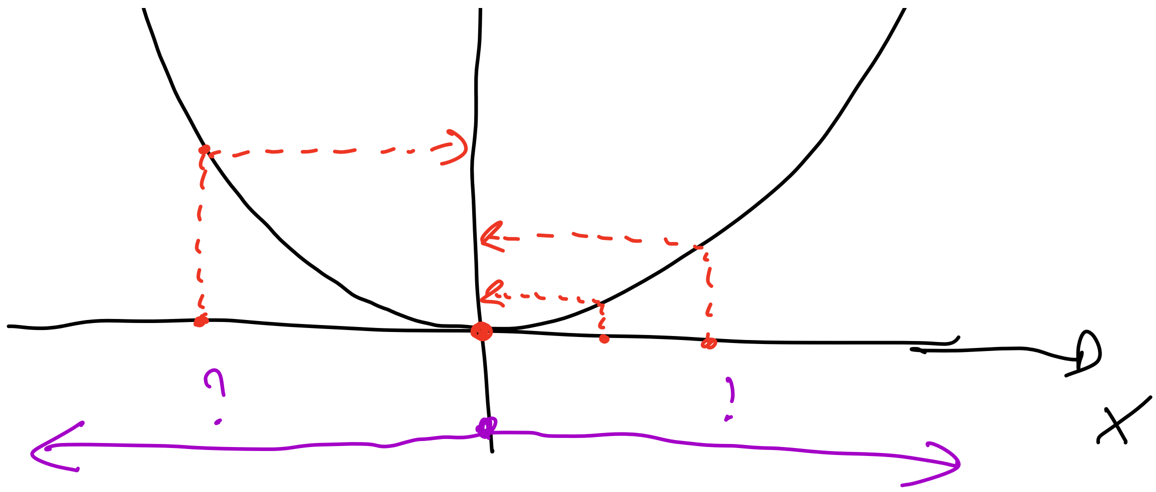
$\cancel{0 = \frac{1}{x}}$ $\quad x = \frac{1}{y}$

$\forall y \in \mathbb{R} \setminus \{0\} \quad \exists \, x \in D : \; y = \frac{1}{x}$

$x = \frac{1}{y}$

4) $f(x) = x^2$ $\quad D = \mathbb{R}$





$$R_f = [0, +\infty) = \{ y \in \mathbb{R} \mid y \geq 0 \}$$

FOR WHICH y I CAN FIND $x \in D$

s.t. $y = x^2$?

5) $f(x) = x^2 + 1 \qquad D = \mathbb{R}$

$$R_f = [1, +\infty) = \{ y \in \mathbb{R} \mid y \geq 1 \}$$

FOR WHICH y I CAN FIND $x \in D$

s.t. $y = x^2 + 1$

$$y = x^2 + 1$$

$$\Updownarrow$$

$$\underbrace{y - 1}_{\geq 0} = x^2$$

$$y - 1 \geq 0 \iff y \geq 1$$

EVEN / ODD FUNCTIONS

A FUNCTION $f : D \subseteq \mathbb{R} \to \mathbb{R}$ IS CALLED **EVEN**

IF

1) $\forall x \in D \Rightarrow -x \in D$

2) $\forall x \in D \Rightarrow f(x) = f(-x)$

f IS EVEN AND $f(2) = 4$

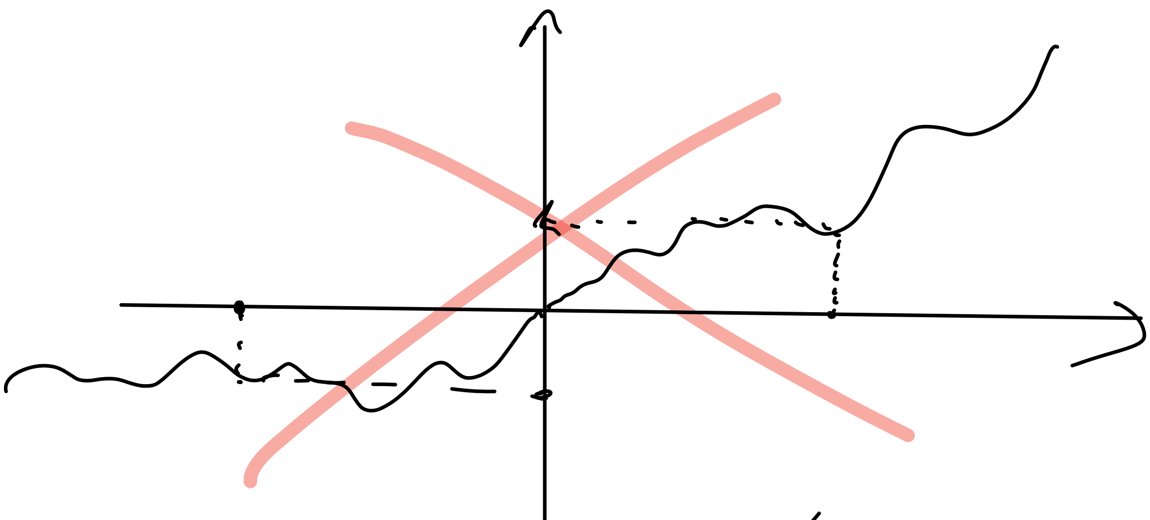
$$f(-2) = 4$$

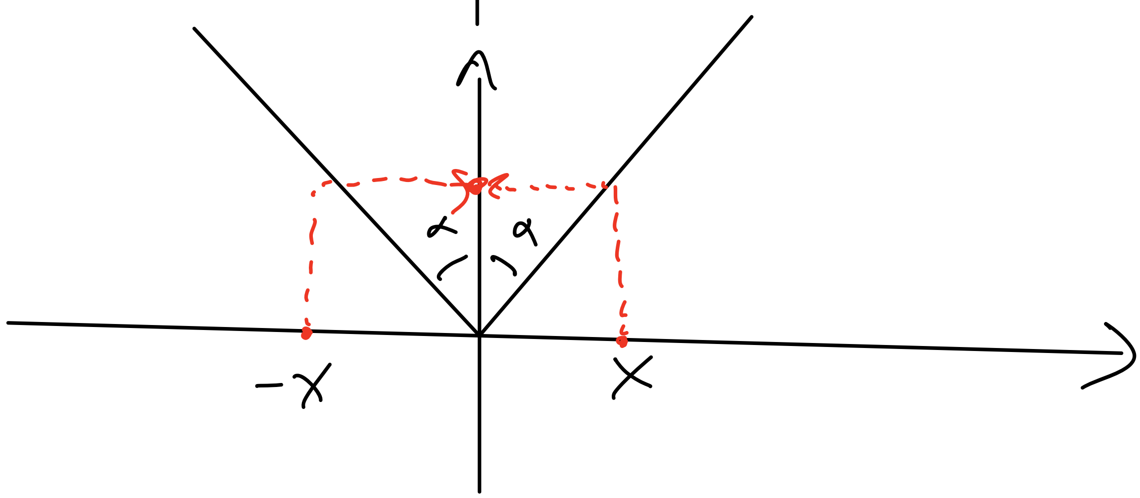
$f(x) = x^2 \qquad f(x) = x^{2m} \qquad m \in \mathbb{N}$

$\Updownarrow$

$f(-x) = (-x)^2 = (-x)\cdot(-x) = +x^2 = f(x)$

IF f IS EVEN THEN THE GRAPH IS SYMMETRIC WITH RESPECT TO THE VERTICAL AXIS





A FUNCTION $f : D \subseteq \mathbb{R} \to \mathbb{R}$ IS

ODD

ODD

IF

1) $\forall x \in D \Rightarrow -x \in D$

2) $\forall x \in D \Rightarrow f(-x) = -f(x)$

$\boxed{0 = -0}$

LET f BE ODD

$$\underbrace{f(0)} = f(-0) \overset{\text{ODD}}{=} \underbrace{-f(0)}$$

$\Rightarrow f(0) = -f(0) \Leftrightarrow f(0) = 0$

LET f BE ODD THEN $f(0) = 0$

$P \Rightarrow Q$

LET f BE A FUNCTION SUCH THAT $f(0) = -1$

IS f ODD ? NO

$f(x) = x$ ODD

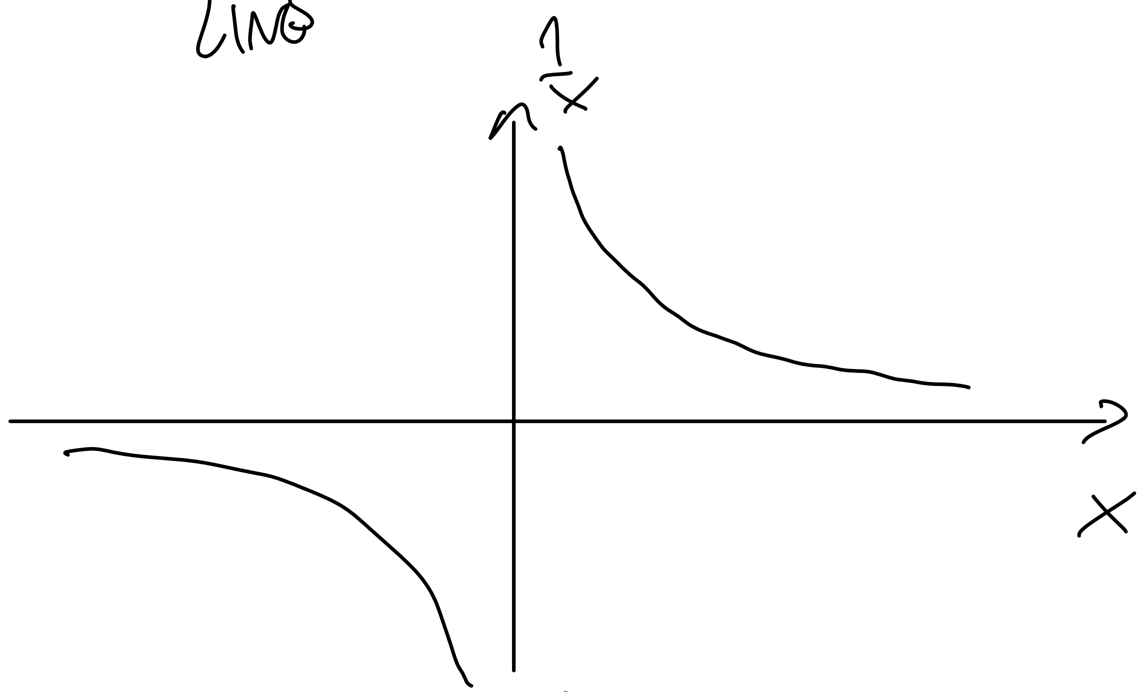
$$f(x) = x^3$$

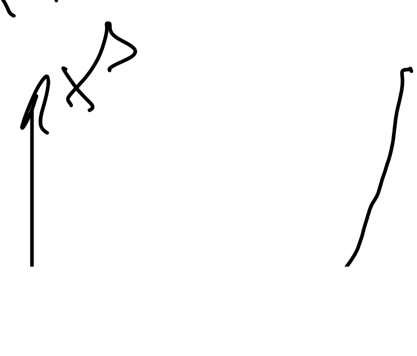
$$\vdots$$

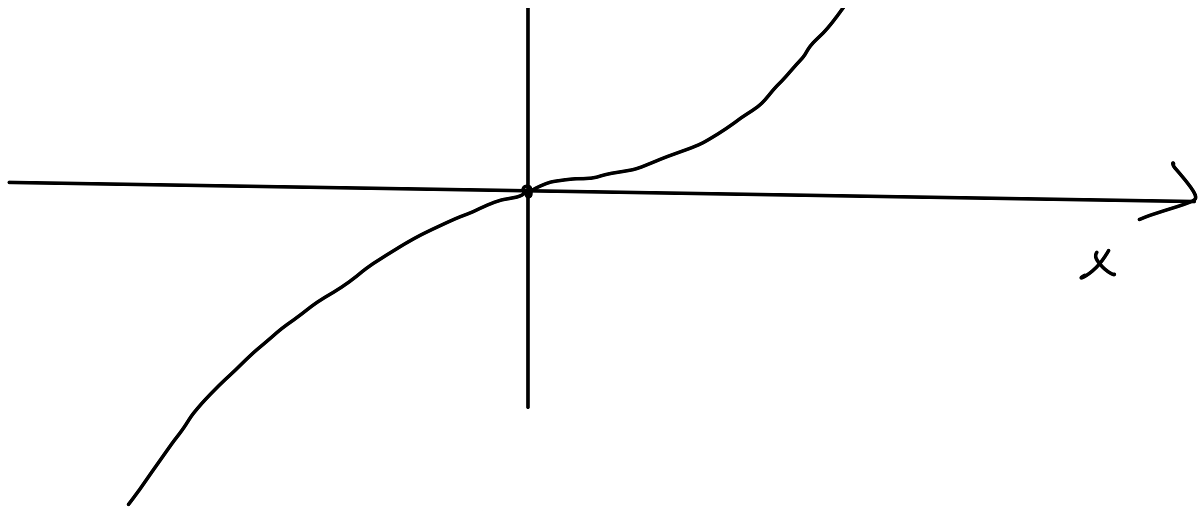
$$f(x) = x^{2n+1} \quad \forall\, n \in \mathbb{N}$$

$f(x) = \dfrac{1}{x}$ is odd

<u>REMARK</u>: IF A FUNCTION IS ODD ITS GRAPH IS ANTI-SYMMETRIC WITH RESPECT TO THE VERTICAL LINE



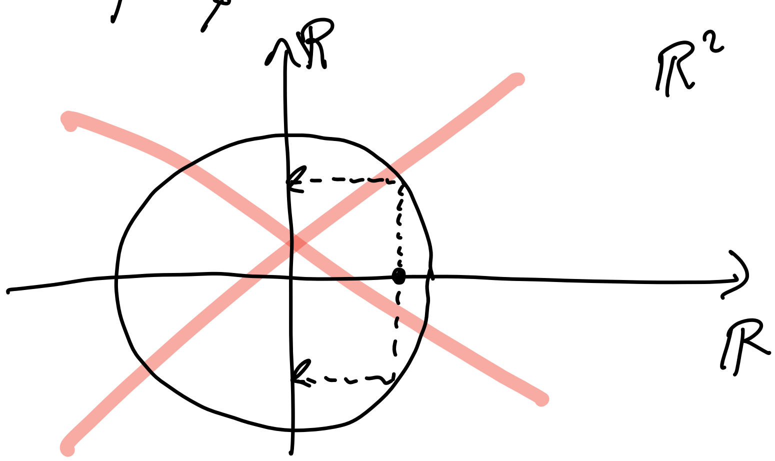


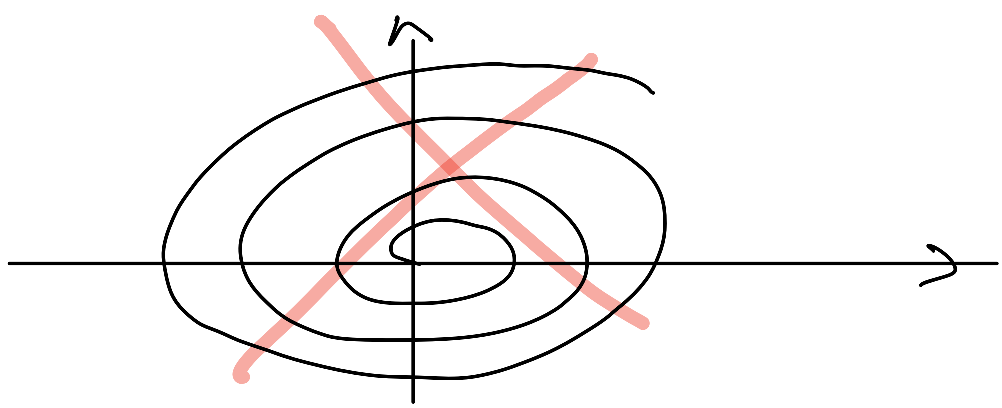


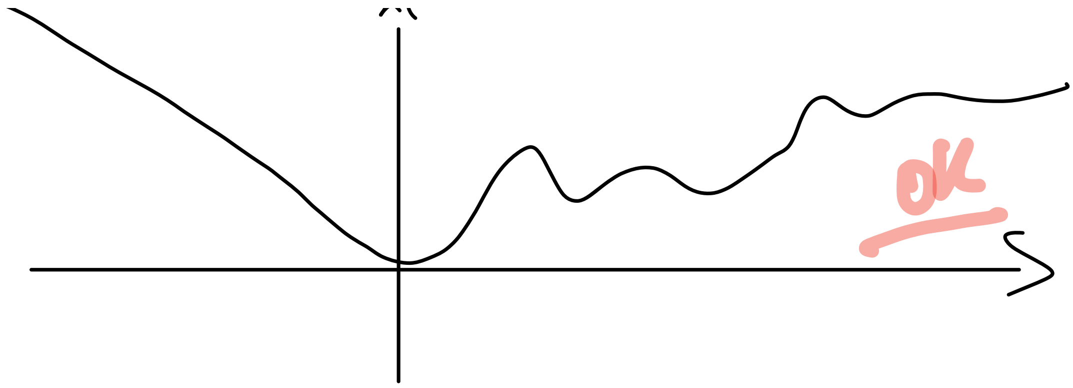
GRAPH OF FUNCTION

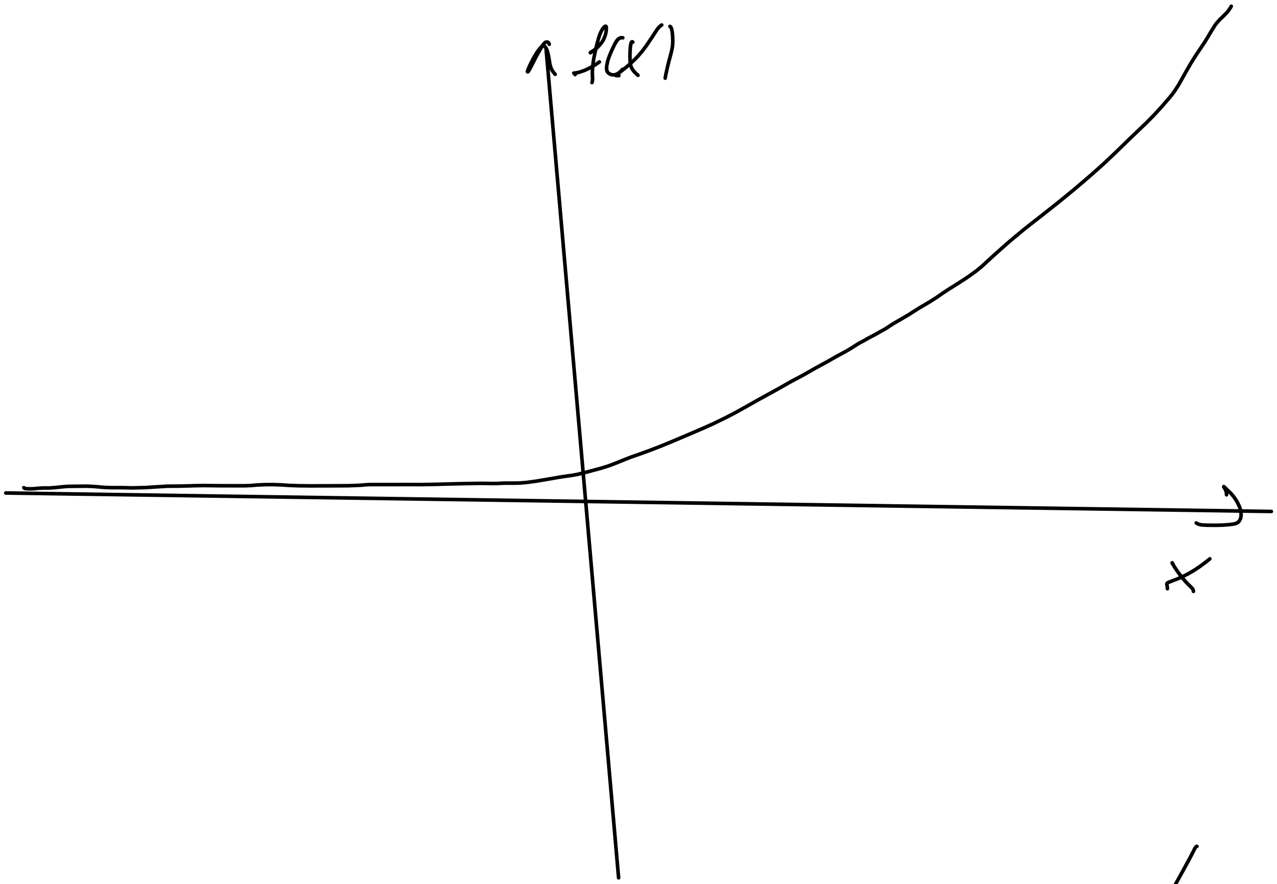
$$f: D \subseteq \mathbb{R} \to \mathbb{R}$$

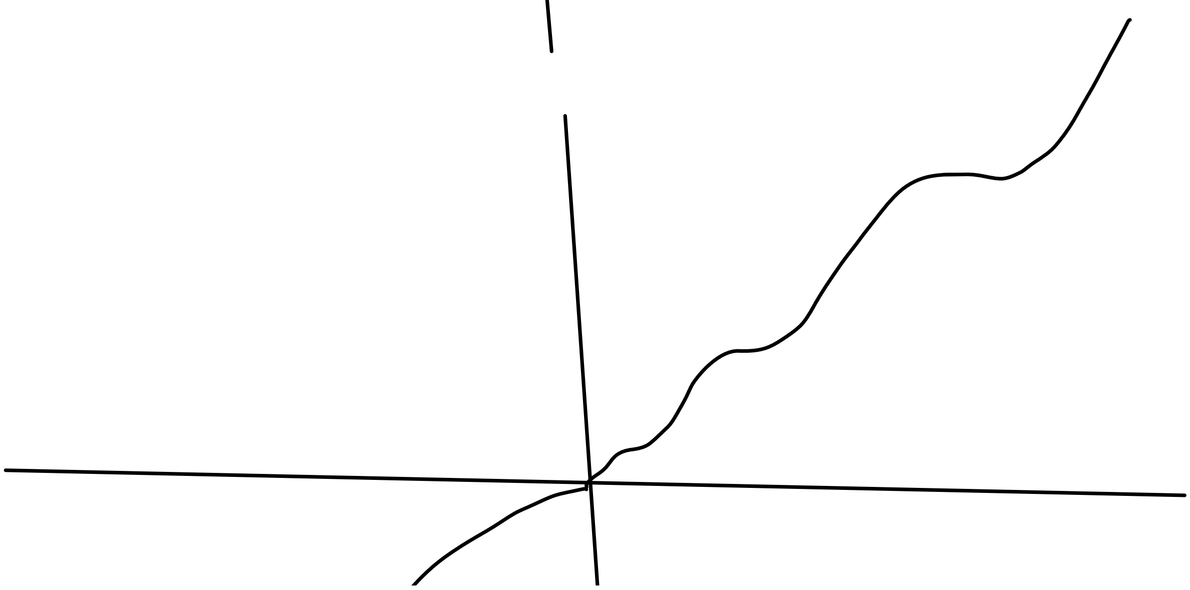
$$G_f = \{ (x, f(x)) \mid x \in D \} \subseteq \mathbb{R}^2$$

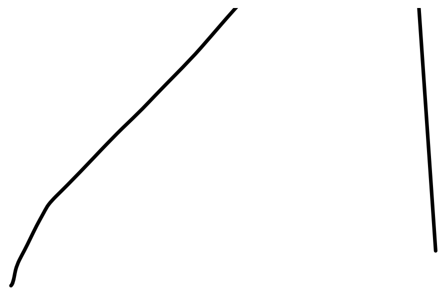





OK


f(x)
x





LET $f: D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

LET $I \subseteq D$

DECREASING

WE SAY THAT f IS INCREASING IN I

IF $\forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) \overset{\geq}{\leq} f(x_2)$

DECREASING

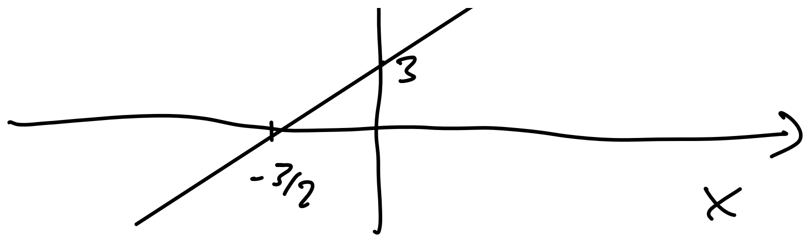
WE SAY THAT IS STRICTLY INCREASING IN I

IF $\forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) \overset{>}{<} f(x_2)$

$f(x) = 2x + 3$ STRICTLY INCREASING EVERYWHERE

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2$$

$$\Rightarrow \underbrace{2x_1 + 3}_{= f(x_1)} < \underbrace{2x_2 + 3}_{= f(x_2)}$$



$f(x) = x^2$ IS STRICTLY INCREASING IN $[0, +\infty)$

IS 〃 DECREASING IN $(-\infty, 0]$

LET $x_1, x_2 \in [0, +\infty)$ $\begin{cases} x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$

$0 \leq x_1 < x_2 \implies 0 \leq \overbrace{x_1 \cdot x_1}^{x_1^2} < x_2 \cdot x_1$

$< x_2 \cdot x_2 = x_2^2$

$0 \leq x_1^2 < x_2^2$

$f(x_1) < f(x_2)$

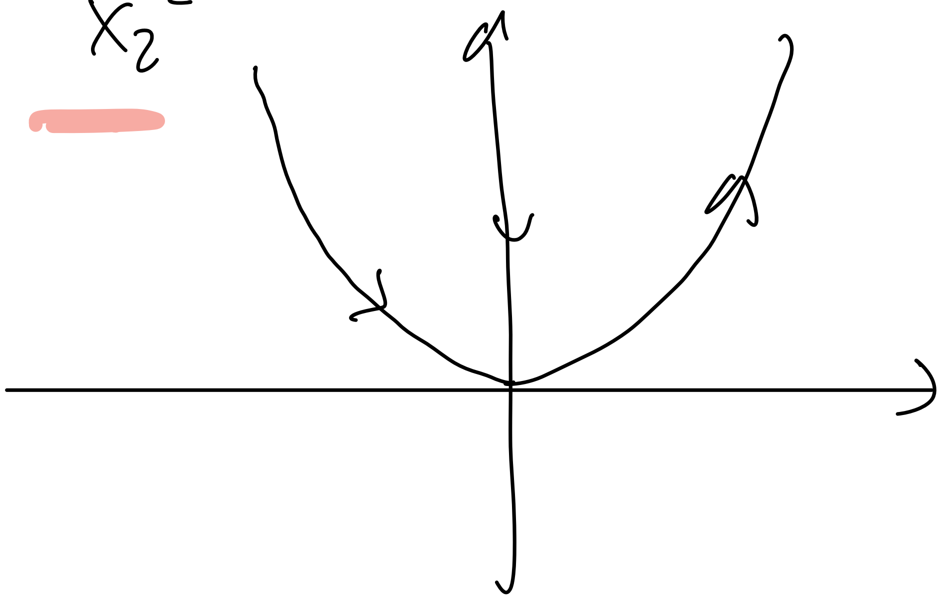
LET $x_1, x_2 \in (-\infty, 0]$

$x_1 < x_2 \leq 0$

$|x_1|^2 > |x_2|^2 \geq 0$

$\shortparallel \qquad \shortparallel$

$x_1^2 > x_2^2$



$0 \leq x_1 < x_2$

$0 \leq x_1 \cdot x_1 < x_2 \cdot x_1$

$0 \leq x_1^2 < x_2 \cdot x_1 < x_2 \cdot x_2 = x_2^2$

$x_1 < x_2$

$x_1 < x_2 \leq 0$

$2x+3$

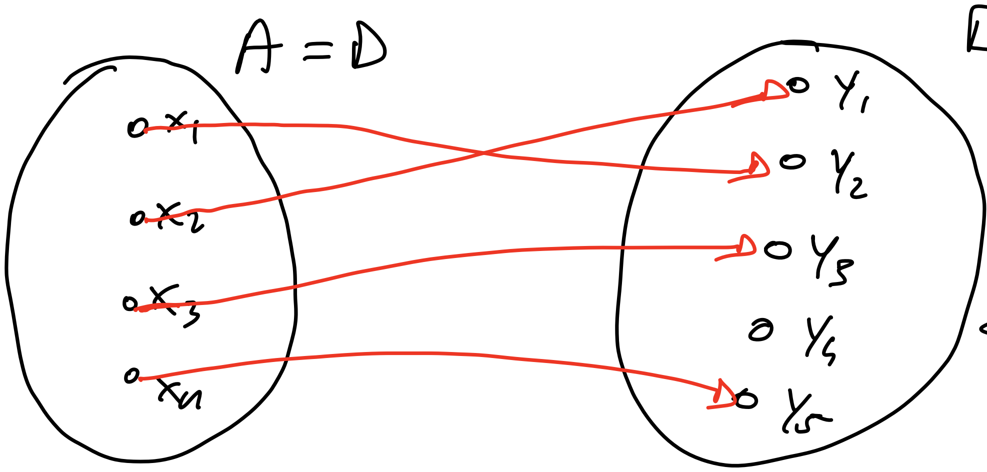
$$|x_1| > |x_2| \geq 0 \qquad x^2$$

$$|x_1|^2 > |x_2|^2$$

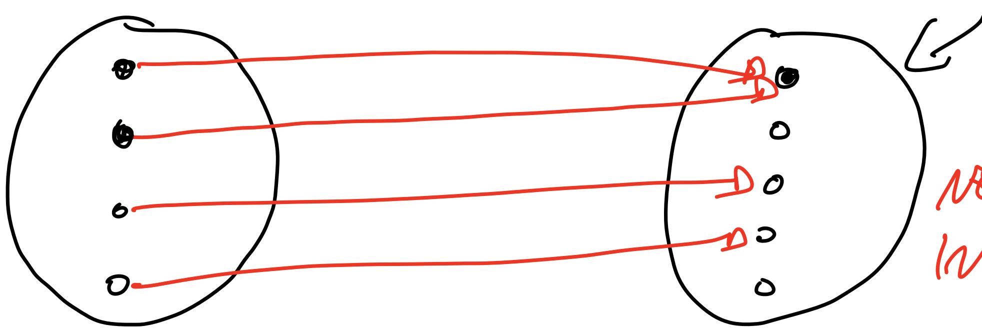
$$x_1^2 > x_2^2$$

INJECTIVE FUNCTIONS

A FUNCTION IS SAID TO BE INJECTIVE IF IMAGES OF DIFFERENT POINTS ARE DIFFERENT

a) 

INJECTIVE

b) 

NOT INJECTIVE

DEF: A FUNCTION IS INJECTIVE IF

- $\forall\, x_1, x_2 \in D: \; x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
- $\forall\, x_1, x_2 \in D: \; f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$$f(x): \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longrightarrow f(x) = x^2$$

$$f(-2) = f(2) = 4$$

NOT INJECTIVE

$$f: [0, +\infty) \longrightarrow [0, +\infty)$$
$$x \longrightarrow f(x) = x^2$$

IS INJECTIVE !

A FUNCTION IS SURJECTIVE IF EVERY POINTS IN THE ARRIVAL SET (CO-DOMAIN) IS REACHED BY AT LEAST AN ELEMENT OF THE DOMAIN THROUGH THE FUNCTION

DEF: LET $f: D \subseteq \mathbb{R} \to E \subseteq \mathbb{R}$

IS SAID TO BE SURJECTIVE IF

$$\forall y \in E \ \exists x \in D : f(x) = y$$

IF $E = R_f$ THEN f IS SURJECTIVE

$$f: \mathbb{R} \to [0, +\infty)$$
$$x \to f(x) = x^2$$

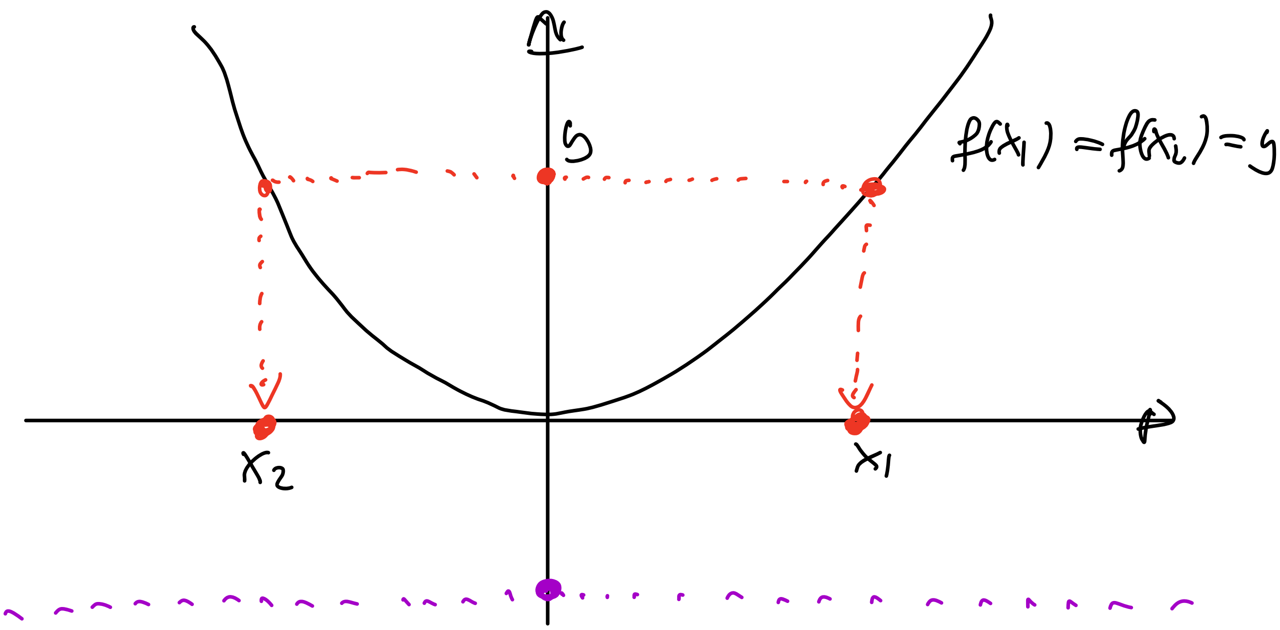
SURJECTIVE

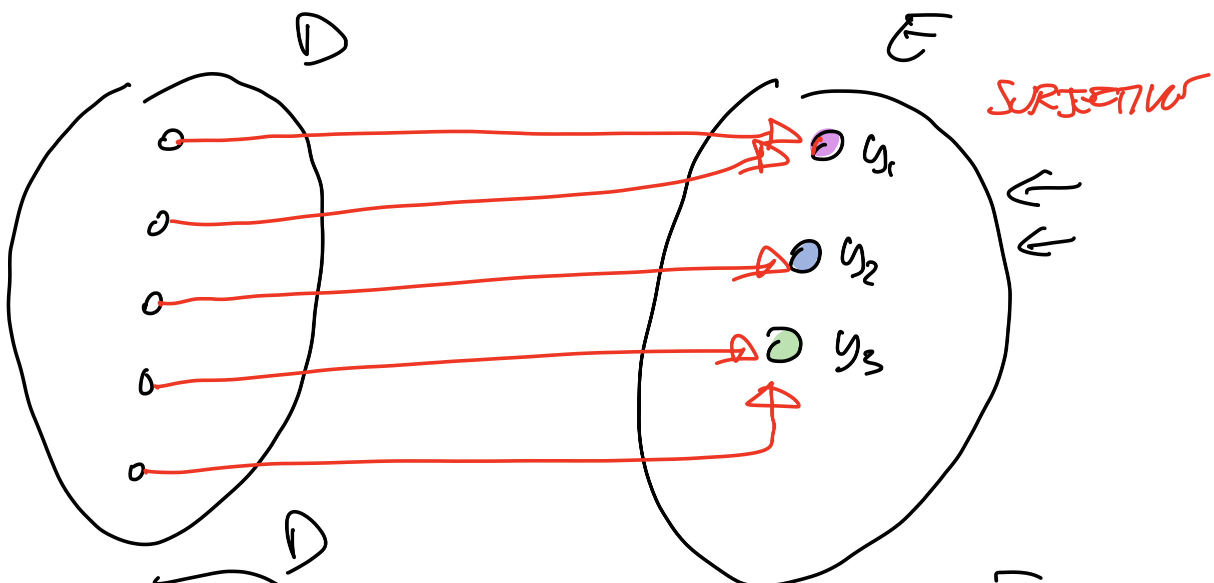
$$\forall y \in [0, +\infty] \quad y = x^2 \Rightarrow x = \pm\sqrt{y}$$

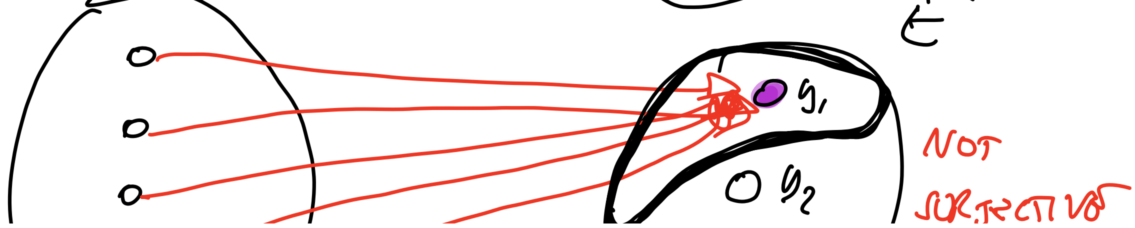
$$R_f = [0, +\infty)$$

$$x = \pm\sqrt{y}$$

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \to f(x) = x^2 \quad \text{NOT SURJECTIVE}$$

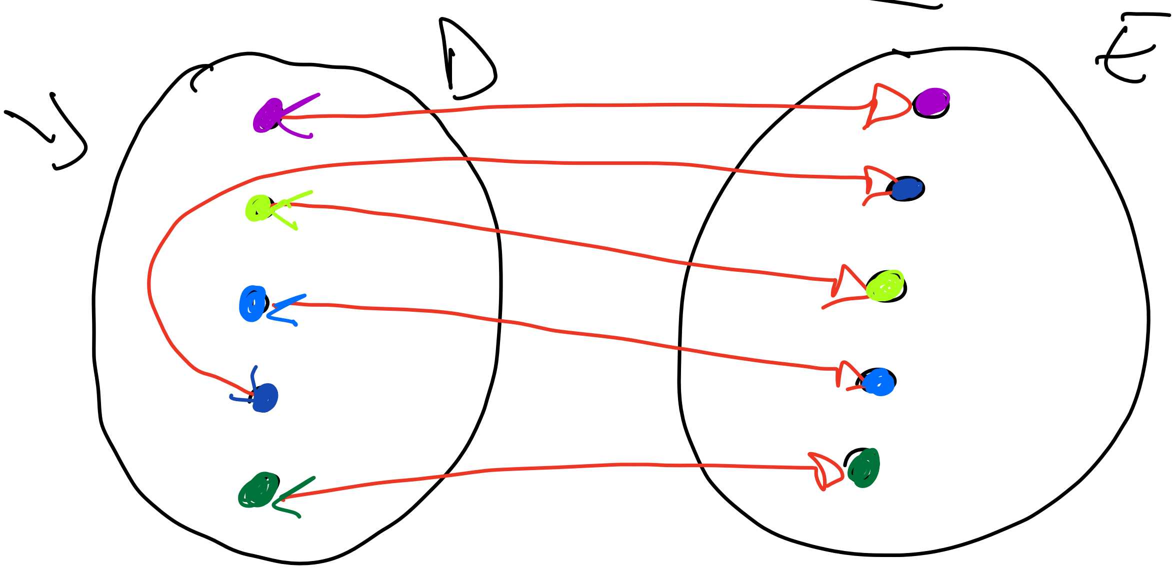






BIJECTIONS:

IF $f : D \subseteq \mathbb{R} \to E \subseteq \mathbb{R}$ IS BOTH INJECTIVE AND SURJECTIVE THEN IT IS CALLED A BIJECTION.



A FUNCTION IS INVERTIBLE IF AND ONLY IF IS A BIJECTION

f $\quad\quad$ $f^{(-1)}$

$$\begin{array}{c} \text{DOMAIN} \qquad\qquad \text{CO-DOMAIN} \\ f:\ [0,+\infty) \longrightarrow [0,+\infty) \\ x \longrightarrow x^2 \end{array} \qquad f(x)=x^2$$

$$f^{(-1)}(y) = +\sqrt{y}$$

$$f^{(-1)}(x) = +\sqrt{x}$$

$$x^2 = y \Rightarrow x = +\sqrt{y}$$

$$f:\ (-\infty,0] \longrightarrow [0,+\infty) \qquad f(x) = x^2$$

$$x \xrightarrow{f} x^2$$

$$f^{(-1)}(y) = -\sqrt{y}$$

$$f^{(-1)}(x) = -\sqrt{x}$$

$$x^2 = y \Rightarrow x = -\sqrt{y}$$

$$f:\ D \subseteq \mathbb{R} \longrightarrow E \subseteq \mathbb{R}$$

$$x \in D \xrightarrow{f} \sqrt{x^2+1} \in E$$

QUADRATIC FUNCTIONS

A QUADRATIC FUNCTION IS ANY FUNCTION

$$f: \mathbb{R} \to \mathbb{R}$$

$a, b, c \in \mathbb{R}$

$$f(x) = ax^2 + bx + c$$

A POLYNOMIAL OF DEGREE $= 2$

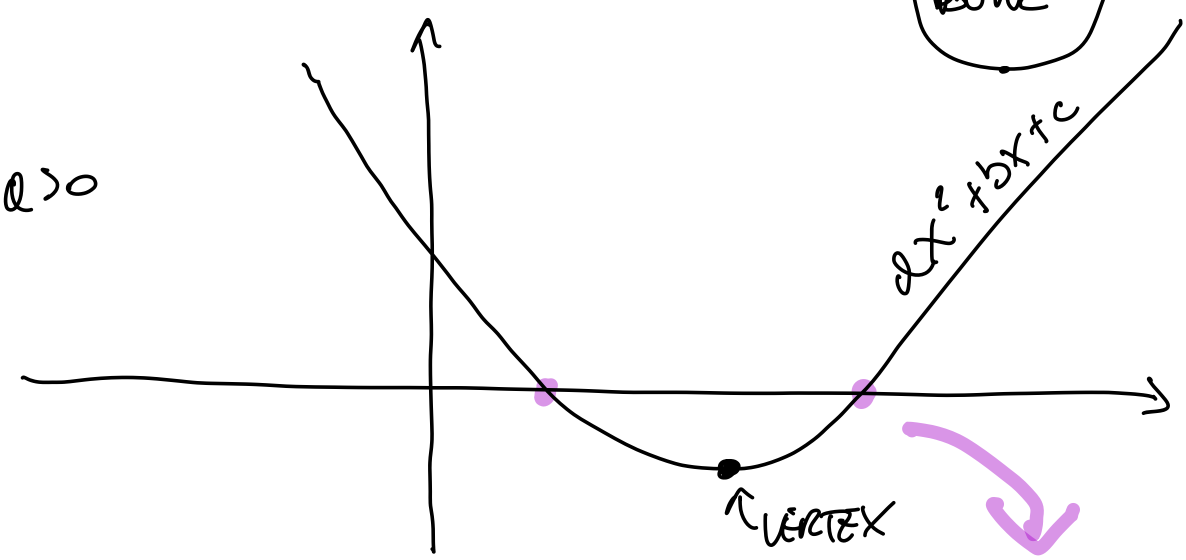
$a = 1 \qquad b = 0 \qquad c = 0 \Rightarrow f(x) = x^2$

THE GRAPH OF SUCH A FUNCTION IS A PARABOLA WHICH IS

CONCAVE IF $a < 0$

CONVEX IF $a > 0$

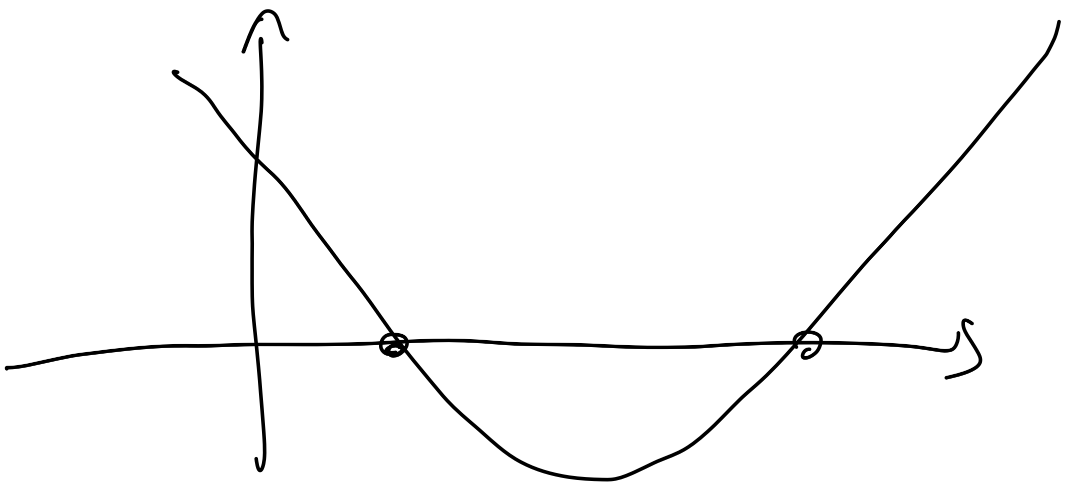
$a > 0$



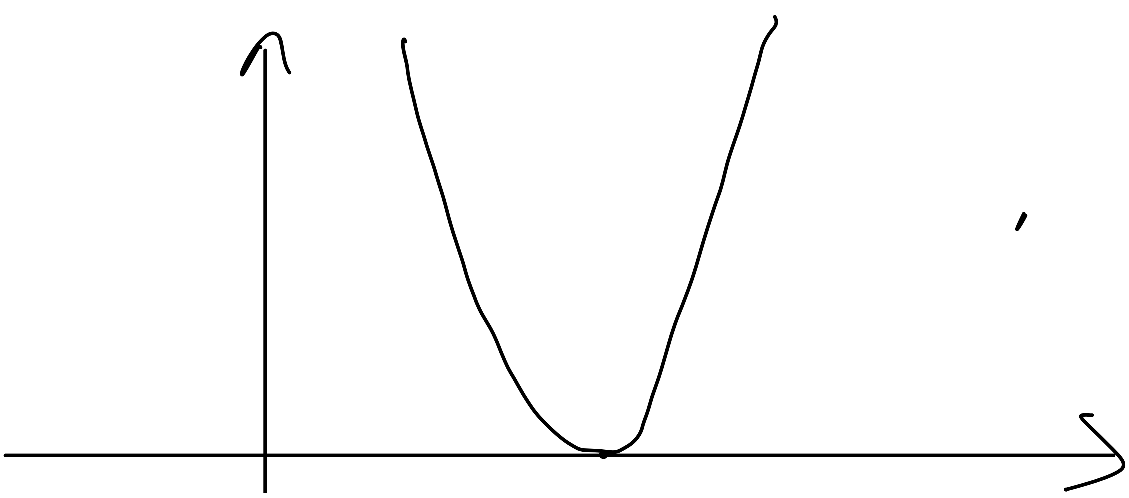
$\Delta = b^2 - 4ac \qquad ax^2 + bx + c = 0$

- $\Delta > 0 \Rightarrow$ 2 DISTINCT REAL SOLUTIONS
- $\Delta = 0 \Rightarrow$ A UNIQUE SOLUTION
- $\Delta < 0 \Rightarrow$ NO SOLUTION

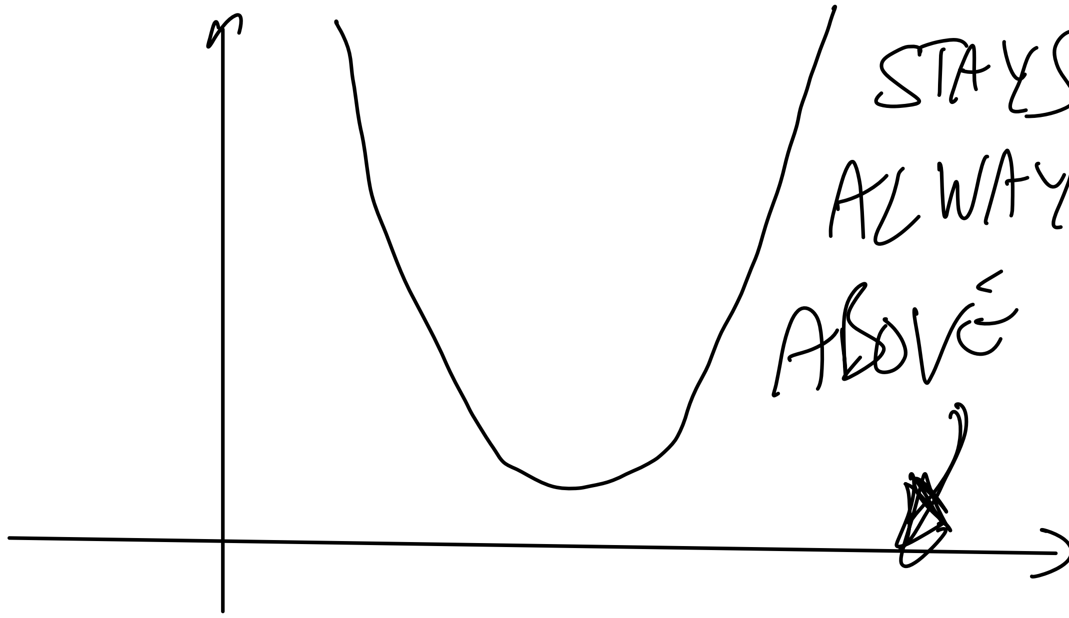
IF $a > 0$ AND $\Delta > 0$



IF $a > 0$ AND $\Delta = 0$



IF $a > 0$ AND $\Delta < 0$



STAYS ALWAYS ABOVE

DEF: FOR EACH $n \in \mathbb{N}$ THE FUNCTION

$$f(x) = x^n = \underbrace{x \cdot x \cdots x}_{n\text{-TIMES}}$$

POWER FUNCTION

$D = \mathbb{R}$

R_f ⟨ $\mathbb{R}$ n IS ODD

$\to [0, +\infty)$ n IS EVEN

f(x) IS ODD WHEN n IS ODD

" " EVEN " n IS EVEN

IF n IS ODD THE FUNCTION IS INVERTIBLE EVERYWHERE

$$f^{(-1)}(x) = x^{\frac{1}{n}}$$

$$f(x) = x^3 \rightsquigarrow f^{(-1)}(x) = x^{\frac{1}{3}}$$

$$f(x) = x^9 \rightsquigarrow f^{(-1)}(x) = x^{\frac{1}{9}}$$

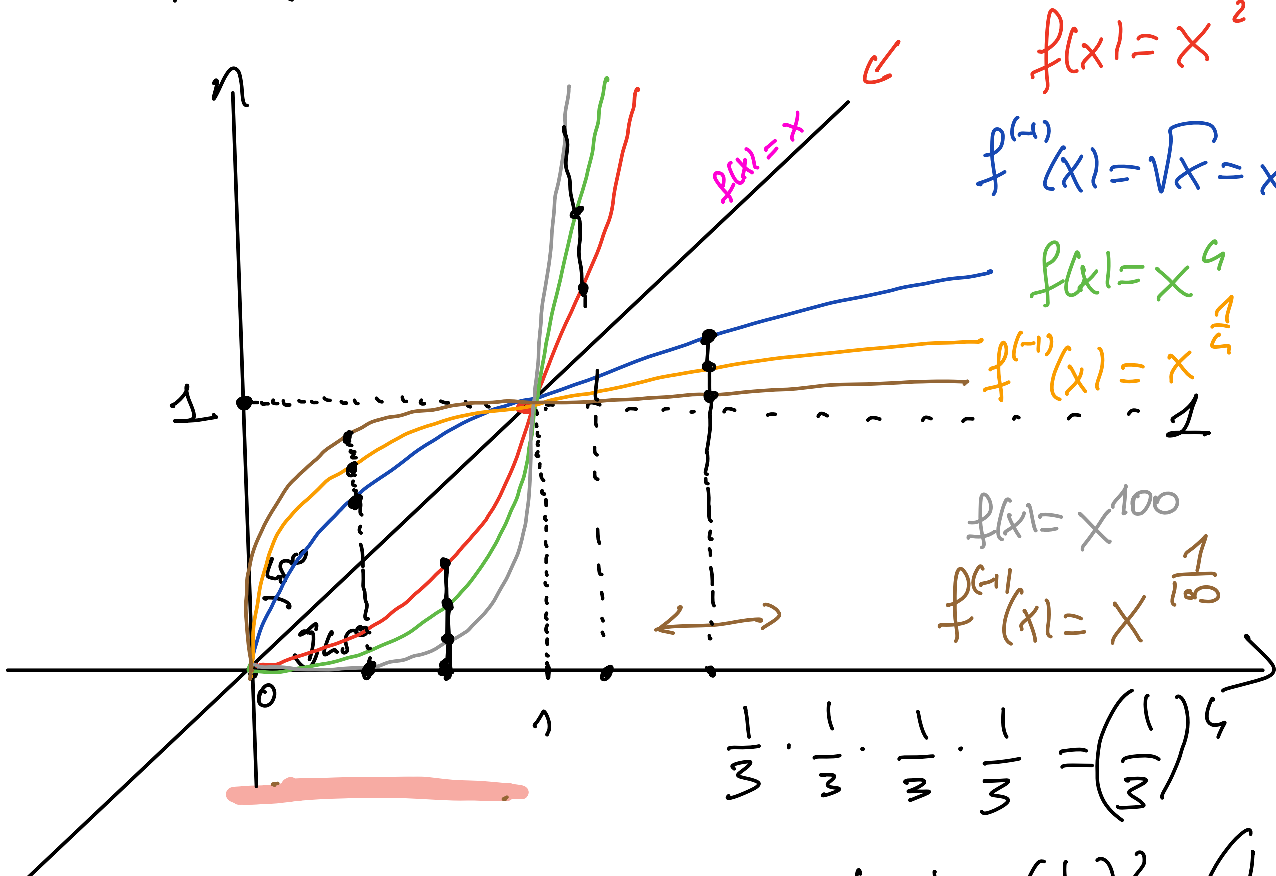
IF n IS EVEN THEN :

$$f(x) : \mathbb{R} \to [0, +\infty)$$

IS NOT INVERTIBLE (NOT INJECTIVE)

HOWEVER THE "RESTRICTIONS"

- $f(x): [0,+\infty) \longrightarrow [0,+\infty)$
- $f^{(-1)}(x) = x^{\frac{1}{n}} \qquad x \geq 0$

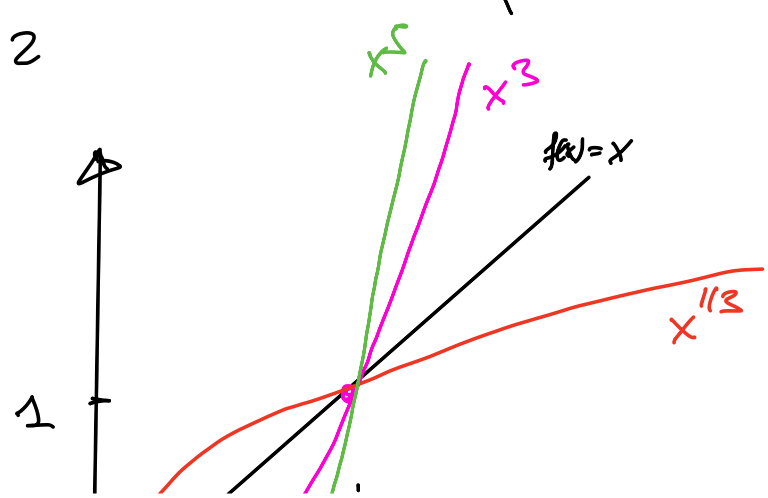


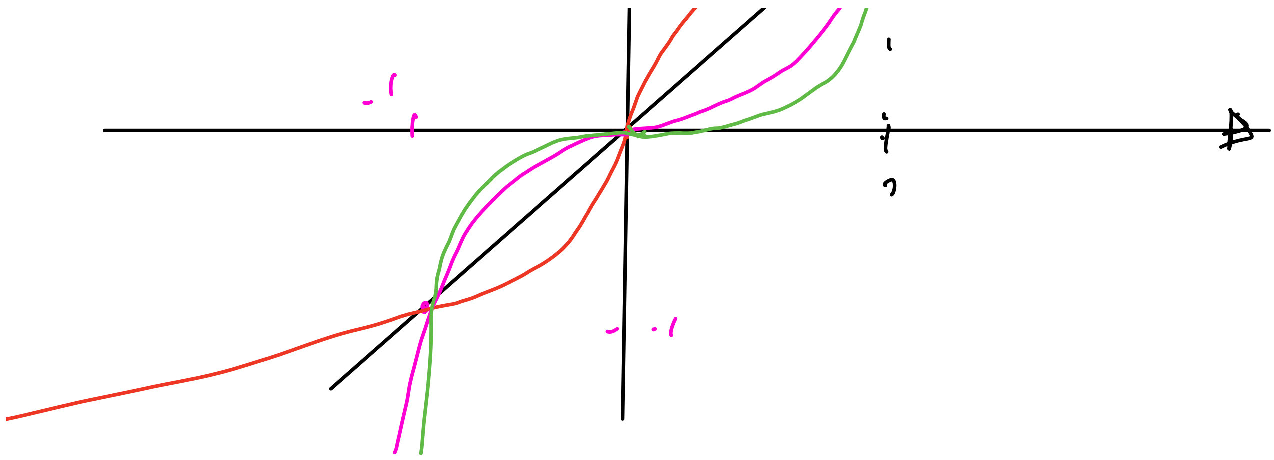
$$\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \left(\frac{1}{3}\right)^4$$

$$\frac{1}{3}\cdot\frac{1}{3} = \left(\frac{1}{3}\right)^2 > \left(\frac{1}{3}\right)^4$$

$$3\cdot 3\cdot 3\cdot 3 > 3\cdot 3$$

$$3^4 > 3^2$$





$$x^7 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$$

$$\boxed{x^{\pi} = \,?} \qquad f(x) = x^n$$

$$\vdots$$

$$f(x) = x^{\alpha} \qquad \alpha \in \mathbb{R}$$

$$\boxed{x > 0}$$

$$f : [0, +\infty) \longrightarrow [0, +\infty)$$

$$f(x) = x^{\alpha} \qquad \alpha \in \mathbb{R}$$

<u>EXPONENTIAL FUNCTION</u> $\quad a > 0$

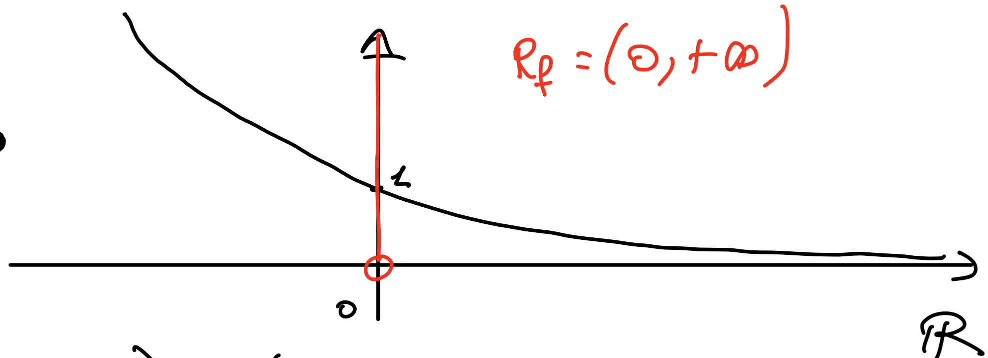
$f(x) = a^x$ IS CALLED THE EXPONENTIAL FUNCTION WITH BASE a

$f: \mathbb{R} \longrightarrow (0, +\infty)$

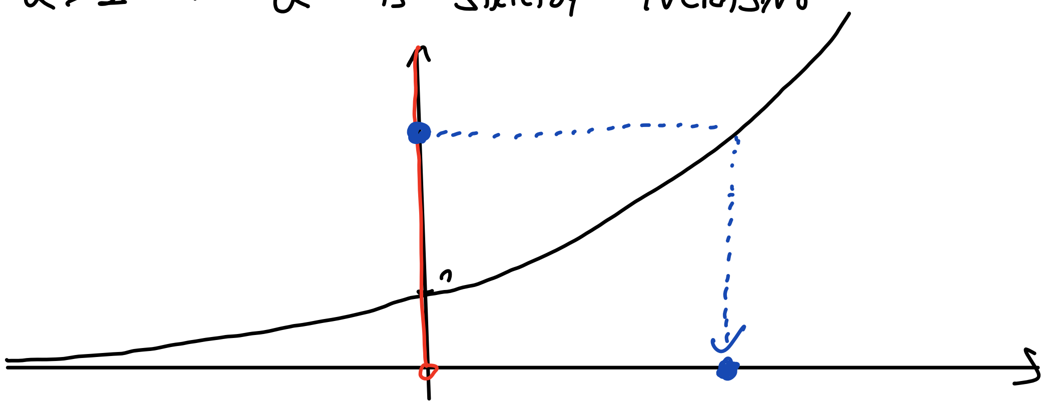
$a^0 = 1$

$a^x > 0$

0 < a < 1 ⇒ a^x IS STRICTLY DECREASING



a > 1 ⇒ a^x IS STRICTLY INCREASING



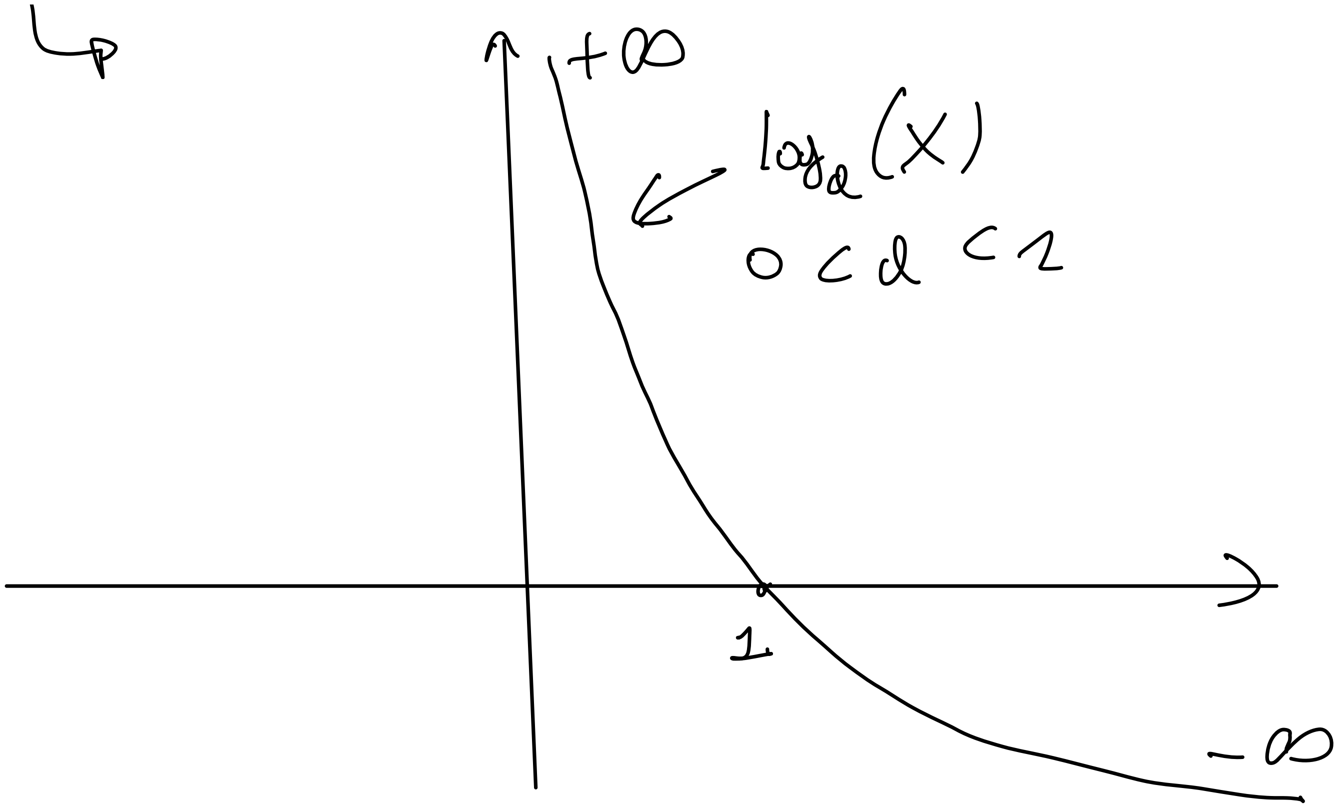
IF $y > 0$ $\exists!\, x \in \mathbb{R}$ s.t. $a^x = y$

THAT x IS CALLED

$$x = \log_a (y)$$

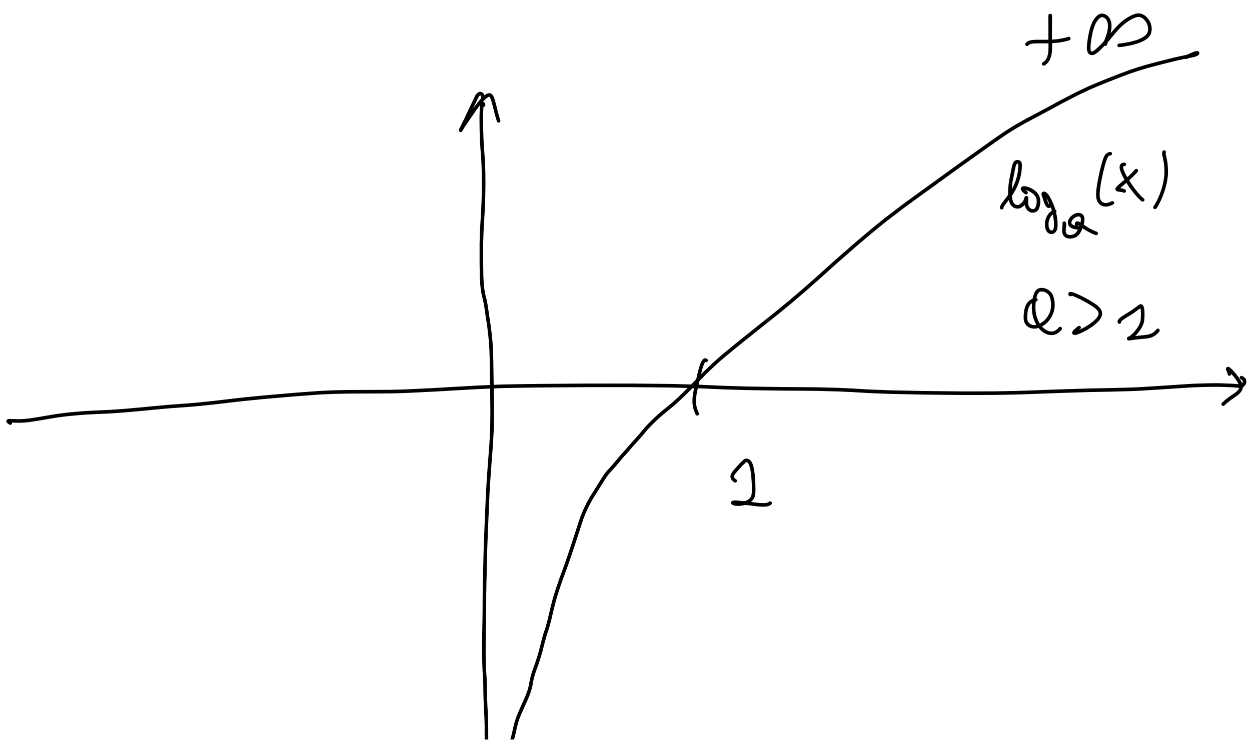
IF $0 < a < 1$

↳



IF $0 < a < 1 \Rightarrow \log_a(x)$ IS STRICTLY DECREASING

IF $a > 1 \Rightarrow \log_a(x)$ IS STRICTLY INCREASING



COMPOSITION OF FUNCTION

$f: D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

$\boxed{R_f = \text{RANGE OF } f}$

$g: D_g \subseteq R_f \longrightarrow \mathbb{R}$

$(g \circ f): D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

$\forall\, x \in D \Rightarrow (g \circ f)(x) = g(f(x))$

1) $h(x) = \sqrt{x+1}$

$x \longrightarrow x+1 \longrightarrow \sqrt{x+1}$

$f(x) = x+1 \qquad g(y) = \sqrt{y}$

$$(g \circ f)(x) = g(f(x)) = g(x+1)$$
$$= \sqrt{\underbrace{x+1}}$$

$$D_{g \circ f} = [-1, +\infty)$$
$$= \{x \in \mathbb{R} \mid x \geq -1\}$$

2) $h(x) = 2^{\frac{1}{x}}$

$$x \longrightarrow \frac{1}{x} \longrightarrow 2^{\frac{1}{x}}$$

$$D_h = \{x \in \mathbb{R} \mid x \neq 0\} = \mathbb{R} \setminus \{0\}$$

3) $h(x) = \log_2(1-x)$

$$x \xrightarrow{f} 1-x \xrightarrow{g} \log_2(1-x)$$

$$D_{f\circ g} = \{x \in \mathbb{R} \mid 1-x>0\}$$

$$= \{x \in \mathbb{R} \mid x<1\}$$

$$= (-\infty, 1)$$

4) $h(x) = 2^{\sqrt{1/x}} = 2^{\sqrt{\frac{1}{x}}}$

$$x \xrightarrow{\frac{1}{x}} \frac{1}{x} \xrightarrow{\sqrt{\ }} \sqrt{\frac{1}{x}} \xrightarrow{2^{(\ )}} 2^{\sqrt{\frac{1}{x}}}$$

$$D_h = \{x \in \mathbb{R} \mid x \neq 0,\ x>0\}$$

$$= (0, +\infty)$$

$$(f \circ g)(x) \neq (g \circ f)(x)$$

$\log_2(x^2 - 1) \geq 0$

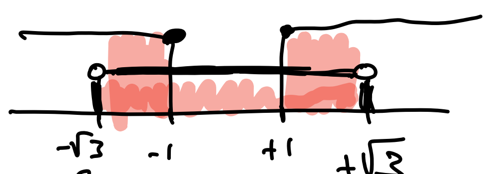
$x^2-1>0 \Leftrightarrow x^2>1$

$x^2-1\geq 1 \Leftrightarrow x^2 \geq 2$

$\log_2(x^2-1) \geq 0 \Leftrightarrow$

$x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)$

$\log_2(3 - x^2) \leq 1$



$\Updownarrow$ $3-x^2>0 \Leftrightarrow 3>x^2$

$-\sqrt{3}<x<+\sqrt{3}$

$3-x^2\leq 2$

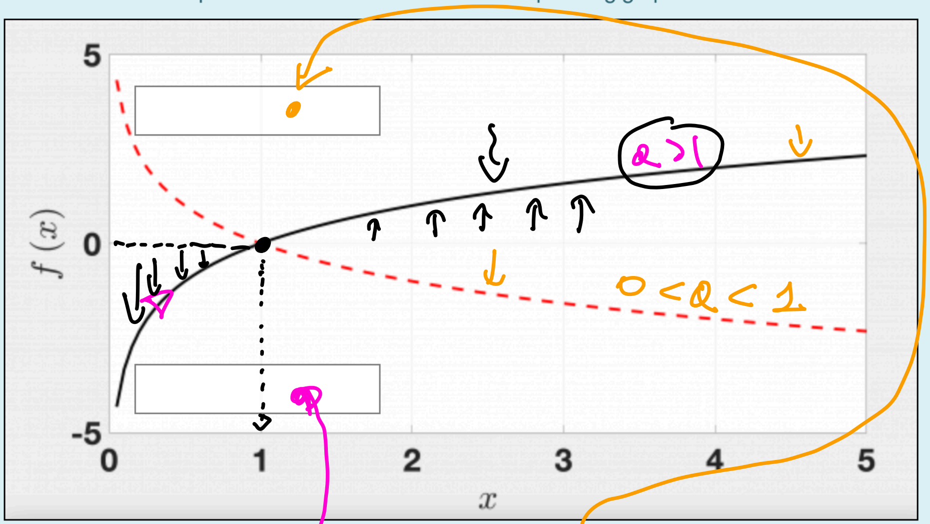
$1 \leq x^2 \Rightarrow \cancel{x \in (-\infty,-1] \cup [+1,+\infty)}$

↳ $-\sqrt{3} < x \leq -1$ OR $+1 \leq x < +\sqrt{3}$

CW9

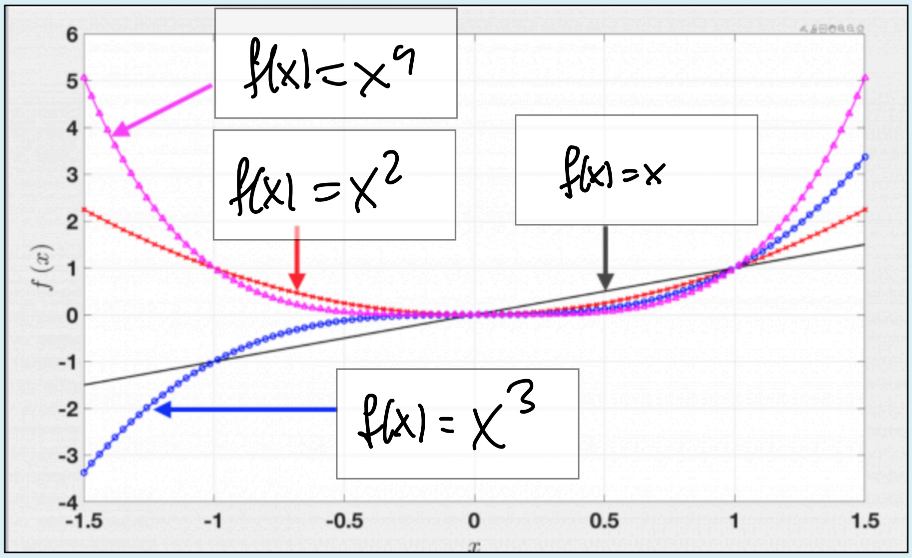
PWD: HELLO1

Drag and drop the two function formulas provided below close to the corresponding graph.



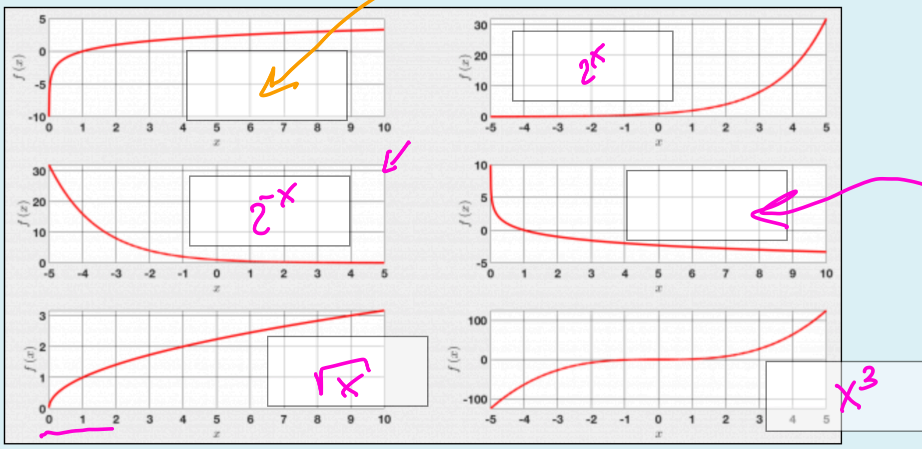
$f(x) = \log_a(x), \quad a > 1$ | $f(x) = \log_a(x), \quad 0 < a < 1$

Associate, using the empty boxes and the arrows in the figure, the function formulas provided below to the corresponding graph.



$f(x) = x$ | $f(x) = x^2$ | $f(x) = x^3$ | $f(x) = x^4$

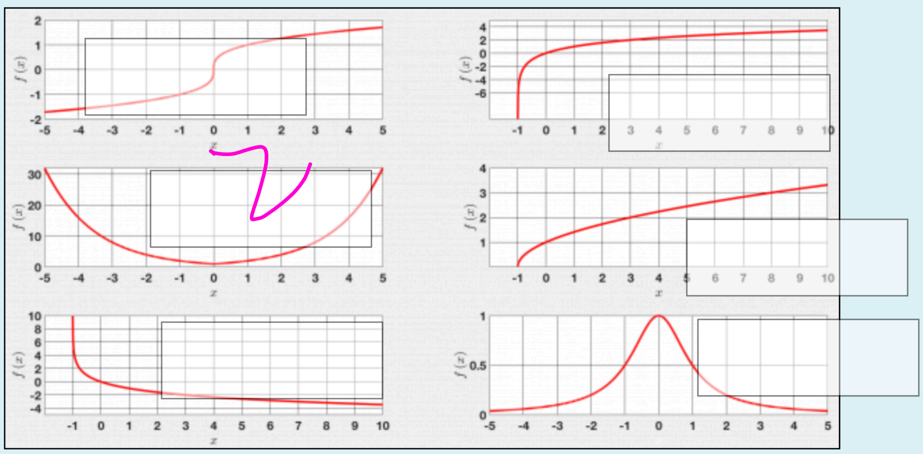
Associate, using the empty boxes, the function formulas provided below to the corresponding graph.



x^3	$\sqrt{x}$	2^{-x}	2^x	$\log_{1/2}(x)$	$\log_2(x)$

Associate, using the empty boxes, the function formulas provided below to the corresponding graph.

WARNING: some of the labels do not correspond to any of the graphs.



$\sqrt{x+1}$	$\log_2(x+1)$	$\log_{1/2}(x+1)$	$2^{\lvert x \rvert}$	$x^{1/3}$
$\log_2(x)$	$\sqrt{x}$	2^x	$(1+x^2)^{-1}$	

$x \longrightarrow \log_2(x) \longrightarrow \log_2(\log_2(x))$

$\log_2(x) > 0 \iff x > 2^0 = 1$ $\quad D = (1, +\infty)$

$\hookrightarrow \log_2(x) \quad 0 \quad 0$

$2 \quad > 2 \iff \underline{x > 2} = 1$

Given the function

$$f(x) = \log_2(\log_2(x))$$

compute its domain D and select the correct answers.

Select one or more:

- [ ] a. $D = [0, +\infty)$
- [ ] b. $D = [1, +\infty)$
- [ ] c. $D = \mathbb{R} \setminus (-\infty, 0)$
- [ ] d. $D = (1, +\infty)$
- [ ] e. $D = \mathbb{R} \setminus (-\infty, 1]$
- [ ] f. $D = \mathbb{R} \setminus (-\infty, 0]$
- [ ] g. $D = (0, +\infty)$
- [ ] h. $D = \{x \in \mathbb{R} | x > 1\}$
- [ ] i. $D = \{x \in \mathbb{R} | x \geq 1\}$
- [ ] j. $D = \mathbb{R} \setminus (-\infty, 1)$
- [ ] k. $D = \{x \in \mathbb{R} | x > 0\}$

Given the function

$\log_2(\log_2(x)) > 0$

$\Leftrightarrow \log_2(x) > 2^0 = 1 \Leftrightarrow \log_2(x) > 1$

$$f(x) = \log_2(\log_2(\log_2(x)))$$

$\Leftrightarrow 2^{\log_2(x)} > 2$

$\Leftrightarrow x > 2$

compute its domain D and select the correct answers.

$D = (2, +\infty)$

Select one or more:

- a. $D = [2, +\infty)$
- b. $D = \{x \in \mathbb{R} | x \geq 1\}$
- c. $D = \mathbb{R} \setminus (-\infty, 0]$
- d. $D = \mathbb{R} \setminus (-\infty, 2]$
- e. $D = \{x \in \mathbb{R} | x > 2\}$
- f. $D = \mathbb{R} \setminus (-\infty, 1)$
- g. $D = \{x \in \mathbb{R} | x > 0\}$
- h. $D = (0, +\infty)$
- i. $D = [2, +\infty)$
- j. $D = (2, +\infty)$
- k. $D = \mathbb{R} \setminus (-\infty, 0)$

Find the solutions of the equation

$$\log_2(1+x^2) = 2 \Leftrightarrow 2^{\log_2(1+x^2)} = 2^2$$

$1+x^2 = 4$ $x^2 = 3$

Select the correct answer. Incorrect answers will be penalised.

$\boxed{x = \pm\sqrt{3}}$

Select one or more:

- [ ] a. $\{x \in \mathbb{R} | x > -\sqrt{3} \text{ and } x < \sqrt{3}\}$
- [ ] b. $x = \pm 1$
- [ ] c. There are no solutions.
- [ ] d. $x = 0$
- [ ] e. $x \in (-1, 1)$
- [ ] f. $x = \pm\infty$
- [ ] g. $x = \pm 3$
- [ ] h. $\mathbb{R} \setminus (-\infty, 0)$
- [ ] i. $x = \pm\sqrt{3}$
- [ ] j. $\{x \in \mathbb{R} | x \leq -1 \text{ or } x \geq -1\}$
- [ ] k. $x = \pm\sqrt{2}$

Find the solutions of the inequality

$x^2-1>0 \Leftrightarrow x^2>1$

$$\log_2(x^2-1) \geq 0$$

More than one answer is correct. Incorrect answers will be

Select one or more:

- a. $x \in (-\infty, -\sqrt{2}] \bigcup [\sqrt{2}, +\infty)$
- b. $x \in \mathbb{R} \setminus (-1, 1)$
- c. $\mathbb{R} \setminus (-\infty, 0)$
- d. $[2, +\infty)$
- e. $x \in (-\infty, 1] \bigcup [1, +\infty)$
- f. $\mathbb{R} \setminus (-\infty, 1)$
- g. $\{x \in \mathbb{R} | x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}\}$
- h. $\mathbb{R} \setminus (-\infty, 0]$
- i. $x \in \mathbb{R} \setminus (-\sqrt{2}, \sqrt{2})$
- j. $\{x \in \mathbb{R} | x \leq -1 \text{ or } x \geq 1\}$
- k. $\{x \in \mathbb{R} | x > 0\}$

Find the solutions of the inequality

$$\log_2(1-x^2) \leq 0$$

More than one answer is correct. Incorrect answers will be penalised.

Select one or more:

- [ ] a. $x \in (-1, 1)$
- [ ] b. $[2, +\infty)$
- [ ] c. $\forall x \in \mathbb{R}$
- [ ] d. $x \in (-\sqrt{2}, \sqrt{2})$
- [ ] e. $\mathbb{R} \setminus (-\infty, 0]$
- [ ] f. $\mathbb{R} \setminus (-\infty, 1)$
- [ ] g. $\{x \in \mathbb{R} | x > 1\}$
- [ ] h. $x \in \mathbb{R} \setminus \{(-\infty, 1] \bigcup [1, +\infty)\}$
- [ ] i. $\{x \in \mathbb{R} | -1 < x < 1\}$
- [ ] j. $\mathbb{R} \setminus (-\infty, 0)$
- [ ] k. $\{x \in \mathbb{R} | -\sqrt{2} \leq x \leq \sqrt{2}\}$

Find the solutions of the inequality

`\{x\in\mathbb{R}|x\leq -1 \text{ or } x\geq -1\}`

$$\log_2(3 - x^2) \leq 1$$

More than one answer is correct. Incorrect answers will be penalised.

Select one or more:

- [ ] a. $x \in (-\infty, -1) \bigcup (1, \infty)$
- [ ] b. $\{x \in \mathbb{R} | -\sqrt{3} < x \leq -1 \text{ or } 1 \leq x < \sqrt{3}\}$
- [ ] c. $x \in [-\sqrt{3}, -1] \bigcup [1, \sqrt{3}]$
- [ ] d. $x \in (-\sqrt{3}, -1] \bigcup [1, \sqrt{3})$
- [ ] e. $\{x \in \mathbb{R} | x < -1 \text{ or } x > -1\}$
- [ ] f. $\{x \in \mathbb{R} | -\sqrt{3} \leq x \leq -1 \text{ or } 1 \leq x \leq \sqrt{3}\}$
- [ ] g. $\{x \in \mathbb{R} | x \leq -1 \text{ or } x \geq -1\}$
- [ ] h. $x \in (-\infty, -1] \bigcup [1, \infty)$
- [ ] i. $\{x \in \mathbb{R} | x > -\sqrt{3} \text{ and } x < \sqrt{3}\}$
- [ ] j. $\mathbb{R} \setminus (-\infty, 0)$
- [ ] k. $\mathbb{R} \setminus (-\infty, 0]$

$f(x) = 2^{1+x^2}$ $\quad f(0) = 2^1 = 2$ ✓

$2^{1+x^2} \geq 2 \quad \forall x \in \mathbb{R}$

For which value of y the following equality?

$$2^{1+x^2} = y$$

has **at least one** solutions in x ?

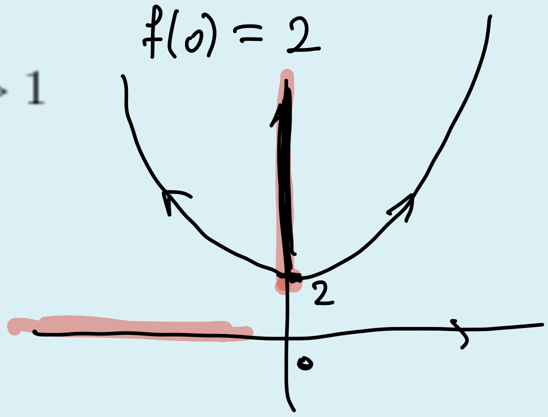
$R_f = [+2, +\infty)$

Select the correct answer.

$f(x) = 2^{1+x^2}$

$f(0) = 2$

- ☐ 1. $y < 0$
- ☐ 2. $y < -1$ OR $y > 1$
- ☒ 3. $y \geq 2$
- ☐ 4. $y \in [-2, 2]$
- ☐ 5. $y \geq 0$



Compute the two solutions x_1 and x_2 of the following equation

$$2^{1+x^2} = 3$$

and report them in the boxes below.

x_1 =

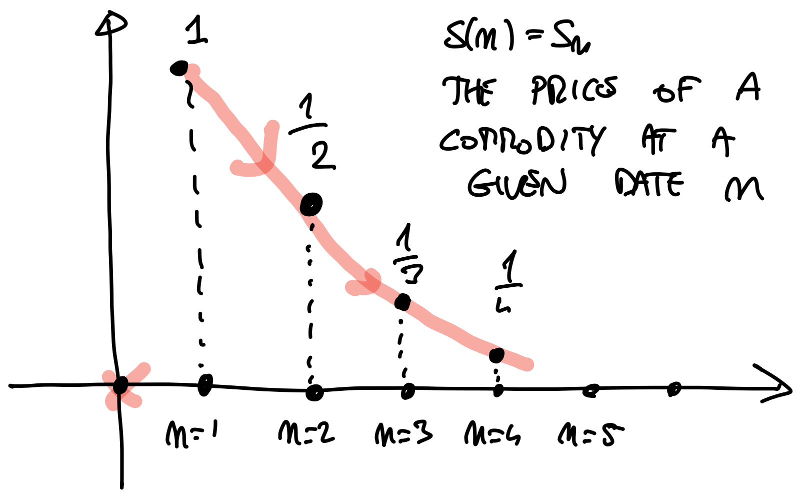
x_2 =

SEQUENCES

$$S : \mathbb{N} \longrightarrow \mathbb{R}$$

$$n \in \mathbb{N} \longrightarrow S(n) = S_n$$

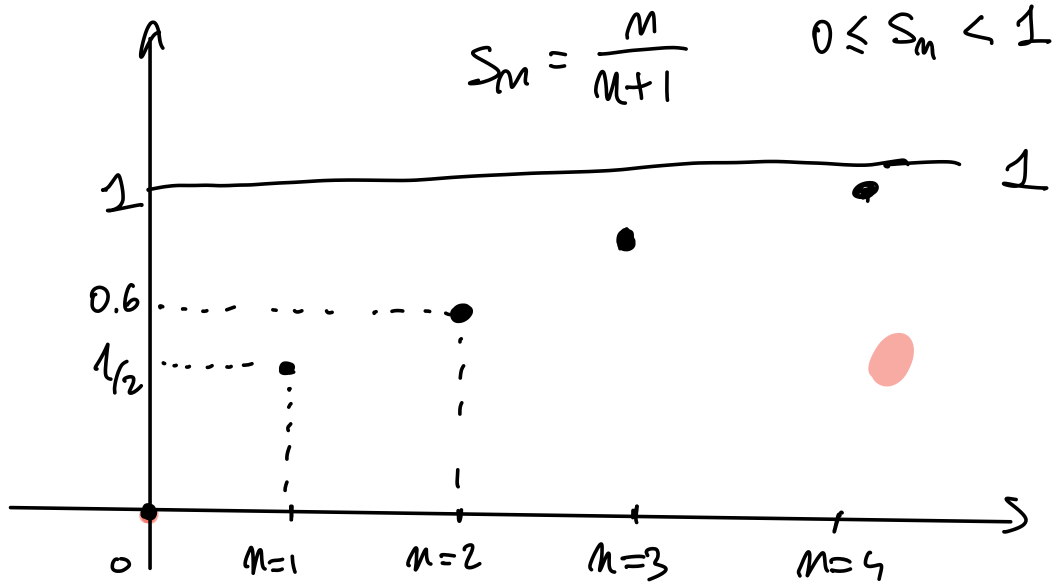
$S(n) = S_n$
THE PRICE OF A COMMODITY AT A GIVEN DATE n



$$S_n = \frac{1}{n}$$

$S_1 = \frac{1}{1} = 1 \qquad S_2 = \frac{1}{2} \qquad S_3 = \frac{1}{3} \quad \dots$

$$S_n = \frac{n}{n+1} \qquad 0 \leq S_n < 1$$



$S_0 = 0 \qquad S_1 = \frac{1}{\dots} = \frac{1}{} \qquad S_2 = \frac{2}{} = \frac{2}{} = 0.6667$

$\ldots \frac{}{1+1} \quad \frac{}{2} \quad \frac{}{2+1} \quad \frac{}{3} \ldots$

$$S_3 = \frac{3}{3+1} = \frac{3}{4} = 0.75$$

$$S_4 = \frac{4}{4+1} = \frac{4}{5} = 0.8$$

$$S_n = (-1)^n$$

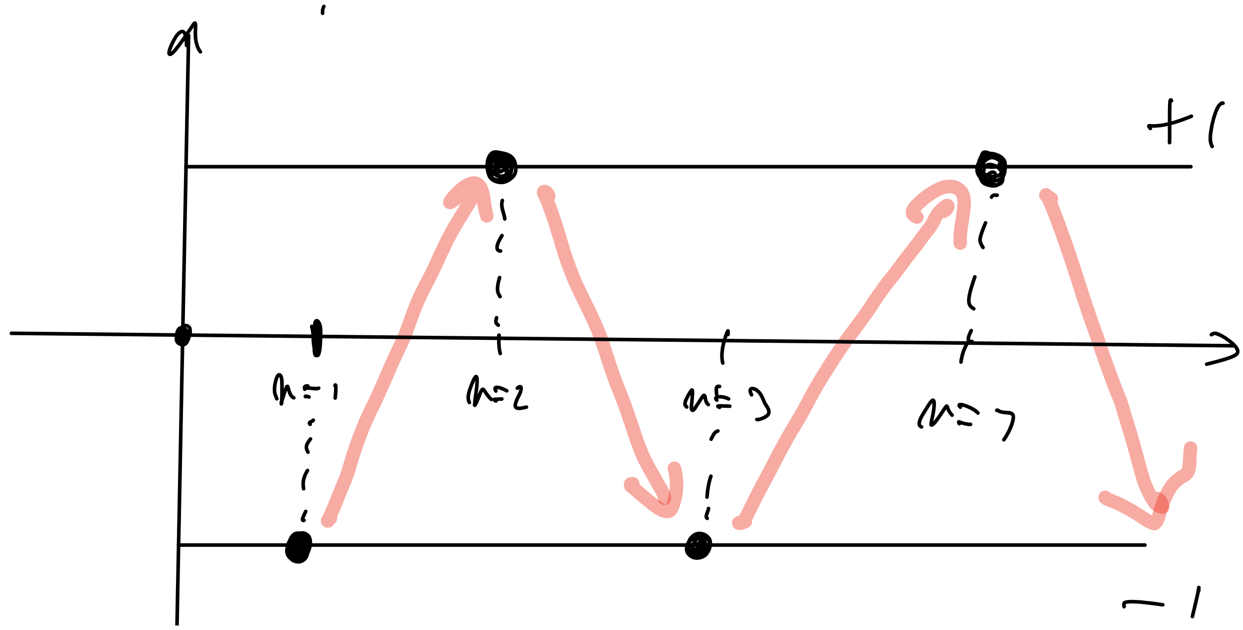
$$S_0 = (-1)^0 = +1 \qquad \boxed{S_{2n}} = +1 \quad \forall n$$

$$S_2 = (-1)^2 = +1 \qquad \boxed{S_{2n+1}} = -1 \quad \forall n$$

$$S_3 = (-1)^3 = -1$$

$$S_4 = (-1)^4 = +1$$

$$\vdots$$



|

DEF: LET $(s_n)_{n \in \mathbb{N}}$ BE A SEQUENCE OF REAL NUMBERS.

WE SAY THAT

$$\lim_{n \to +\infty} s_n = \ell$$

IF $\exists\, \ell \in \mathbb{R}$ SUCH THAT

$$\forall\, \varepsilon > 0 \quad \exists\, n_\varepsilon \in \mathbb{N} : \quad \forall\, n \geq n_\varepsilon \Rightarrow$$

$$|s_n - \ell| < \varepsilon$$

$$\Updownarrow$$

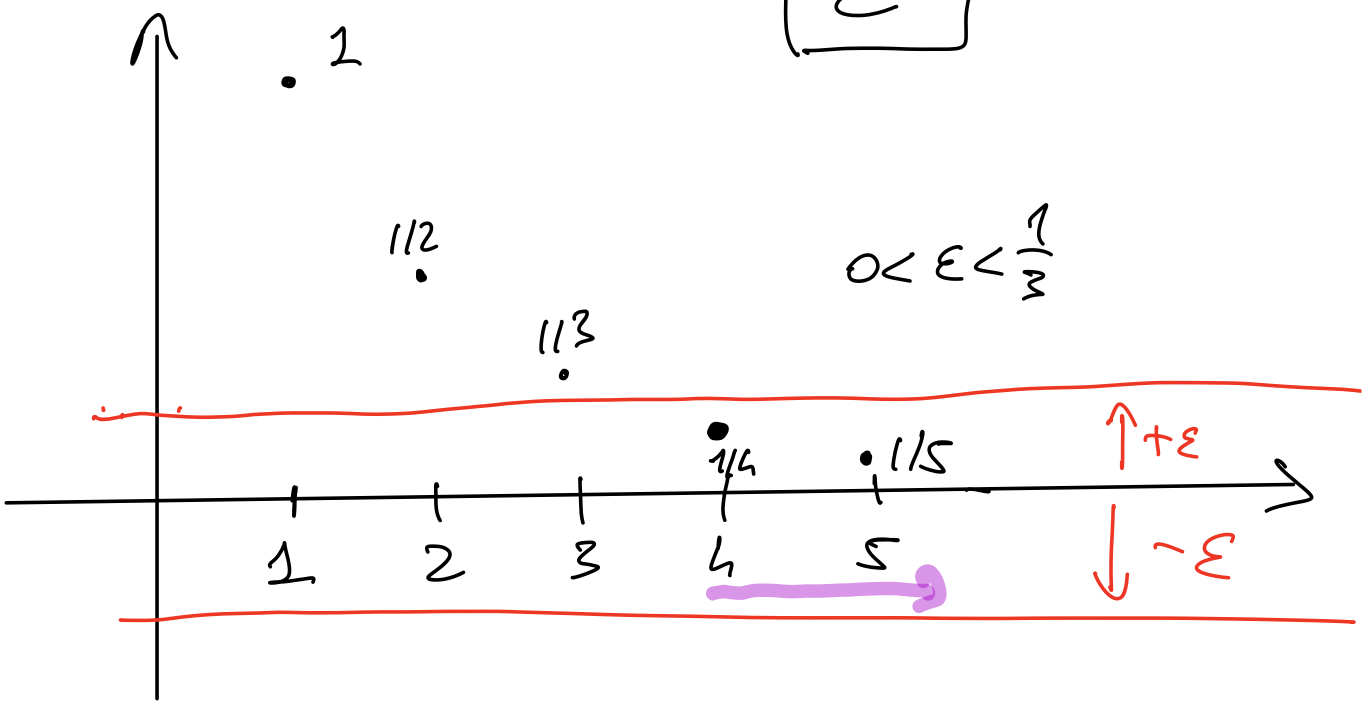
$$\ell - \varepsilon < s_n < \ell + \varepsilon$$

IN THIS CASE WE ALSO SAY

$$s_n \longrightarrow \ell$$

$$S_n = \frac{1}{n}$$

$$\boxed{\varepsilon}$$

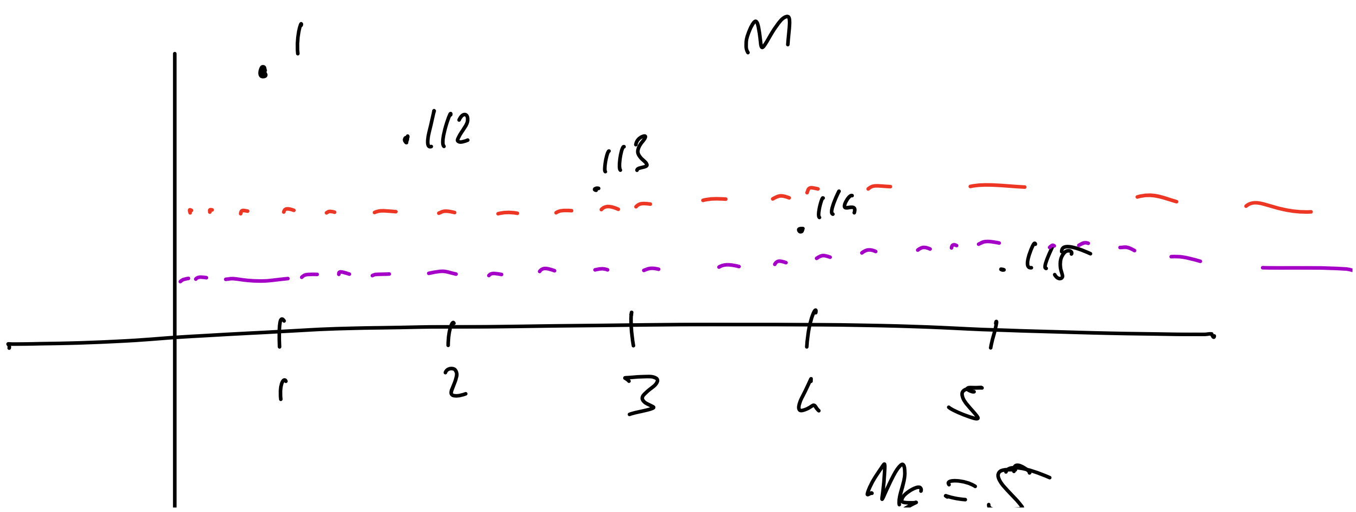


$0 < \varepsilon < \frac{1}{3}$

$n \geq 4$ $\qquad$ $n_\varepsilon = 4$

$$n \geq 4 \qquad \left|\frac{1}{n} - 0\right| < \varepsilon < \frac{1}{3}$$

$$\frac{1}{n}$$



$n_\varepsilon = 5$

$$\lim_{n\to+\infty} s_n = 0 \iff \forall \varepsilon>0\ \exists n_\varepsilon : \forall n\ge n_\varepsilon \Rightarrow |s_n - \ell| < \varepsilon$$

$$\forall \varepsilon>0\ \exists n_\varepsilon : \forall n \ge n_\varepsilon \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\iff \left|\frac{1}{n}\right| < \varepsilon$$

$$\iff \boxed{\frac{1}{n} < \varepsilon}$$

$\forall \varepsilon>0$ I TAKE n_ε AS ANY INTEGER

$$n_\varepsilon > \frac{1}{\varepsilon}$$

IF $\quad n \ge n_\varepsilon > \dfrac{1}{\varepsilon}$

$$\Rightarrow \boxed{\frac{1}{n} \le \frac{1}{n_\varepsilon} < \varepsilon}$$

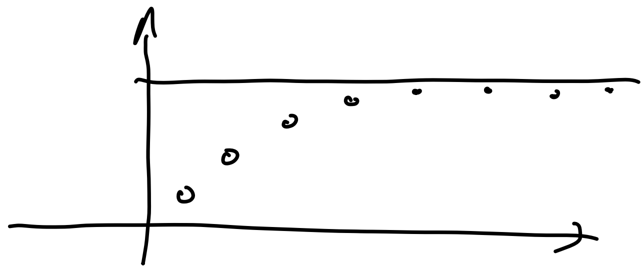
$$\lim_{n\to+\infty} \frac{1}{n} = 0$$

$$S : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longrightarrow S(n) = S_n$$

DEF: WE SAY $\lim\limits_{n \to +\infty} S_n = \ell \qquad \ell \in \mathbb{R}$

$$\Leftrightarrow \forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} : \forall n \geq n_\varepsilon \Rightarrow |S_n - \ell| < \varepsilon$$

$$S_n = \frac{n}{n+1} \qquad 0 \leq S_n < 1$$



IS IT TRUE THAT $\lim\limits_{n \to +\infty} \frac{n}{n+1} = 1$?

WE HAVE TO PROVE THAT

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} : \forall n \geq n_\varepsilon \Rightarrow \left| \frac{n}{n+1} - 1 \right| < \varepsilon$$

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{n - n - 1}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1}$$

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} : \forall n \geq n_\varepsilon \Rightarrow \frac{1}{n+1} < \varepsilon$$

I CHOOSE ANY INTEGER $m_\varepsilon > \frac{1}{\varepsilon} - 1$

$$\Rightarrow \quad m_\varepsilon + 1 > \frac{1}{\varepsilon}$$

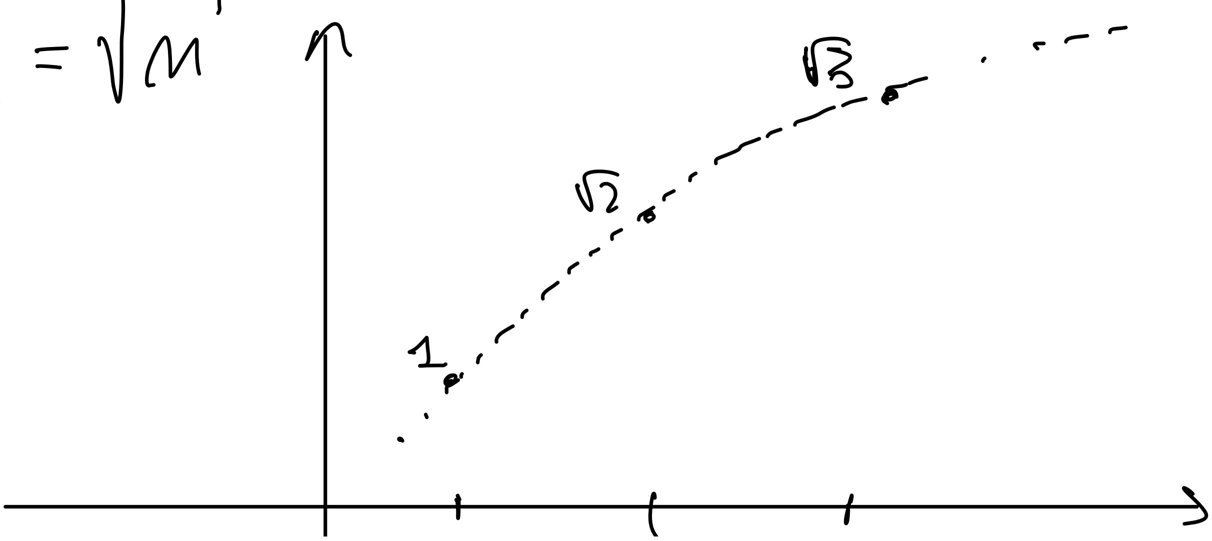
$$\Rightarrow \quad \frac{1}{m_\varepsilon + 1} < \varepsilon$$

IF $m \geq m_\varepsilon \Rightarrow m+1 \geq m_\varepsilon + 1$

$$\Rightarrow \quad \frac{1}{m+1} \leq \frac{1}{m_\varepsilon+1} < \varepsilon$$

I HAVE PROVED $\lim_{m \to +\infty} \frac{m}{m+1} = 1$

$S_m = \sqrt{m}$



| 1 2 3

DEF: GIVEN A SEQUENCE $(S_n)_{n \in \mathbb{N}}$

WE SAY THAT

$$\lim_{n \to +\infty} S_n = +\infty$$

IF $\forall M > 0 \ \exists n_M \in \mathbb{N} : \forall n \geq n_M \Rightarrow S_n \geq M$

WE SAY THAT

$$\lim_{n \to +\infty} S_n = -\infty$$

IF $\forall M > 0 \ \exists n_M \in \mathbb{N} : \forall n \geq n_M \Rightarrow S_n \leq -M$

I WANT TO PROVE THAT

$$\lim \sqrt{n} = +\infty$$

$n \to +\infty$

SO I HAVE TO PROVE THAT

$$\forall M > 0 \Rightarrow \exists n_M : \forall n \geq n_M \Rightarrow \sqrt{n} \geq M$$

FOR ANY GIVEN M I CHOOSE

$$n_M \geq M^2$$

$$n_M = M^2 + 1 \geq M^2$$

IF $n \geq n_M \geq M^2$

$$\Rightarrow \sqrt{n} \geq \sqrt{n_M} \geq \sqrt{M^2} = M$$

DEF: GIVEN $k_n : \mathbb{N} \longrightarrow \mathbb{N}$

SUCH THAT

$$k_n < k_{n+1}$$

THEN FOR ANY SEQUENCE

$$S_n : \mathbb{N} \to \mathbb{R}$$

I DEFINE A NEW SEQUENCE

$$\left(S_{k_n} \right)_{n \in \mathbb{N}}$$

THIS IS CALLED A SUB-SEQUENCE

FOR EXAMPLE

$$e_n = S_{2 \cdot n} \to \text{EVEN SEQUENCE}$$

$$O_n = S_{2n+1} \to \text{ODD SEQUENCE}$$

$$S_n = (-1)^n$$

$$S_{2n} = (-1)^{2n} = ((-1)^2)^n = 1^n = 1$$

$$S_{2n+1} = (-1)^{2n+1} = (-1)^{2n} \cdot (-1) = -1$$

$$S_{2n} \longrightarrow 1$$

$$S_{2n+1} \longrightarrow -1$$

Theo: A SEQUENCE $(S_n)_{n \in \mathbb{N}}$

IS SUCH THAT

$$S_n \longrightarrow \ell$$

IF AND ONLY IF

$$S_{n_k} \longrightarrow \ell$$

FOR ALL THE POSSIBLE SUB-SEQUENCES

$\to M_K$

THIS SEQUENCE $S_n = (-1)^n$

DOES NOT HAVE A LIMIT

BECAUSE $|S_n| = +1 \longrightarrow +1$

$$S_{2n} = +1 \longrightarrow +1$$

$$S_{2n+1} = -1 \longrightarrow -1$$

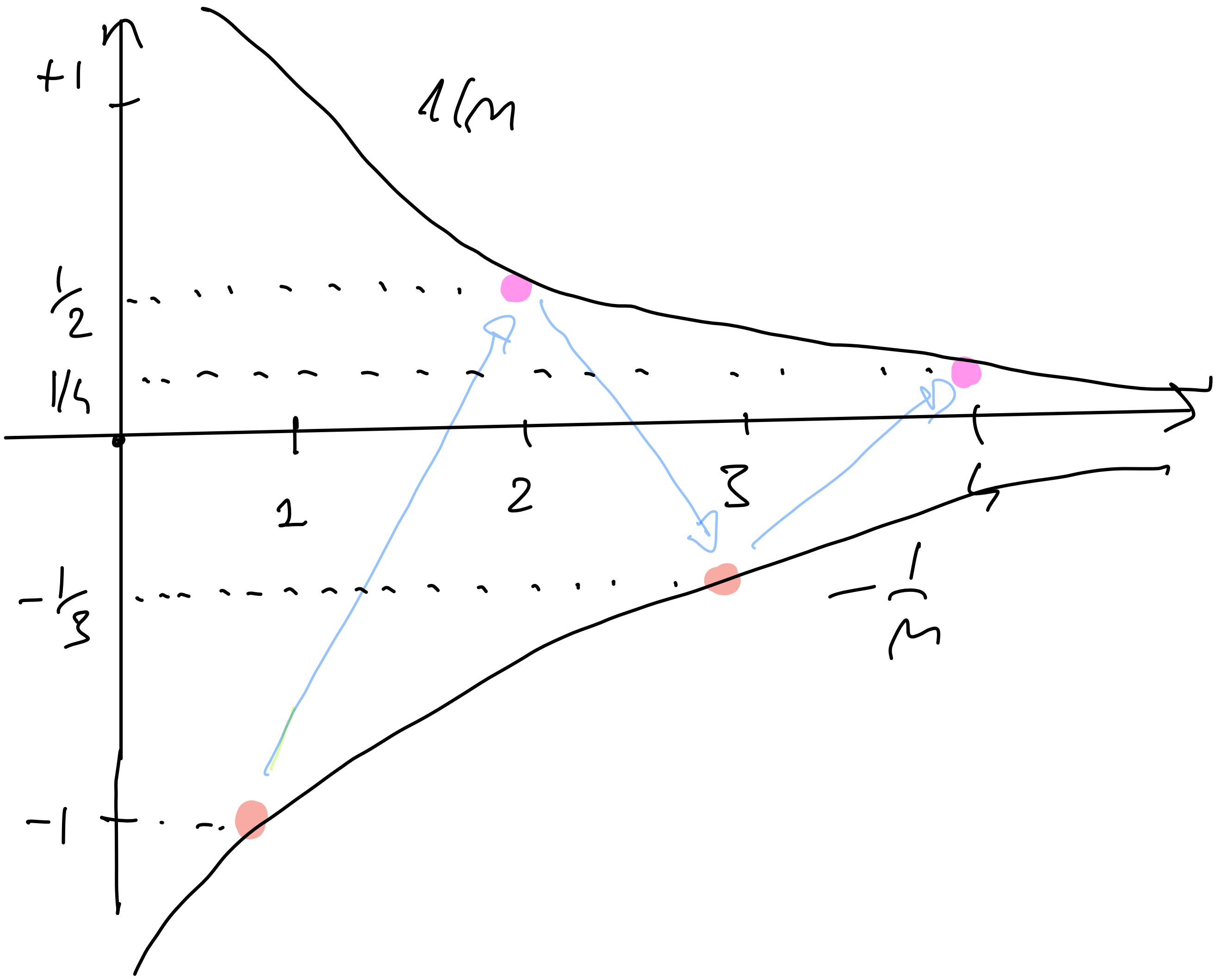
Theo:

IF $|S_n| \longrightarrow 0$ THEN $S_n \longrightarrow 0$

$$S_n = \frac{(-1)^n}{n}$$

$S_1 = -1 \quad S_2 = +\frac{1}{2} \quad S_3 = -\frac{1}{3}$

$S_4 = +\frac{1}{4} \quad S_5 = -\frac{1}{5}$



$$|s_n| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \longrightarrow 0$$

$$\Rightarrow s_n \longrightarrow 0$$

$(-1)^n$

$$\lim_{n \to +\infty} \frac{(-1)}{n} = 0$$

COMPARISON THEOREM.

$(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$, $(c_n)_{n \in \mathbb{N}}$

THREE SEQUENCES SUCH THAT

$$\underline{b_n \leq a_n \leq c_n}$$

$$\forall\, n \geq n_0$$

THEN IF $b_n \longrightarrow \ell$ AND

$c_n \longrightarrow \ell$

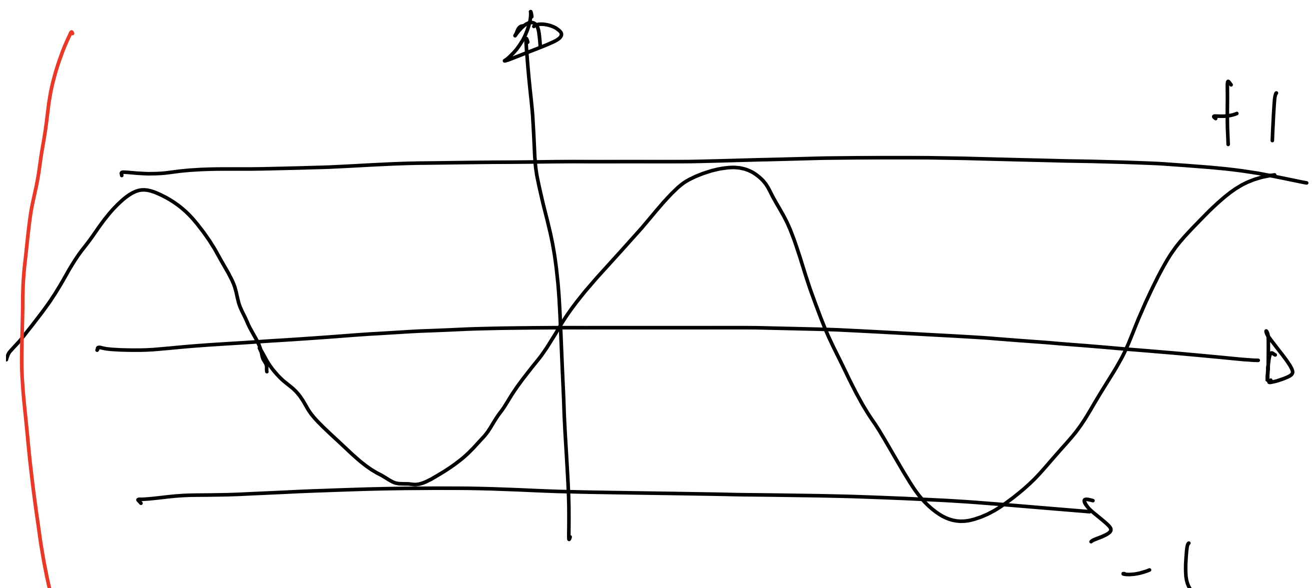
WE HAVE $a_n \longrightarrow \ell$

IF $b_n \longrightarrow +\infty \Rightarrow a_n \longrightarrow +\infty$

IF $c_n \longrightarrow -\infty \Rightarrow a_n \longrightarrow -\infty$

$$S_n = \frac{\sin(n)}{n}$$

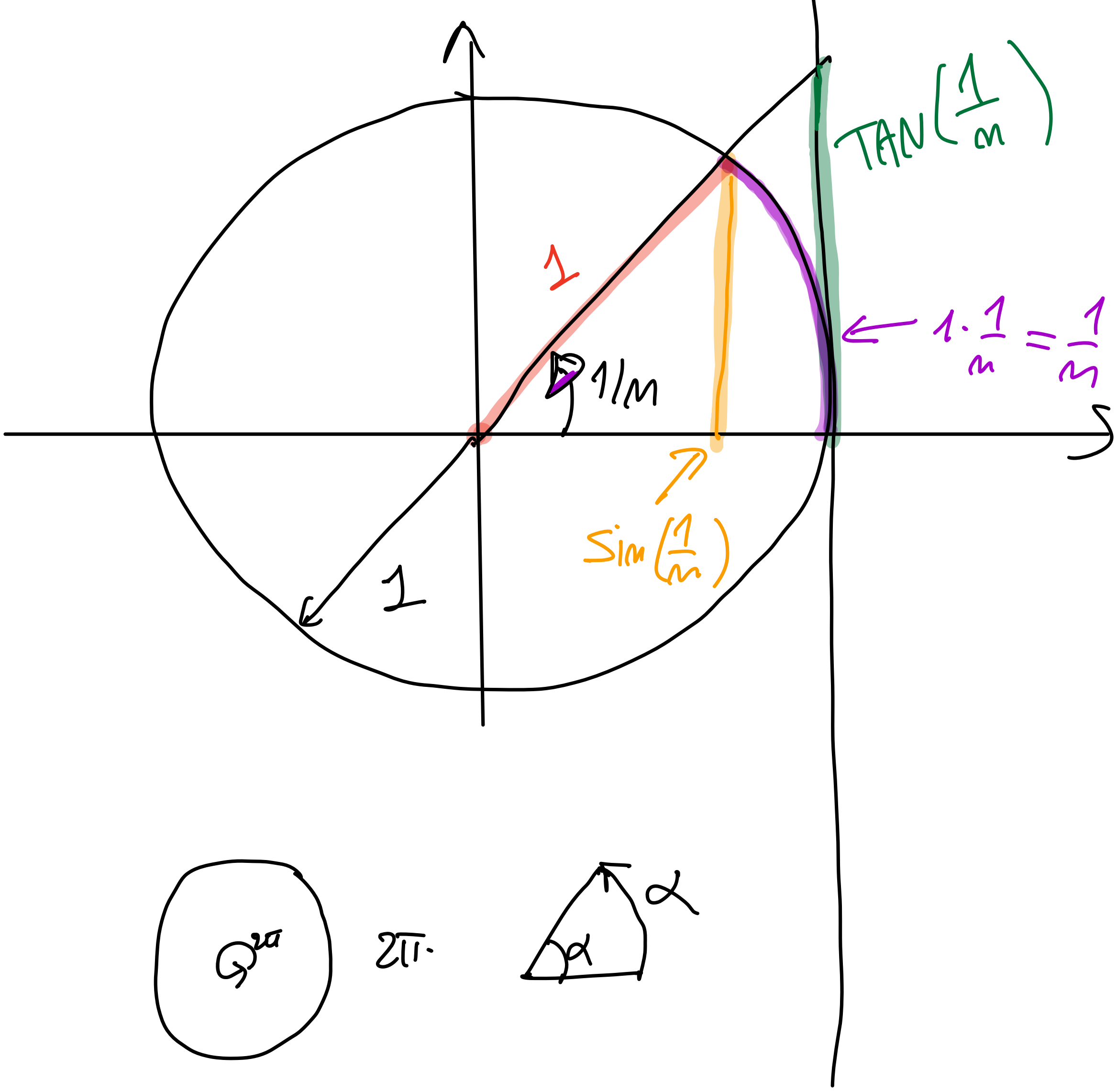
$-1 \le \sin(n) \le +1 \qquad \forall\, n \in \mathbb{N}$



$$\underbrace{-\frac{1}{n}}_{\to 0} \le \frac{\sin(n)}{n} \le \underbrace{\frac{1}{n}}_{\to 0}$$

$$\Rightarrow \lim_{n \to +\infty} \frac{\sin(n)}{n} = 0$$

$$S_n = n \cdot \sin\left(\frac{1}{n}\right)$$



$$0 \leq \sin\left(\frac{1}{n}\right) \leq \frac{1}{n} \leq \tan\left(\frac{1}{n}\right)$$

$$0 \leq \sin\left(\frac{1}{n}\right) \leq \frac{1}{n} \leq \frac{\sin(1/n)}{\cos(1/n)}$$

$$\frac{1}{\sin\left(\frac{1}{n}\right)} \geq n \geq \frac{\cos(1/n)}{\sin\left(\frac{1}{n}\right)} \geq 0$$

$$\underbrace{\frac{\cancel{\sin\left(\frac{1}{n}\right)}}{\cancel{\sin\left(\frac{1}{n}\right)}}}_{=1} \geq n \cdot \sin\left(\frac{1}{n}\right) \geq \frac{\cos\left(\frac{1}{n}\right)}{\cancel{\sin\left(\frac{1}{n}\right)}} \cancel{\sin\left(\frac{1}{n}\right)}$$

$$\rightarrow \quad 1 \geq n \cdot \sin\left(\frac{1}{n}\right) \geq \cos\left(\frac{1}{n}\right)$$

$$\cos\left(\frac{1}{n}\right) \leq n \cdot \sin\left(\frac{1}{n}\right) \leq 1 \qquad \forall n \geq 1$$

$$n \to +\infty \Rightarrow \frac{1}{n} \longrightarrow 0$$

$$\cos\left(\frac{1}{n}\right) \longrightarrow \cos(0) = 1$$

$$\underbrace{\cos\left(\frac{1}{n}\right)}_{\downarrow\, 1} \leq \underbrace{n \cdot \sin\left(\frac{1}{n}\right)}_{\downarrow\, 1} \leq 1 \longrightarrow 1$$

$$\lim_{n \to +\infty} n \cdot \sin\left(\frac{1}{n}\right) = 1$$

1) IF $s_n \to l$ AND $p_n \to l'$

$$s_n + p_n \longrightarrow l + l'$$

THE CONVERSE IS NOT TRUE!!

$$(-1)^n - (-1)^n = 0 \longrightarrow 0$$

$$s_n \cdot p_n \to l \cdot l'$$

MOREOVER IF $l' \neq 0$ THEN

$$\frac{S_n}{P_n} \to \frac{l}{l'}$$

MORE GENERALLY I CAN SUBSTITUTE $+\infty$ IN PLACE OF n IN THE DEFINITION OF THE SEQUENCE PROVIDED THAT I FALL IN ONE OF THESE CASES:

1) $c + \infty = +\infty$ $\quad$ c A CONSTANT

$c - \infty = -\infty$

2) $+\infty + \infty = +\infty$

$$-\infty - \infty = -\infty$$

3) $(+\infty) \cdot (+\infty) = +\infty$

$(+\infty) \cdot (-\infty) = -\infty$

4) $\dfrac{c}{+\infty} = \dfrac{c}{-\infty} = 0$ IF c IS A CONSTANT

5) $0^{+\infty} = 0$

6) $|c| \cdot (+\infty) = +\infty$ $\boxed{c \neq 0}$

$|c| \cdot (-\infty) = -\infty$

CASES NOT DEFINED

OR

INDETERMINATE FORMS

$+\infty - \infty$?

$-\infty + \infty$?

$0 \cdot (+\infty)$?

$0 \cdot (-\infty)$?

$1^{+\infty}$?

$$\left(1+\frac{1}{n}\right)^{n} = 1^{+\infty} \ ?$$

$(\pm\infty)^{0}$? 0^{0} ?

$$S_n = \frac{1}{n}$$

$$\lim_{n \to +\infty} \frac{1}{n} = \frac{1}{+\infty} = 0$$

$$S_n = \sqrt{n^2+n} - n$$

$$\sqrt{+\infty^2+\infty} - \infty = +\infty - \infty \; ??$$

$$\lim_{n\to+\infty} \frac{\left(\sqrt{n^2+n} - n\right)\left(\sqrt{n^2+n} + n\right)}{\left(\sqrt{n^2+n} + n\right)}$$

$$= \lim \frac{\left(\sqrt{n^2+n}\right)^2 - n^2}{}$$

$$n \to +\infty \quad \sqrt{n^2+n} + n$$

$$= \lim_{n \to +\infty} \frac{\cancel{n^2} + n - \cancel{n^2}}{\sqrt{n^2+n} + n}$$

$$= \lim_{n \to +\infty} \frac{n}{\underline{\sqrt{n^2+n}} + n} = \frac{+\infty}{+\infty} \; ?$$

$$= \lim_{n \to +\infty} \frac{n}{\sqrt{n^2\left(1+\frac{1}{n}\right)} + n}$$

$$= \lim_{n \to +\infty} \frac{n}{n\sqrt{1+\frac{1}{n}} + n}$$

$$= \lim_{n\to+\infty} \frac{\cancel{n}}{\cancel{n}\left[\sqrt{1+\frac{1}{n}}+1\right]}$$

$$= \lim_{n\to+\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

$$\lim_{n\to+\infty} \frac{n^7-6n^8+\sin(n)}{n^8} =$$

$$\lim_{n\to+\infty} \frac{\cancel{n^8}\left(\frac{1}{n}-6+\frac{\sin(n)}{n^8}\right)}{\cancel{n^8}}$$

$$=\lim_{n\to+\infty}\left(\underbrace{\frac{1}{n}}_{0}\underbrace{-6}_{-6}+\underbrace{\frac{\sin(n)}{n^8}}_{0}\right)=-6$$

$\log_a(x)$ IS THE INVERSE OF a^x

$0 < a < 1 \qquad a > 1$

$a^{x+y} = a^x \cdot a^y \Rightarrow \log_a(x \cdot y) = \log_a(x) + \log_a(y)$

$2^{3+5} = 2^3 \cdot 2^5 = 2^8$

$a^{xy} = (a^x)^y \Rightarrow \log_a(x^y) = y \log_a(x)$

$\log_a\left(\frac{x}{y}\right) = \log_a(x \cdot y^{-1}) =$

$= \log_a(x) + \log_a(y^{-1})$

$= \log_a(x) - \log_a(y)$

$\log_a(\sqrt{x}) = \log_a(x^{1/2}) = \frac{1}{2}\log_a(x)$

$S_n = q^n \qquad q \in \mathbb{R} \qquad q \neq 1$

$$\lim_{n \to +\infty} q^n = \begin{cases} +\infty & q > 1 \\ 0 & -1 < q < 1 \\ \nexists & q \leq -1 \end{cases}$$

$0 \leq q < 1 \Rightarrow 0 \leq q^n < 1$

$q = \frac{1}{100} \qquad q^2 = \frac{1}{100^2} \qquad q^5 = \frac{1}{100^5}$

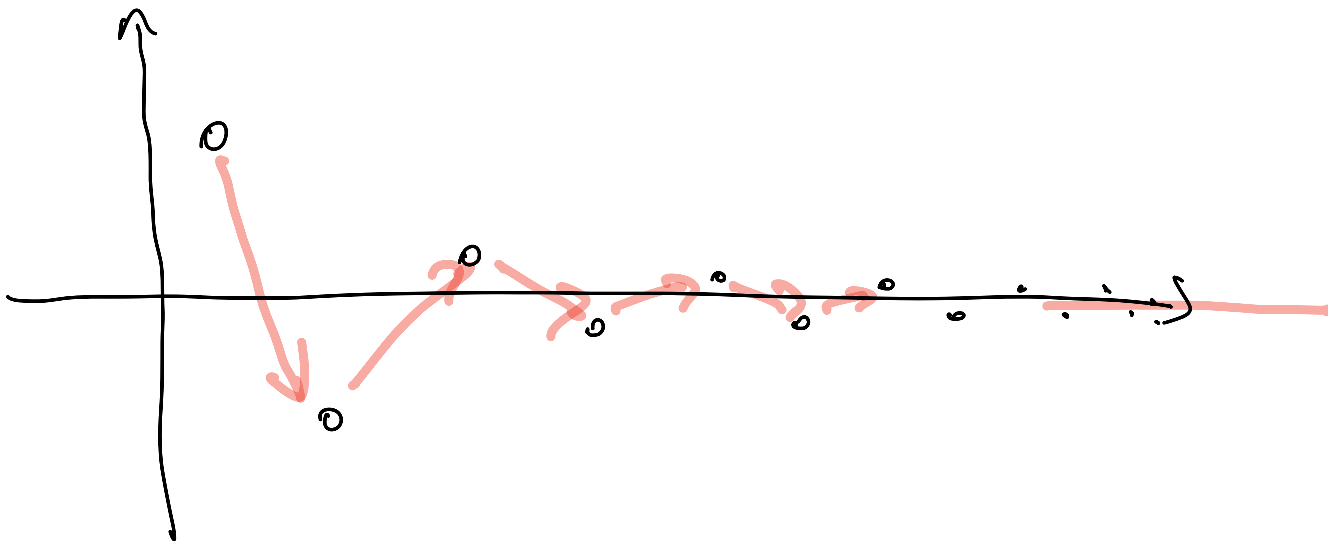
$\to q^n \longrightarrow 0$

$-1 < q < 0 \Rightarrow q = -|q| \qquad 0 < |q| < 1$

$q^n = (-1)^n \underbrace{|q|^n}$

$\hookrightarrow 0$

$$e_n = \left(-\frac{1}{2}\right)^n$$



$$\lim_{n\to+\infty} (-2)^n = \nexists$$

$$s_n = (-2)^n$$

$$s_{2n} = (-2)^{2n} = +(4)^n \longrightarrow +\infty$$

$$s_{2n+1} = (-2)^{2n+1} = -2\,(4)^n \longrightarrow -\infty$$

$$\lim_{n \to +\infty} (-1)^n = \nexists$$

$$\lim_{n \to +\infty} \left(-\frac{1}{10}\right)^n = 0$$

$$(-2)^{2n+1} = (-2)^{2n} \cdot (-2)^{+1} \to -\infty$$

$$\begin{cases} n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 \\ 0! \equiv 1 \end{cases}$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$\lim_{n \to +\infty} n! = +\infty$$

2) $\lim\limits_{n\to+\infty} n^b = +\infty$ $\quad b>0$

$\sqrt{n} \quad n^{1/3} \quad n^2 \quad n^3 \quad n^{100}$ $\quad a>1$

1) $\lim\limits_{n\to+\infty} \log_a(n) = +\infty$

3) $\lim\limits_{n\to+\infty} a^n = +\infty$

4) $\lim\limits_{n\to+\infty} n! = +\infty$

5) $\lim\limits_{n\to+\infty} n^n = +\infty$

$$\lim_{n\to+\infty} \frac{\log_a(n)}{n^b} = 0 = \frac{\infty}{\infty}$$

$0 < a < 1 \qquad b > 0$

$a > 1$

n

$$\lim_{n \to +\infty} \frac{n}{a^n} = 0 = \frac{\infty}{\infty}$$

$$\lim_{n \to +\infty} \frac{a^n}{n!} = 0 = \frac{\infty}{\infty}$$

$$\lim_{n \to +\infty} \frac{n!}{n^n} = 0 = \frac{\infty}{\infty}$$

$$\lim_{n \to +\infty} \frac{4^n + \log_7(n)}{n! + n^2}$$

$$\lim_{n \to +\infty} \frac{4^n \left[1 + \frac{\log_7(n)}{4^n}\right]}{n! \left[1 + \frac{n^2}{n!}\right]}$$

$$= \lim_{n \to +\infty} \frac{4^n}{n!} = 0$$

$$\lim_{n\to+\infty} \frac{n! + n^{1000000000000000}}{n^n} = 0$$

DEF: A SEQUENCE IS CALLED STRICTLY INCREASING IF

$$s_n \overset{<}{\leq} s_{n+1} \quad \forall n$$

A SEQUENCE IS CALLED STRICTLY DECREASING IF

$$s_n \overset{>}{\geq} s_{n+1} \quad \forall n$$

Theo: LET s_n BE AN INCREASING SEQUENCE.

THEN

$$\lim_{n\to+\infty} s_n = \ell \iff s_n \text{ IS BOUNDED FROM ABOVE}$$

$$\exists M : S_n \leq M \quad \forall n$$

LET S_n BE A DECREASING SEQUENCE. THE

$$\lim_{n \to +\infty} S_n = l \iff S_n \text{ IS BOUNDED FROM BELOW}$$

$$\exists M : S_n \geq M \quad \forall n$$

$$S_n = \left(1 + \frac{1}{n}\right)^n$$

$$S_1 = 2 \quad S_2 = \left(1 + \frac{1}{2}\right)^2 \quad S_3 = \left(1 + \frac{1}{3}\right)^3 \quad \dots\dots$$

IT CAN BE PROVED THAT

$$\boxed{2 \leq \left(1 + \frac{1}{n}\right)^n < 3}$$

IT CAN BE PROVED THAT

$$S_n < S_{n+1}$$

$$\left(1+\frac{1}{n}\right)^{n} < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\exists \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n} = e = 1^{\infty}$$

EULER NUMBER

$$e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n} = \lim_{n\to+\infty}\boxed{\sum_{k=0}^{n}\frac{1}{k!}}$$

$$= \sum_{k=0}^{+\infty}\frac{1}{k!} = 2.78\ldots$$

$$1^{\infty}$$

$$\|$$

$$\lim_{n\to+\infty}\left(1+\frac{7}{n^2}\right)^{n^2} = \lim_{n\to+\infty}\left(1+\frac{1}{\frac{n^2}{7}}\right)^{n^2}$$

$$= \lim_{n\to+\infty}\left[\left(1+\frac{1}{\frac{n^2}{7}}\right)^{\frac{n^2}{7}}\right]^{7}$$

$\lim_{n\to+\infty} \left[\left(\quad \frac{n}{7} \right) \right]$

$m = \frac{n^2}{7}$

$$= \lim_{m\to+\infty} \left[\left(1+\frac{1}{m}\right)^m \right]^7$$

$$= \lim_{m\to+\infty} \left[\left(1+\frac{1}{m}\right)^m \right]^7 = e^7$$

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = 1^{+\infty} = e$$

$$\lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^n = \qquad n = (-)(-n)$$

$$\lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^{(-n)(-1)} =$$

$$= \lim_{n \to +\infty} \left(1 + \frac{1}{(-n)}\right)^{(-n)(-1)} = e^{-1} = \frac{1}{e}$$

$$\lim_{n \to +\infty} \left(1 + \frac{2}{n}\right)^{n} =$$

$$= \lim_{n \to +\infty} \left[\left(1 + \frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right]^{2} = e^{2}$$

$\boxed{\text{SERIES}}$

$$\sum_{k=0}^{3} k^2 = 0^2 + 1^2 + 2^2 + 3^2$$

$$(a_k)_{k \in \mathbb{N}}$$

n

$$S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \dots + a_n$$

SEQUENCE OF THE PARTIAL SUMS

$$S_0 = a_0$$
$$S_1 = a_0 + a_1$$
$$S_2 = a_0 + a_1 + a_2$$
$$\vdots$$
$$S_n = a_0 + a_1 + a_2 + \dots + a_n$$

IF $n \to +\infty$ $S_n \to \ell$ WE SAY THAT THE SERIES

$$\sum_{k=0}^{+\infty} a_k = \ell$$

OR THAT THE SERIES $\sum_{k=0}^{+\infty} a_k$ CONVERGES TO ℓ

$Q_k = \frac{1}{k} \quad Q_1 = 1 \quad Q_2 = \frac{1}{2} \quad Q_3 = \frac{1}{3}$

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$$\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$$

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

$$\sum_{k=0}^{+\infty} \frac{1}{k!} = e$$

$S_3(x) = 1 + x + x^2 + x^3$

$\forall\, x \in \mathbb{R}$

$S_{10}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots + x^{10}$

$$S_n(x) = x^0 + x^1 + x^2 + \dots + x^n$$

$$= 1 + x + x^2 + \dots + x^n$$

n

$$= \sum_{k=0}^{n} x^k$$

GEOMETRIC SEQUENCE

$$x \cdot S_n(x) = x \cdot (1 + x + x^2 + \dots + x^n)$$

$$x \cdot S_n(x) = x + x^2 + x^3 + \dots + x^{n+1}$$

$$x S_n(x) - S_n(x) = \cancel{x} + \cancel{x^2} + \dots + x^{n+1} - (1 + \cancel{x} + \cancel{\dots} + x^n)$$

$$x S_n(x) - S_n(x) = x^{n+1} - 1$$

$$S_n(x)(x-1) = x^{n+1} - 1$$

$$\boxed{S_n(x) = \frac{1 - x^{n+1}}{1 - x}}$$

COMPUTE THE NUMBER

$$S_7(2) = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$$

$$= \frac{1-2^8}{1-2} = \frac{1-2^8}{(-1)} = 2^8 - 1$$

SUMMARY:

$\forall x \in \mathbb{R} \quad \forall n \in \mathbb{N}$

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1-x}$$

FOR WHICH $x \in \mathbb{R}$ WE HAVE THAT

$$\exists \lim_{n \to +\infty} \sum_{k=0}^{n} x^k = ?$$

FOR WHICH $x \in \mathbb{R}$ WE HAVE THAT

$$\exists \lim_{n \to +\infty} \frac{1-x^{n+1}}{1-x} = \begin{cases} +\infty & x > 1 \\ \frac{1}{1-x} & -1 < x < 1 \\ \nexists & x \leq -1 \end{cases}$$

$x^{n+1} \to 0$

$$\sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots = \frac{1}{1-\frac{1}{2}}$$

$$x = \frac{1}{2} \in (-1, 1) \qquad = \frac{1}{\frac{1}{2}} = 2$$

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots = 2$$

$$\sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

$$x = -\frac{1}{2} \in (-1, 1)$$

$$\sum_{n=0}^{+\infty} (-2)^k = \nexists \qquad x = -2$$

$$\sum_{n=0}^{+\infty} 2^k = +\infty$$

GEOMETRIC SERIES

$$S_n(x) = 1 + x + x^2 + \dots + x^n = \sum_{k=0}^{n} x^k$$

$$S_n(x) = \frac{1 - x^{n+1}}{1-x} \qquad \forall x \neq 1 \quad \forall n$$

$$\Rightarrow \lim_{n \to +\infty} S_n(x) = \lim_{n \to +\infty} \sum_{k=0}^{n} x^k$$

$$= \begin{cases} +\infty & x > 1 \\ \dfrac{1}{1-x} & |x| < 1 \\ \nexists & x \leq -1 \end{cases}$$

$$\lim_{n \to +\infty} \sum_{k=0}^{n} x^k = \sum_{k=0}^{+\infty} x^k$$

$$x = \frac{1}{2} \cdot \quad |x| < 1$$

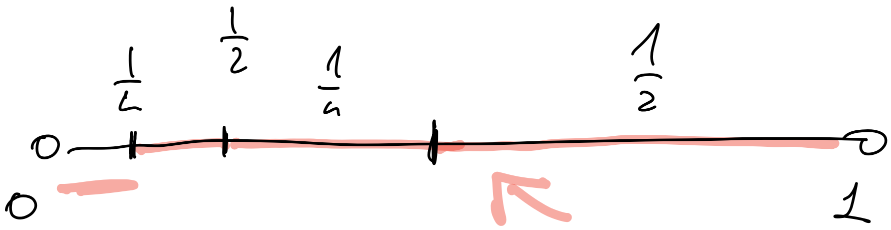
$$\sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1 - \frac{1}{2}} = 2$$

$$\sum_{k=1}^{+\infty} \left(\frac{1}{2}\right)^k = \sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^k - \overbrace{\left(\frac{1}{2}\right)^0}^{=1}$$

$$= \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n - 1$$

$$= 2 - 1 = 1$$

$$\sum_{n=1}^{+\infty} \left(\frac{1}{2}\right)^n = 1$$



$$l_n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$$

$$\sum_{n=1}^{+\infty} \left(\frac{1}{2}\right)^n = 1$$

$$0{,}999999\ldots\ldots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \ldots$$

LIMITS OF FUNCTIONS

$f: D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \frac{2x^2 - 8}{x - 2}$$

$D = \{x \in \mathbb{R} \mid x \neq 2\}$

$$\nexists f(2) = \frac{2 \cdot 4 - 8}{2 - 2} = \frac{0}{0}$$

x	$f(x)$
2.1	8.2
2.01	8.02
2.001	8.002
2.0004	8.0008

x	$f(x)$
1.9	7.8
1.95	7.9
1.995	7.99
1.9991	7.9982

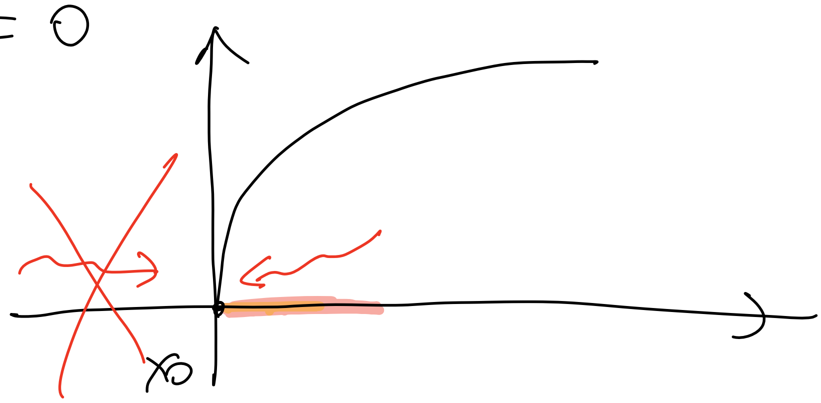
X APPROCHES 2 FROM ABOVE THIS FUNCTION GETS CLOSER TO 8

X APPROCHES 2 FROM BELOW STILL THE FUNCTION GETS CLOSER TO 8

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon : \underset{x \neq 2}{|x-2| < \delta} \Rightarrow |f(x) - 8| < \varepsilon$

I MUST BE ABLE TO APPROACH THE NUMBER 2 FROM POINTS IN THE DOMAIN

$f(x) = \sqrt{x}$ WE WANT TO ANALYZE THE BEHAVIOUR OF f AROUND $x_0 = 0$



THE FINITE LIMIT OF A FUNCTION IN A POINT.

GIVEN $f: D \subseteq \mathbb{R} \to \mathbb{R}$ WE SAY THAT

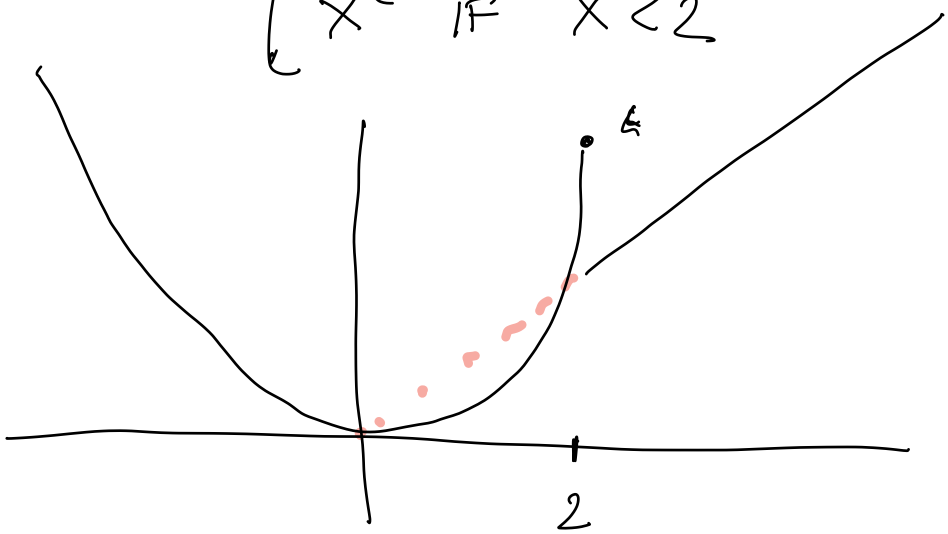
$$\lim_{x \to x_0} f(x) = L \Leftrightarrow$$

$$\forall \varepsilon > 0 \Rightarrow \exists \delta_\varepsilon : \begin{cases} |x - x_0| < \delta_\varepsilon \\ x \neq x_0 \end{cases} \Rightarrow |f(x) - L| < \varepsilon$$

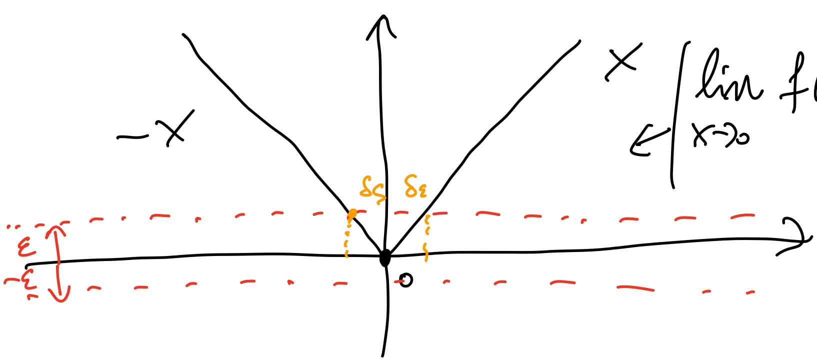
$$\forall \varepsilon > 0 \Rightarrow \exists \delta_\varepsilon : 0 < |x - x_0| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$$

PIECE-WISE DEFINED FUNCTIONS

$$f(x) = \begin{cases} x & \text{IF } x \geq 2 \\ x^2 & \text{IF } x < 2 \end{cases}$$

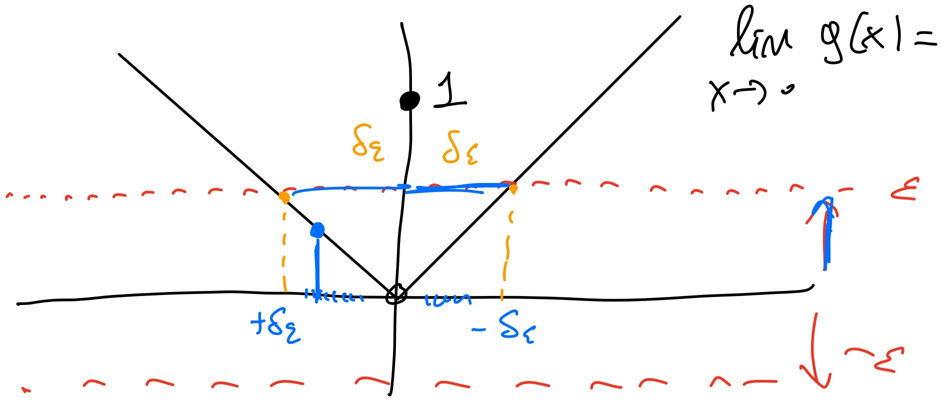


$$f(x) = \begin{cases} -x & \text{IF } x \leq 0 \\ +x & \text{IF } x \geq 0 \end{cases} = |x|$$



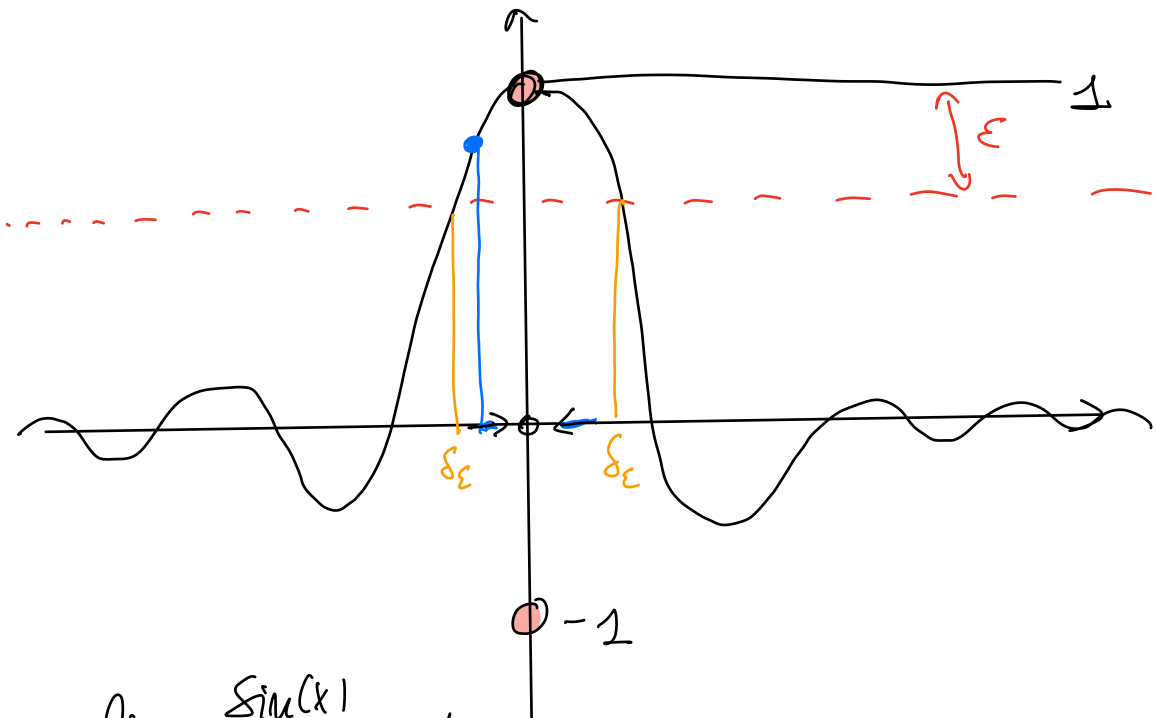
$\lim_{x \to 0} f(x) = 0$

$$g(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ +x & \text{if } x > 0 \end{cases}$$



$$\lim_{x \to 0} g(x) = 0$$

$$f(x) = \frac{\sin(x)}{x} \qquad D = \mathbb{R} \setminus \{0\}$$



$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

RIGHT AND LEFT LIMITS.

WE SAY THAT

$$\lim_{x \to x_0^+} f(x) = L$$

$$\Updownarrow$$

$$\forall \varepsilon > 0 \Rightarrow \exists \delta_\varepsilon : \underset{x \in D}{0 < x - x_0} < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$$

WE SAY THAT

$$\lim_{x \to x_0^-} f(x) = L$$

$$\Updownarrow$$

$$\forall \varepsilon > 0 \Rightarrow \exists \delta_\varepsilon : \underset{x \in D}{0 < x_0 - x} < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$$

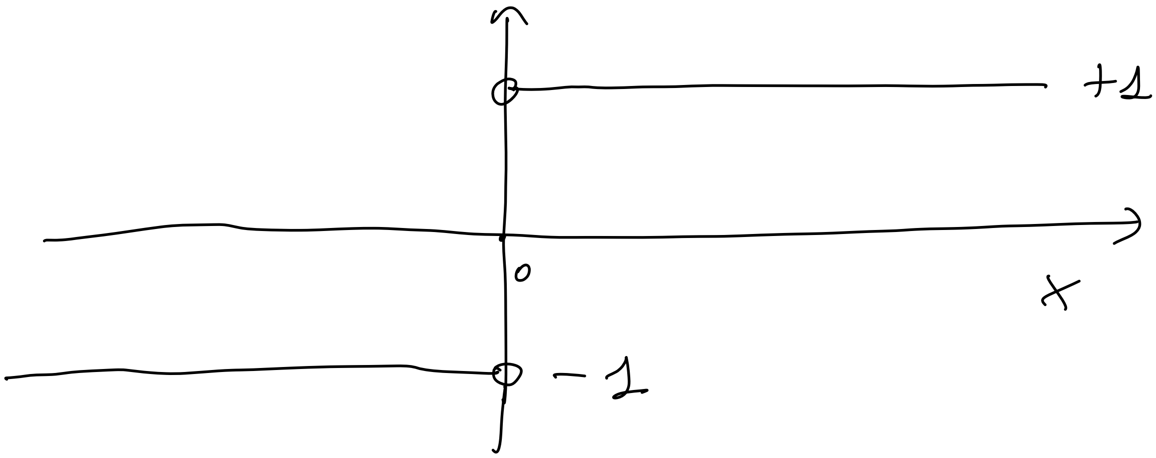
Theo: $\lim_{x \to x_0} f(x) = L \Longleftrightarrow \begin{cases} \lim_{x \to x_0^+} f(x) = L \\ \lim_{x \to x_0^-} f(x) = L \end{cases}$

$$f(x) = \qquad = \frac{|x|}{x} = \frac{x}{|x|} \qquad D = \mathbb{R} \setminus \{0\}$$

$|-2|$?

$$f(-2) = \frac{|-2|}{-2} = \frac{2}{-2} = -1$$

$$f(+3) = \frac{|3|}{3} = \frac{3}{3} = 1$$



if $x > 0 \Rightarrow \frac{|x|}{x} = \frac{+x}{x} = 1$

if $x < 0 \Rightarrow \frac{|x|}{x} = \frac{-x}{x} = -1$

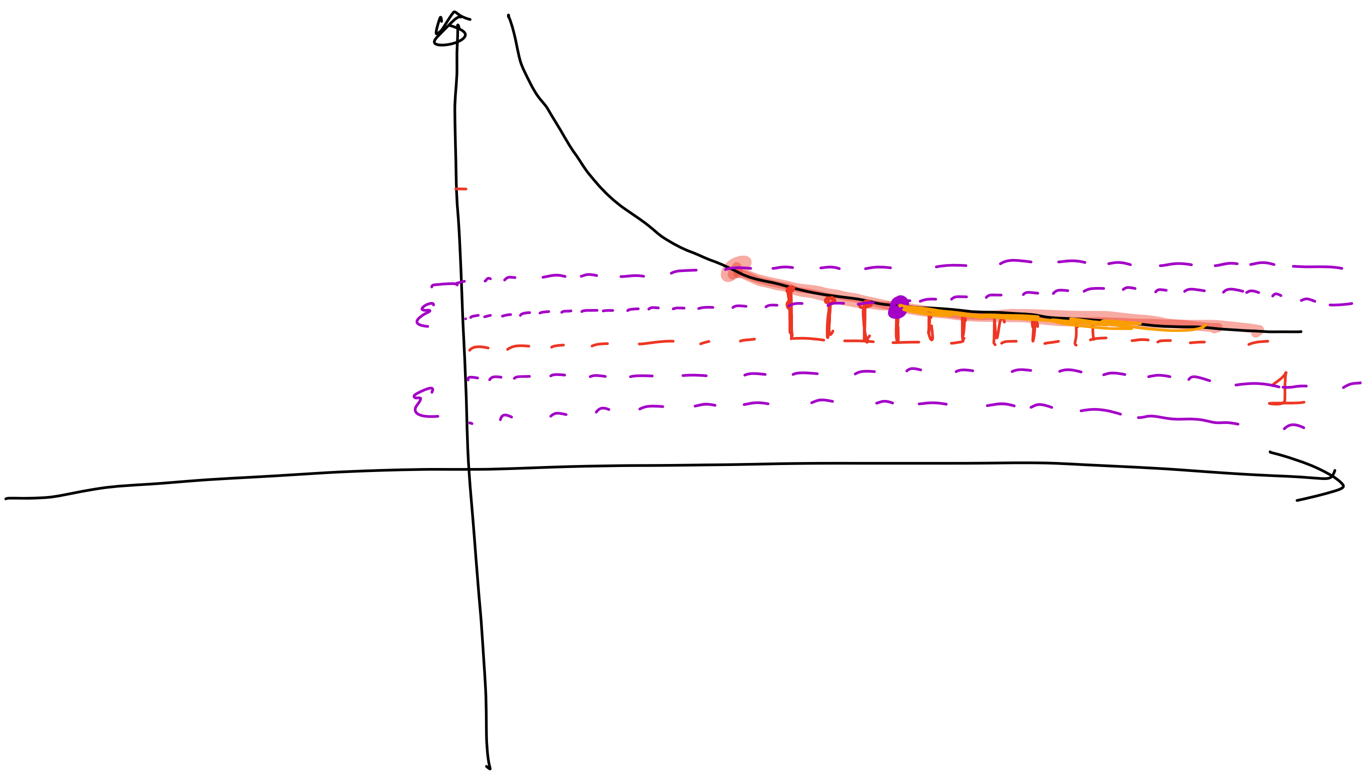
$$\lim_{x \to 0^+} \frac{|x|}{x} = +1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1$$

$$\lim_{x \to 0} \cancel{\frac{|x|}{x}} = ?$$

$$\boxed{\lim_{x \to x_0} f(x) = L}$$

$$f(x) = \frac{x+1}{x} \qquad D = \mathbb{R} \setminus \{0\}$$



FINITE LIMITS AT INFINITY

LET $f : D \subseteq \mathbb{R} \to \mathbb{R}$

$D = [a, +\infty)$ OR $(a, +\infty)$

THE DOMAIN OF THE FUNCTION MUST BE UNBOUNDED FROM ABOVE

WE SAY THAT

$$\lim_{x \to +\infty} f(x) = L$$

HORIZONTAL ASYMPTOTE

$$\Updownarrow$$

$$\forall \varepsilon > 0 \Rightarrow \exists M_\varepsilon > 0 : \forall x \geq M_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$$

SIMILARLY .....

$D = (-\infty, a)$ OR $D = (-\infty, a]$

UNBOUNDED FROM BELOW

WE SAY THAT

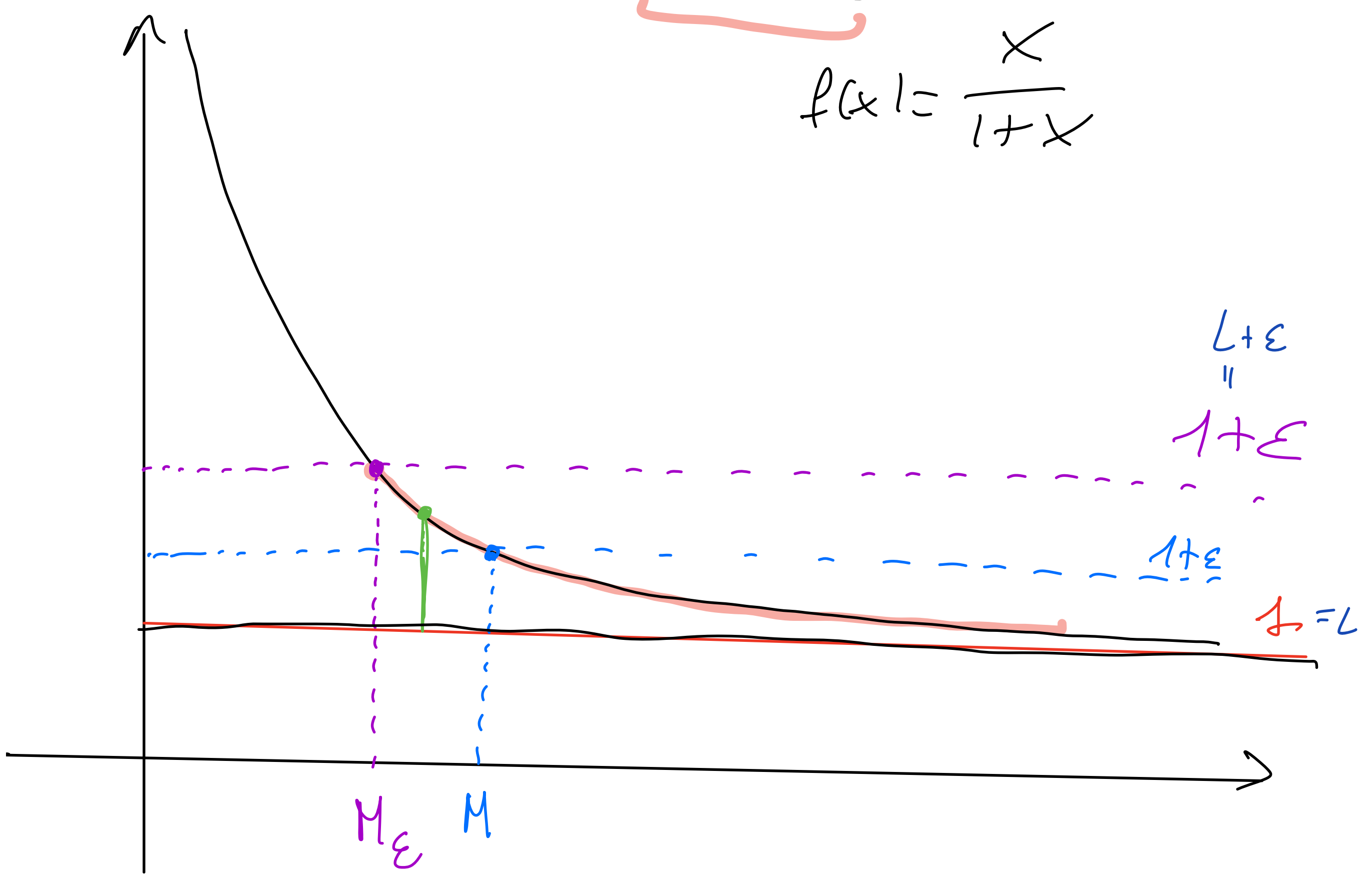
$$\lim_{x \to -\infty} f(x) = L$$

HORIZONTAL ASYMPTOTE

$$\Updownarrow$$

$$\forall \varepsilon>0 \Rightarrow \exists M_\varepsilon>0 : \forall \underbrace{x \le -M_\varepsilon} \Rightarrow |f(x)-L|<\varepsilon$$

$$f(x)=\frac{x}{1+x}$$



$$|f(x)-L|<\varepsilon \Leftrightarrow L-\varepsilon<f(x)<L+\varepsilon$$

$$f(x)=\frac{1}{(x+5)^2} \qquad D=\mathbb{R}\setminus\{-5\}$$

x	f(x)

x	f(x)

-4.9	100		-5.1	100
-4.95	400		-5.05	400
-4.99	1000		-5.005	40000
-4.999	100000		-5.0005	4000000

INFINITE LIMIT IN A POINT.

LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$

WE SAY THAT

$$\lim_{x \to x_0} f(x) = \overset{-\infty}{+\infty}$$

$$\Updownarrow$$

$$\forall K > 0 \Rightarrow \exists \delta_k > 0 : \forall \overset{x \in D}{x} : 0 < |x - x_0| < \delta_k$$

$$\Rightarrow f(x) \geq K$$

$$f(x) \leq -K$$

IN THIS CASE WE SAY THAT $f(x)$ HAS A VERTICAL ASYMPTOTE w x_0

$$\lim_{x\to x_0^{+}} f(x) = +\infty \quad (-\infty) \iff$$

$$\forall K>0 \Rightarrow \exists\, \delta_K>0 : \forall x : 0 < \overset{x_0 - x}{x - x_0} < \delta_K$$

$$\Downarrow$$

$$f(x) \geq K$$

$$f(x) \leq -K$$

$$f(x) = \frac{1}{x}$$

$$\lim_{x\to 0^{+}} \frac{1}{x} = +\infty$$

$$\lim_{x\to 0^{-}} \frac{1}{x} = -\infty \quad \Rightarrow \nexists \lim_{x\to 0} \frac{1}{x}$$

$$f(x) = \frac{1}{x^2}$$

$$\lim_{x \to 0^+} \frac{1}{x^2} = +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x^2} = +\infty \quad \Rightarrow \lim_{x \to 0} \frac{1}{x} = +\infty$$

$$\lim_{\substack{x \to +\infty \\ -\infty}} f(x) = +\infty \quad -\infty$$

$x \le -M_k$

$$\forall K > 0 \Rightarrow \exists M_k > 0 : \forall x : x \ge M_k \Rightarrow f(x) \ge K$$

$f(x) \le -K$

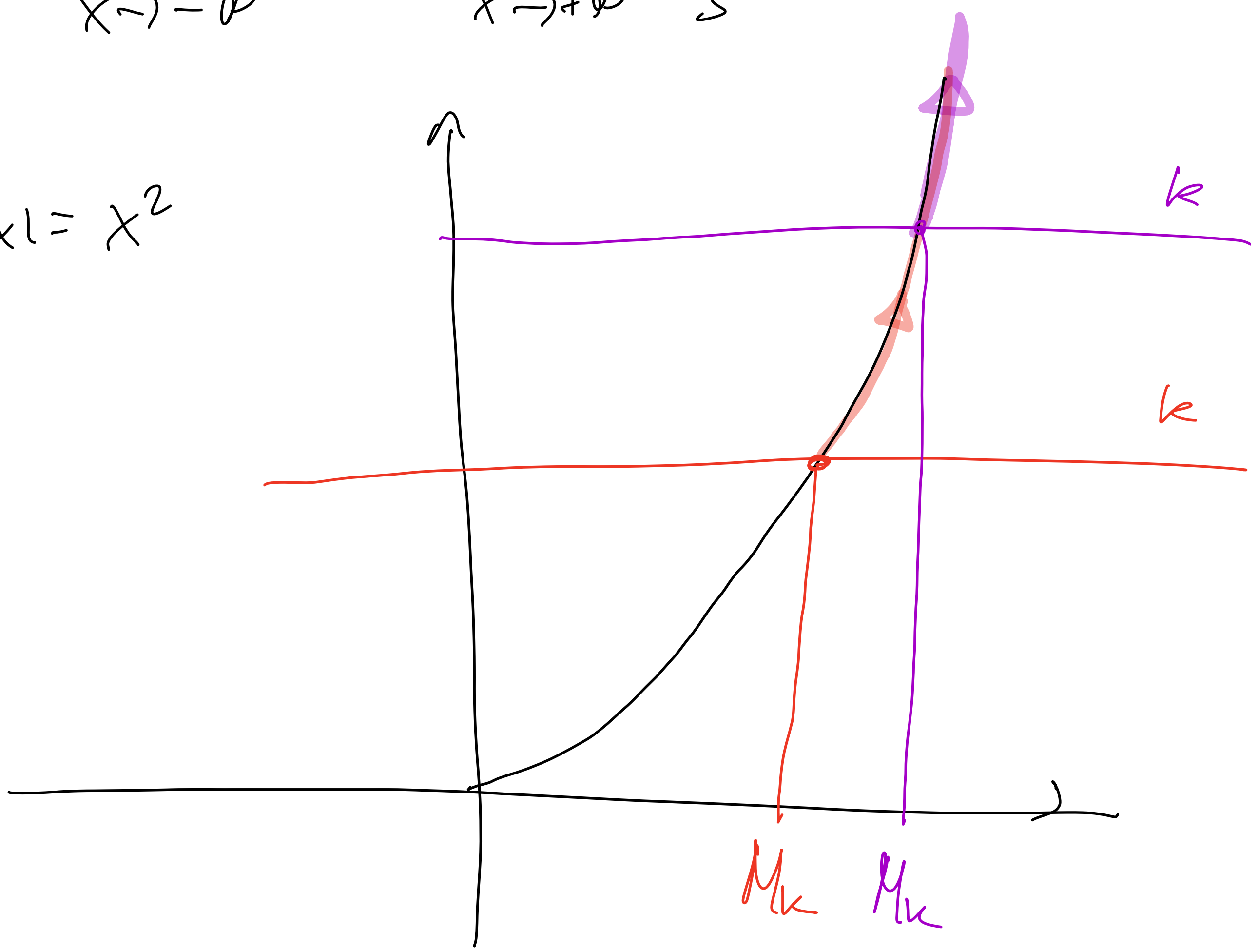
$$f(x) = 3^x$$

$$\lim_{x \to +\infty} 3^x = +\infty$$

$\nearrow$

$$\lim_{x \to -\infty} 3^x = \lim_{x \to +\infty} \frac{1}{3^x} = 0$$

$f(x) = x^2$



$$\lim_{x \to +\infty} x^2 = +\infty$$

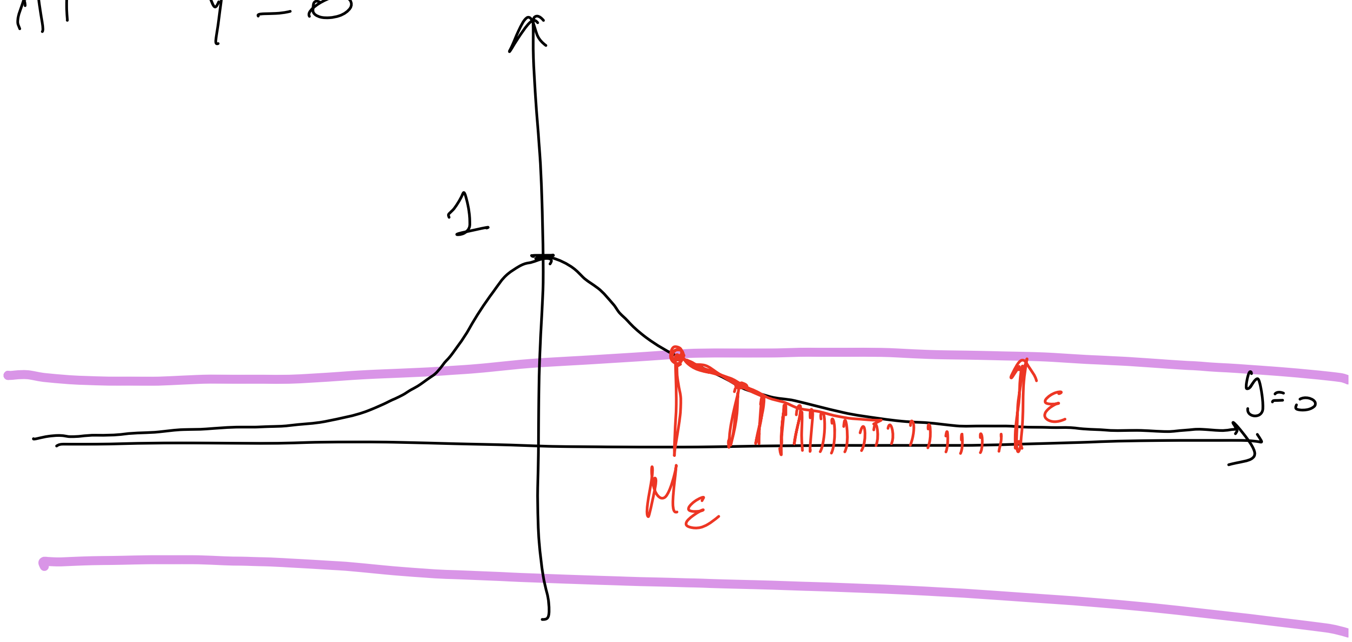
$$\lim_{x \to -\infty} x^2 = +\infty$$

$$f(x) = \frac{1}{1+x^2}$$

$$\lim_{x \to +\infty} \frac{1}{1+x^2} = 0$$

WE HAVE AN HORIZONTAL ASYMPTOTE

AT $y = 0$



HOW TO COMPUTE LIMITS IN PRACTICE

OR

PRACTICAL RULES FOR LIMITS.

SUPPOSE THAT I HAVE TO COMPUTE

$$\lim_{x \to x_0} f(x)$$

x_0 CAN BE ALSO $\pm \infty$

I CAN TRY BY SUBSTITUTION

$$f(x_0)$$

IF WHAT I GET IS NOT AN INDETERMINATE FORM

I AM DONE.

REFER TO THE LECTURE ON LIMITS OF SEQUENCES

NOTABLE LIMITS

1) $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n} = e$

$\quad = 1^{\infty}$

$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}$ if $x<0 \Rightarrow x=-|x|$

$$= \lim_{x\to-\infty}\left(1-\frac{1}{|x|}\right)^{-|x|} = \lim_{x\to-\infty}\left(\frac{|x|-1}{|x|}\right)^{-|x|}$$

$$= \lim_{x\to-\infty}\left(\frac{|x|}{|x|-1}\right)^{+|x|}$$

$$= \lim_{x\to-\infty}\left(\frac{|x|-1+1}{|x|-1}\right)^{+|x|}$$

$$= \lim_{x \to -\infty} \left(1 + \frac{1}{|x|-1}\right)^{|x|} \quad 1)$$

$$= \lim_{x \to -\infty} \left(1 + \frac{1}{|x|-1}\right)^{|x|-1+1} \quad 2)$$

$$= \lim_{x \to -\infty} \underbrace{\left(1+\frac{1}{|x|-1}\right)^{|x|-1}} \cdot \underbrace{\left(1+\frac{1}{|x|-1}\right)^{+1}} \quad 3)$$

$\boxed{|x|-1 = y} \to +\infty$

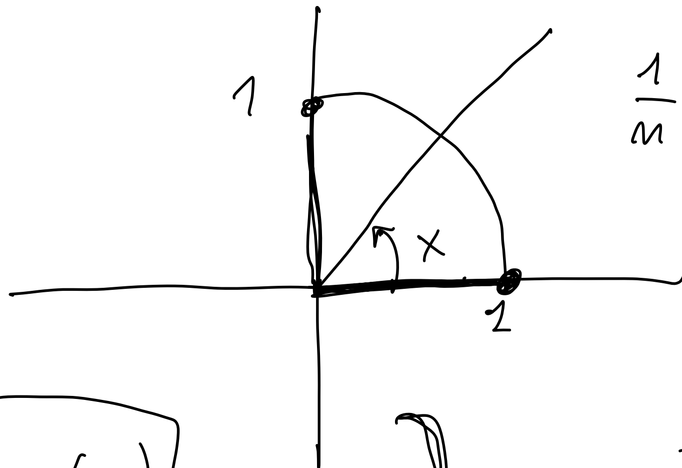
$$= \lim_{y \to +\infty} \left(1+\frac{1}{y}\right)^{y} \cdot \left(1+\frac{1}{y}\right)^{1} \quad 4)$$

$x = y$

$$= \lim_{x \to +\infty} \underbrace{\left(1+\frac{1}{x}\right)^{x}}_{\downarrow\, e} \cdot \underbrace{\left(1+\frac{1}{x}\right)}_{\downarrow\, 1}$$

$$= e \cdot 1 = e$$

$$\lim_{n\to+\infty} n\cdot \sin\left(\frac{1}{n}\right) = 1$$



$$\frac{1}{n} = x \to 0$$

$$\lim_{x\to 0} \boxed{\frac{\sin(x)}{x}} = 1$$

$$(a+b)(a-b) = a^2 - b^2$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \frac{1-\cos(0)}{0} = \frac{1-1}{0} = \frac{0}{0} = 0$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x}\left(\frac{1+\cos(x)}{x}\right)\left(\frac{x}{1+\cos(x)}\right)$$

$$= \lim_{x\to 0} \frac{(1-\cos(x))(1+\cos(x))}{x^2}\,\frac{x}{1+\cos(x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2(x)}{x^2}\,\frac{x}{1+\cos(x)}$$

1)

$$= \lim_{x\to 0} \frac{\sin^2(x)}{x^2} \frac{x}{1+\cos(x)} \quad 2)$$

$$= \lim_{x\to 0} \underbrace{\left(\frac{\sin(x)}{x}\right)^2}_{1} \frac{x^{\to 0}}{\underbrace{1+\cos(x)}_{2}} = 1 \cdot \frac{0}{2} = 0$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1-\cos(0)}{0^2} = \frac{1-1}{0} = \frac{0}{0} = \frac{1}{2}$$

$$\shortparallel$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} \frac{1+\cos(x)}{1+\cos(x)} \qquad (a-b)(a+b) = a^2-b^2$$

$$= \lim_{x\to 0} \frac{(1-\cos(x))(1+\cos(x))}{x^2} \cdot \frac{1}{1+\cos(x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2(x)}{x^2} \frac{1}{1+\cos(x)}$$

$$= \lim_{x \to 0} \frac{\sin^2(x)}{x^2} \quad \frac{1}{1+\cos(x)}$$

$$= \lim_{x \to 0} \underbrace{\left(\frac{\sin(x)}{x}\right)^2}_{1} \underbrace{\frac{1}{1+\cos(x)}}_{\frac{1}{2}} = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \frac{\sin(x)}{\underbrace{\cos(x)}_{1} \cdot x} = 1$$

2

$\log_e(x) = \ln(x)$ NATURAL LOGARITHM

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \frac{\ln(1+0)}{0} = \frac{\ln(1)}{0} = \frac{0}{0} = 1$$

$\frac{1}{}$

$$\frac{\ln(1+x)}{x} = \frac{1}{x}\cdot\ln(1+x) = \ln(1+x)^{x}$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} \qquad t = \frac{1}{x}$$

$$= \lim_{t\to\infty} \ln\left(1+\frac{1}{t}\right)^{t} = \ln(e) = 1$$

$$\lim_{t\to\infty} \left(1+\frac{1}{t}\right)^{t} = e$$

$$\lim_{x\to 0} 3\cdot\frac{\sin(3x)}{3x} = \lim_{y\to 0} 3\cdot\frac{\sin(y)}{y} = 3$$

$$y = 3x$$

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^3} = \lim_{x\to 0^{\pm}} \underbrace{\frac{\sin(x^2)}{x^2}}_{1}\cdot\frac{1}{x} = \pm\infty$$

$$\lim_{x\to 0} \frac{\sin(x^3)}{x^2} = \lim_{x\to 0} x \cdot \underbrace{\boxed{\frac{\sin(x^3)}{x^3}}}_{1} = 0$$

$b>1$

$$\boxed{\lim_{x\to +\infty} \frac{\log_b(x)}{x^\alpha} = 0 = \frac{\infty}{\infty}}$$

$\alpha>0$

$$\lim_{x\to +\infty} \frac{x^\alpha}{b^x} = 0$$

$$\lim_{x\to +\infty} x^{\frac{1}{x}} = \lim_{x\to +\infty} e^{\ln\left(x^{\frac{1}{x}}\right)} =$$

$$f(x) = e^{\ln(f(x))}$$

$$= \lim_{x \to +\infty} e^{\frac{1}{x}\ln(x)} = e^{0} = 1$$

$$\frac{\ln(x)}{x} \xrightarrow[x \to +\infty]{} 0$$

$$\lim_{n \to +\infty} \left(\frac{n^2+3}{n^2} \right)^{2n^2} \qquad \left(1+\frac{1}{n}\right)^n$$

$$\left(\frac{n^2+3}{n^2} \right)^{2n^2} = \left(1+\frac{3}{n^2}\right)^{2n^2} = \left(1+\frac{1}{\frac{n^2}{3}}\right)^{\frac{2n^2}{3}\cdot 3}$$

$$= \left[\left(1+\frac{1}{\frac{n^2}{3}}\right)^{\frac{n^2}{3}}\right]^6 = \Big[\underbrace{\left(1+\frac{1}{m}\right)^m}_{e}\Big]^{6} \to e^6$$

$$m = \frac{n^2}{3} \qquad \left(1+\frac{1}{m}\right)^m$$

$$\lim_{n \to +\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \qquad (a-b)(a+b) = a^2 - b^2$$

$$\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$= \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\underbrace{\sqrt{n+1}+\sqrt{n}}_{+\infty}} \to 0$$

$$\lim_{n \to +\infty} \left(1+\frac{1}{n^2}\right)^n \qquad \left(1+\frac{1}{m}\right)^m$$

$$\left(1+\frac{1}{n^2}\right)^n = \left(1+\frac{1}{n^2}\right)^{\frac{n^2}{n}} \qquad n = n\cdot\frac{n}{n} = \frac{n^2}{n}$$

$$= \Big[\underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}_{e}\Big]^{\frac{1}{n}} \to e^0 = 1$$

$$\lim_{n\to+\infty} \frac{\sin(n)+2n^2}{3n-n^2} \Rightarrow$$

$$\frac{\sin(n)+2n^2}{3n-n^2} = \frac{\cancel{n^2}\left[\underbrace{\frac{\sin(n)}{n^2}}_{\to 0}+2\right]}{\cancel{n^2}\left(\underbrace{\frac{3}{n}}_{\to 0}-1\right)}$$

$$\to -2$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{-n^2} \qquad e^{-n} = \frac{1}{e^n}$$

$$n^2 = n\cdot n$$

$$\left(1+\frac{1}{n}\right)^{-n^2} = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}} =$$

$$= \frac{1}{\left[\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e}\right]^{n}} \to \frac{1}{e^{+\infty}} = 0$$

$$\lim_{n\to+\infty}\left(\frac{n^2+2}{n^2}\right)^{n^2}$$

$$\left(\frac{n^2+2}{n^2}\right)^{n^2} = \underbrace{\left(1+\frac{1}{\frac{n^2}{2}}\right)^{\frac{n^2}{2}}}_{e}{}^{\cdot 2} \to e^2$$

$$\left(\frac{n^2+5}{n^2}\right)^{\frac{3}{5}n^2} =$$

$$= \left[\left(1+\frac{1}{\frac{n^2}{5}}\right)^{\frac{n^2}{5}}\right]^{3} \to e^3$$

$$\left(\frac{n^2+2}{n^2}\right)^{2n^2} = \left(1+\frac{1}{n}\right)^{n}$$

$$= \left(1+\frac{2}{n^2}\right)^{2n^2} = \left(1+\frac{1}{\boxed{\frac{n^2}{2}}}\right)^{\boxed{\frac{2n^2}{2}}\cdot 2}$$

$$= \left[\left(1+\frac{1}{\frac{n^2}{2}}\right)^{\frac{n^2}{2}}\right]^{4} \longrightarrow e^{4}$$

$$n\sqrt{\frac{1}{n+1}} = \sqrt{\frac{n^2}{n+1}} \to \sqrt{+\infty} = +\infty$$

$$\left(1+\frac{2}{\sqrt{n}}\right)^{n} \qquad \sqrt{n} = n^{1/2}$$

$$m = \sqrt{n}$$

$$\Rightarrow n = m^2$$

$$\left(1+\frac{2}{m}\right)^{m^2}$$

$$\left[\left(1+\frac{2}{m}\right)^{m}\right]^{m} \to e^{+\infty} = +\infty$$

$$\left(1+\frac{2}{\sqrt{n}}\right)^{n} = \left[\left(1+\frac{1}{\frac{\sqrt{n}}{2}}\right)^{\sqrt{n}}\right]^{\sqrt{n}}$$

$$n = \sqrt{n}\cdot\sqrt{n}$$

$$\left(1+\frac{2}{\sqrt{n}}\right)^{n} = \left[\left(1+\frac{1}{\frac{\sqrt{n}}{2}}\right)^{\frac{\sqrt{n}}{2}}\right]^{2\sqrt{n}}$$

$$\left(1+\frac{2}{\sqrt{n}}\right)^{n}=\left(1+\frac{1}{\frac{\sqrt{n}}{2}}\right)^{n}$$

$$=\underbrace{\boxed{\left(1+\frac{1}{\frac{\sqrt{n}}{2}}\right)^{\frac{\sqrt{n}}{2}}}}_{e}{}^{\cdot\sqrt{n}\cdot 2}$$

$$\longrightarrow e^{+\infty}=+\infty$$

$$n\log\left(1+\frac{1}{n}\right)=\log\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e}\to\log(e)=1$$

$$\left(\frac{\sqrt{n}-1}{\sqrt{n}}\right)^{2\sqrt{n}}$$

$$(-1)\cdot(-1)=+1$$

$$=\left(1-\frac{1}{\sqrt{n}}\right)^{2\sqrt{n}}=$$

$$2\sqrt{n}=(-\sqrt{n})\cdot(-2)$$

$$=\left(1+\frac{1}{-\sqrt{n}}\right)^{2\sqrt{n}}=$$

$$2=-(-2)$$

$$=\underbrace{\boxed{\left(1+\frac{1}{-\sqrt{n}}\right)^{(-\sqrt{n})}}}_{e}{}^{\cdot(-2)}\to e^{-2}=\frac{1}{e^{2}}$$

$$\lim_{n\to+\infty} \frac{5^n - n^5}{4^n + n^6} =$$

$$= \lim_{n\to+\infty} \frac{5^n\left[1-\frac{n^5}{5^n}\right]}{4^n\left[1+\frac{n^6}{4^n}\right]} = \lim_{n\to+\infty} \frac{5^n}{4^n}$$

$\frac{n^5}{5^n} \to 0$, $\frac{n^6}{4^n} \to 0$

$$= \lim_{n\to+\infty} \left(\frac{5}{4}\right)^n$$

$$a = \frac{5}{4} > 1$$

$$= +\infty$$

$$\lim_{n\to+\infty} a^n = \begin{cases} +\infty & a > 1 \\ 0 & 0 < a < 1 \end{cases}$$

$$\frac{\left(\sqrt{2n+1} - \sqrt{3n+5}\right)\left(\sqrt{2n+1} + \sqrt{3n+5}\right)}{\sqrt{2n+1} + \sqrt{3n+5}}$$

$$(a+b)(a-b) = a^2 - b^2$$

$$= \frac{2n+1-3n-5}{\sqrt{2n+1} + \sqrt{3n+5}}$$

$$= \frac{-n-4}{\sqrt{2n+1} + \sqrt{3n+5}} \sim \frac{-n}{\sqrt{n}} \to -\infty$$

$$= \frac{-n\left[1+\frac{4}{n}\right]}{\sqrt{n^2\left[\frac{2}{n}+\frac{1}{n^2}\right]} + \sqrt{n^2\left[\frac{3}{n}+\frac{5}{n^2}\right]}}$$

$$= \frac{-\cancel{n}\left[1+\frac{4}{n}\right]}{\cancel{n}\sqrt{\frac{2}{n}+\frac{1}{n^2}}+\cancel{n}\sqrt{\frac{3}{n}+\frac{5}{n^2}}}$$

$$\longrightarrow -\frac{1}{0^+} = -\infty$$

$$\sum_{n=0}^{+\infty} \frac{(-1)^n + 4^n - 6^n}{9^n}$$

$$= \sum_{n=0}^{+\infty} \Big[\underbrace{\left(-\frac{1}{9}\right)^n}_{A} + \underbrace{\left(\frac{4}{9}\right)^n}_{B} - \underbrace{\left(\frac{6}{9}\right)^n}_{C} \Big]$$

$$A = \sum_{n=0}^{+\infty} \left(-\frac{1}{9}\right)^n = \frac{1}{1-\left(-\frac{1}{9}\right)} =$$

$\left|-\frac{1}{9}\right|<1$

$$= \frac{1}{1+\frac{1}{9}} = \frac{9}{9+1}$$

$\left|\frac{4}{9}\right|<1$

$$= \frac{9}{10}$$

$$B = \sum_{n=0}^{+\infty} \left(\frac{4}{9}\right)^n = \frac{1}{1-\frac{4}{9}} = \frac{9}{9-4} = \frac{9}{5}$$

$$C = \sum_{n=0}^{+\infty} \left(\frac{6}{9}\right)^n = \frac{1}{1-\frac{6}{9}} = \frac{9}{9-6} = \frac{9}{3}$$

$$\sum_{n=0}^{+\infty} \frac{(-1)^n + 4^n - 6^n}{9^n} = \frac{9}{10} + \frac{9}{5} - \frac{9}{3} = \dots$$

VERTICAL HORIZONTAL ASYMPTOTES

VERTICAL ASYMPTOTES

1) COMPUTE THE DOMAIN OF THE FUNCTION
2) I COMPUTE THE LEFT AND/OR RIGHT LIMIT AT ALL THE BOUNDARY POINTS OF D

$$D = (-\infty, 0) \cup (0, +\infty)$$

$$D = (1, +\infty)$$

3) IF ONE BETWEEN THE LEFT OR THE RIGHT LIMIT IN x_0 IS EQUAL TO + OR − INFINITY THEN THE FUNCTION HAS A VERTICAL ASYMPTOTE IN x_0

EX: FIND THE VERTICAL ASYMPTOTES OF

$$f(x) = \frac{5-x^2}{x+3}$$

1) FIND THE DOMAIN OF f.

$$D = \mathbb{R} \setminus \{-3\} = (-\infty, -3) \cup (-3, +\infty)$$

2) ISOLATE THE BOUNDARY POINTS

$$x_0 = -3$$

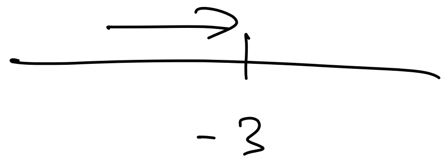
3) COMPUTE LEFT AND RIGHT LIMITS IN ALL THE BOUNDARY POINTS.

$$\lim_{x \to -3^+} \frac{5-x^2}{x+3} = \frac{5-(-3^+)^2}{-3^+ + 3} = \frac{-4}{0^+} = -\infty$$

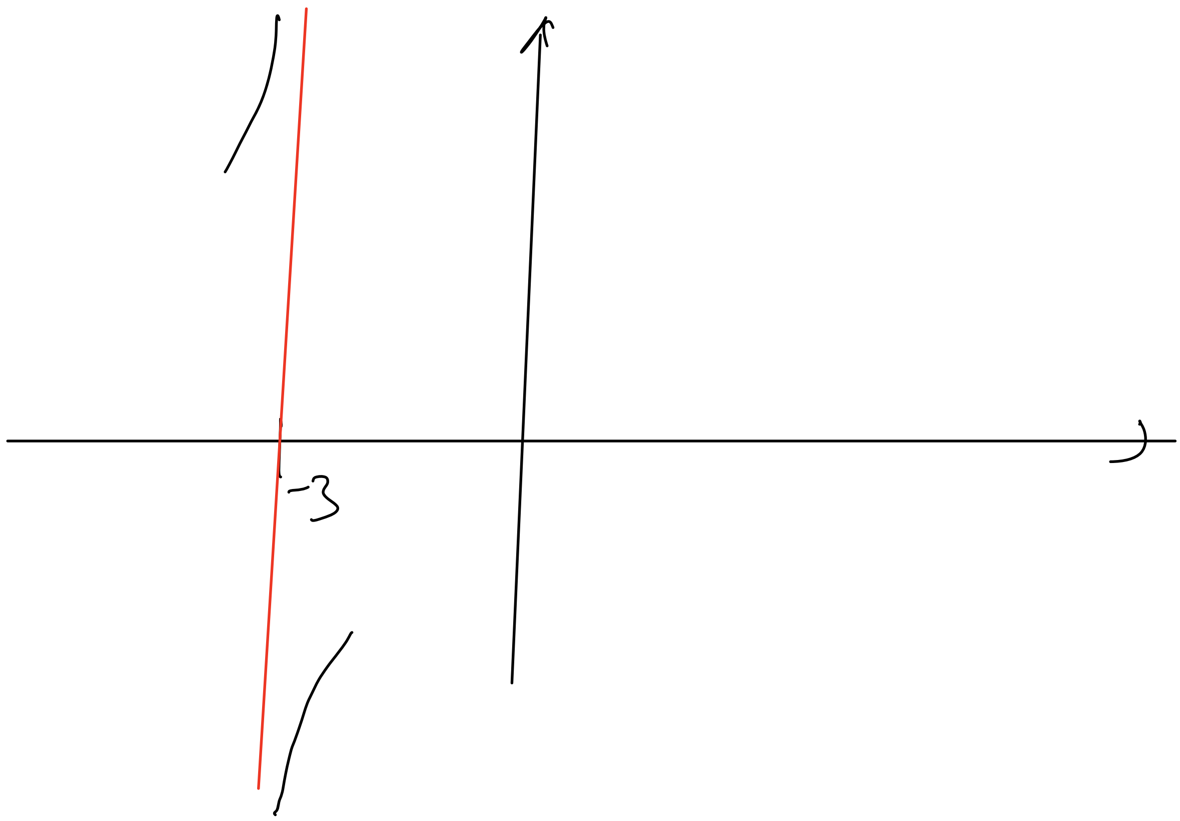
$$-3^+ = -2{,}9999$$

$$\lim_{x \to -3^-} \frac{5-x^2}{x+3} = \frac{-4}{-3^- + 3} = \frac{-4}{0^-} = +\infty$$

$x \to -3$



$$-3{,}000001 + 3 = 0^-$$



HORIZONTAL ASYMPTOTE

1) THE DOMAIN MUST BE UNBOUNDED

$D = (-\infty, a)$ OR $(a, +\infty)$

2) $\lim\limits_{x \to \pm\infty} f(x) = L$ FINITE

$$f(x) = \frac{5 - x^2}{x+3}$$

$$D = (-\infty, -3) \cup (-3, +\infty)$$

$$\lim_{x \to +\infty} \frac{5-x^2}{x+3} = \lim_{x \to +\infty} \frac{-x^2\left(-\frac{5}{x^2} + 1\right)}{x\left(1 + \frac{3}{x}\right)}$$

$$= \lim_{x \to +\infty} \frac{-x^{\cancel{2}}\left(1 - \frac{5}{x^2}\right)}{\cancel{x}\left(1 + \frac{3}{x}\right)}$$

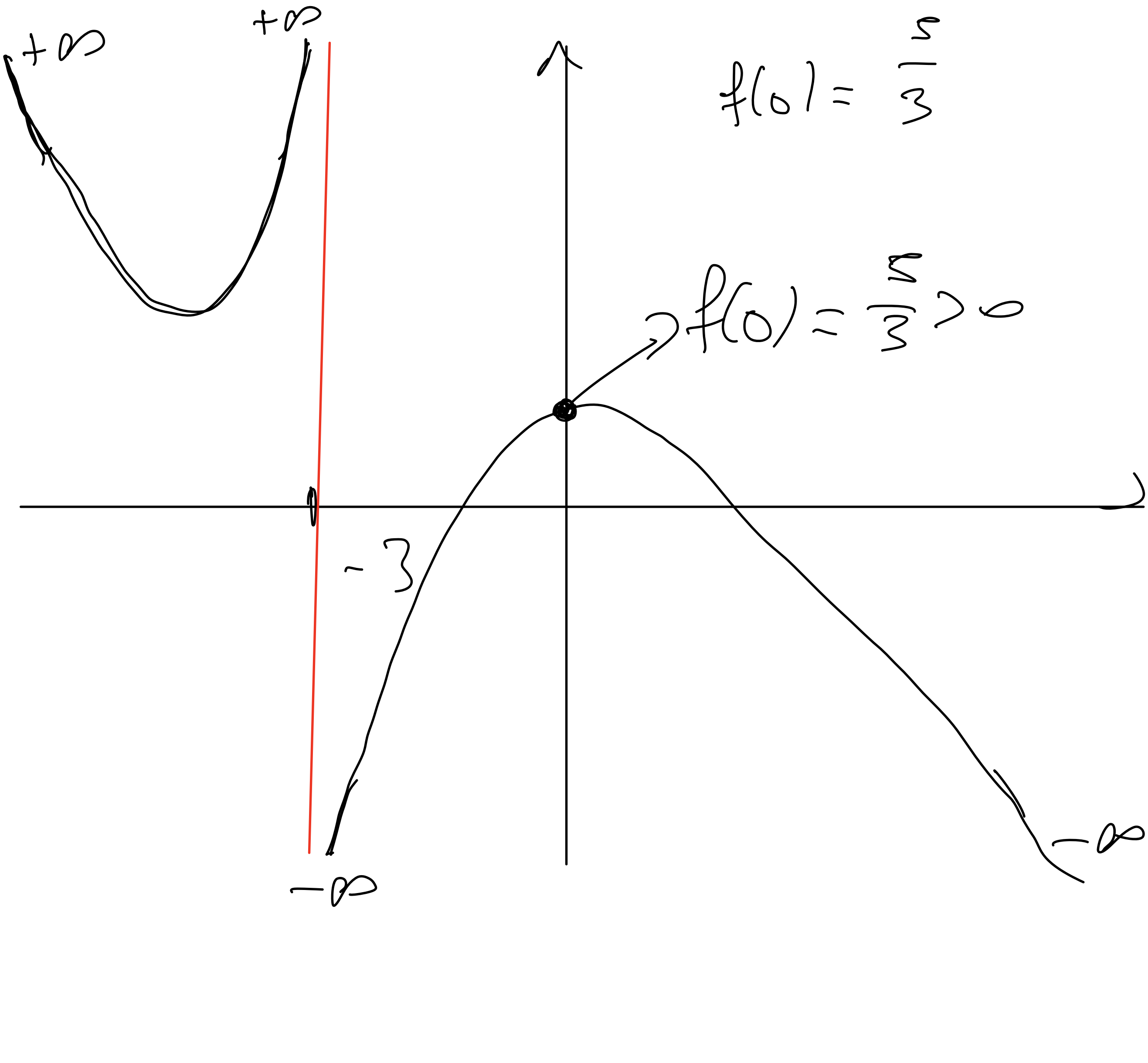
$$= \lim \frac{-x\left(1 - \frac{5}{x^2}\right)}{} \to -\infty$$

$x \to +\infty \qquad 1 + \frac{3}{x}$

NO HORIZONTAL ASYMPTOTE AT $+\infty$

$$\lim_{x \to -\infty} \frac{5 - x^2}{x + 3} = \ldots = +\infty$$

NO HORIZONTAL ASYMPTOTE AT $-\infty$



$$f(x) = \frac{\sqrt{1-x^2}}{x} \qquad D = [-1,0) \cup (0,1]$$

$$f(x) = \frac{\sqrt{x^2+5}}{x+1} \qquad D = (-\infty,-1) \cup (-1,+\infty)$$

$$\lim_{x\to -1^+} \frac{\sqrt{x^2+5}}{x+1} = \frac{\sqrt{(-1)^2+5}}{-1^+ + 1} = \frac{\sqrt{6}}{0^+} = +\infty$$

$$\lim_{x\to -1^-} \frac{\sqrt{x^2+5}}{x+1} = -\infty \qquad \sqrt{x^2} = |x|$$

ONE VERTICAL ASYMPTOTE IS FOUND AT $x_0 = -1$

$$\lim_{x\to+\infty} \frac{\sqrt{x^2+5}}{x+1} = \lim_{x\to+\infty} \frac{|x|\sqrt{1+\frac{5}{x^2}}}{x\left(1+\frac{1}{x}\right)} = \lim_{x\to+\infty} \frac{x\sqrt{1+\frac{5}{x^2}}}{x\left(1+\frac{1}{x}\right)}$$

$$= 1$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+5}}{x+1} = \lim_{x\to-\infty} \frac{|x|\sqrt{1+\frac{5}{x^2}}}{x\left(1+\frac{1}{x}\right)} = \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{5}{x^2}}}{x\left(1+\frac{1}{x}\right)}$$

$|x| = -x$

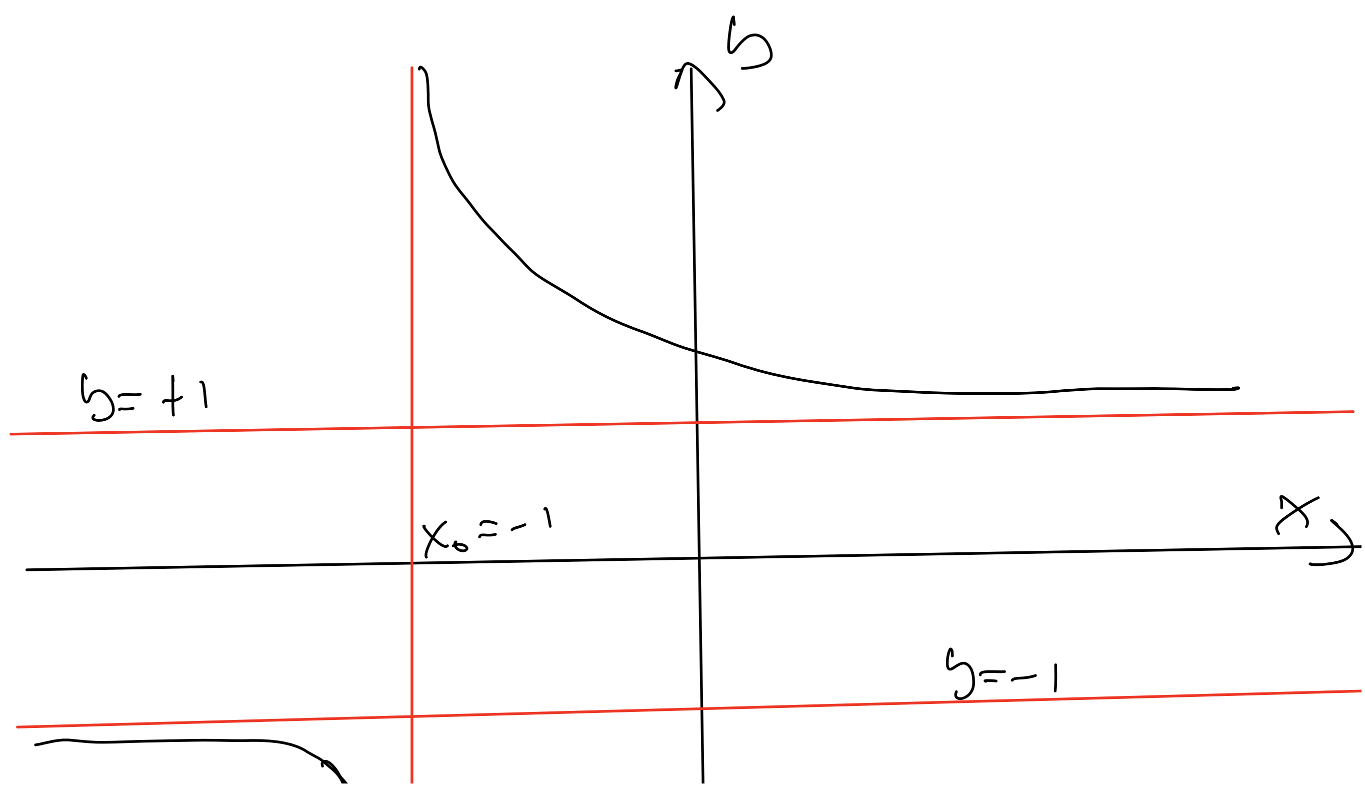
$$= -1$$

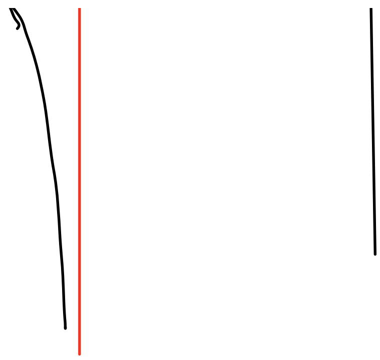
$$\sqrt{x^2+5} = \sqrt{x^2\left(1+\frac{5}{x^2}\right)} = \sqrt{x^2}\sqrt{1+\frac{5}{x^2}}$$

$$= |x|\sqrt{1+\frac{5}{x^2}}$$

if $x \to +\infty \Rightarrow |x| = x$

if $x \to -\infty \Rightarrow |x| = -x$





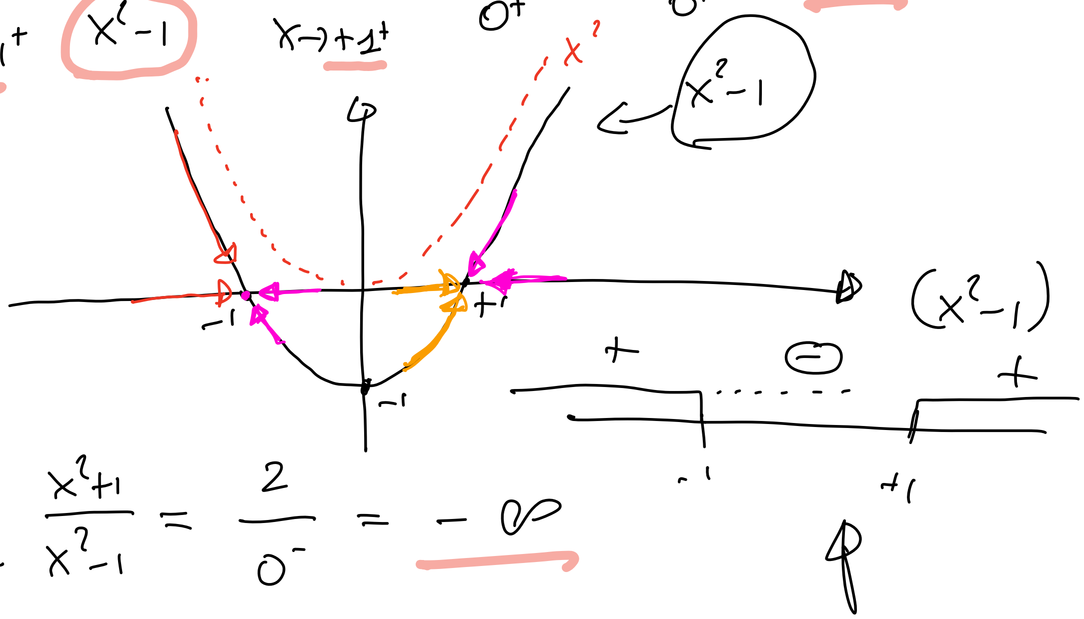
ASYMPTOTES :

ex: FIND ALL THE ASYMPTOTES OF

$$f(x) = \frac{x^2+1}{x^2-1} \qquad x^2-1=0 \iff x=\pm 1$$

1) FIND OUT THE DOMAIN $D=(-\infty,-1)\cup(-1,1)\cup(+1,+\infty)$

2) $$\lim_{x\to+1^+}\frac{x^2+1}{x^2-1} = \lim_{x\to+1^+}\frac{(1)^2+1}{0^+} = \frac{2}{0^+} = +\infty$$



$$\lim_{x\to+1^-}\frac{x^2+1}{x^2-1} = \frac{2}{0^-} = -\infty$$

$$\lim_{x\to-1^+}\frac{x^2+1}{x^2-1} = \frac{(-1)^2+1}{0^-} = \frac{2}{0^-} = -\infty$$

$$\lim_{x \to -1^-} \frac{x^2+1}{x^2-1} = \frac{(-1)^2+1}{0^+} = \frac{2}{0^+} = +\infty$$

X = +1 AND X = -1 ARE BOTH VERTICAL ASYMPTOTES

3) $$\lim_{x \to +\infty} \frac{x^2+1}{x^2-1} = 1 = \lim_{x \to +\infty} \frac{\cancel{x^2}\left(1+\frac{1}{x^2}\right)}{\cancel{x^2}\left(1-\frac{1}{x^2}\right)} = 1 = \frac{\infty}{\infty}$$

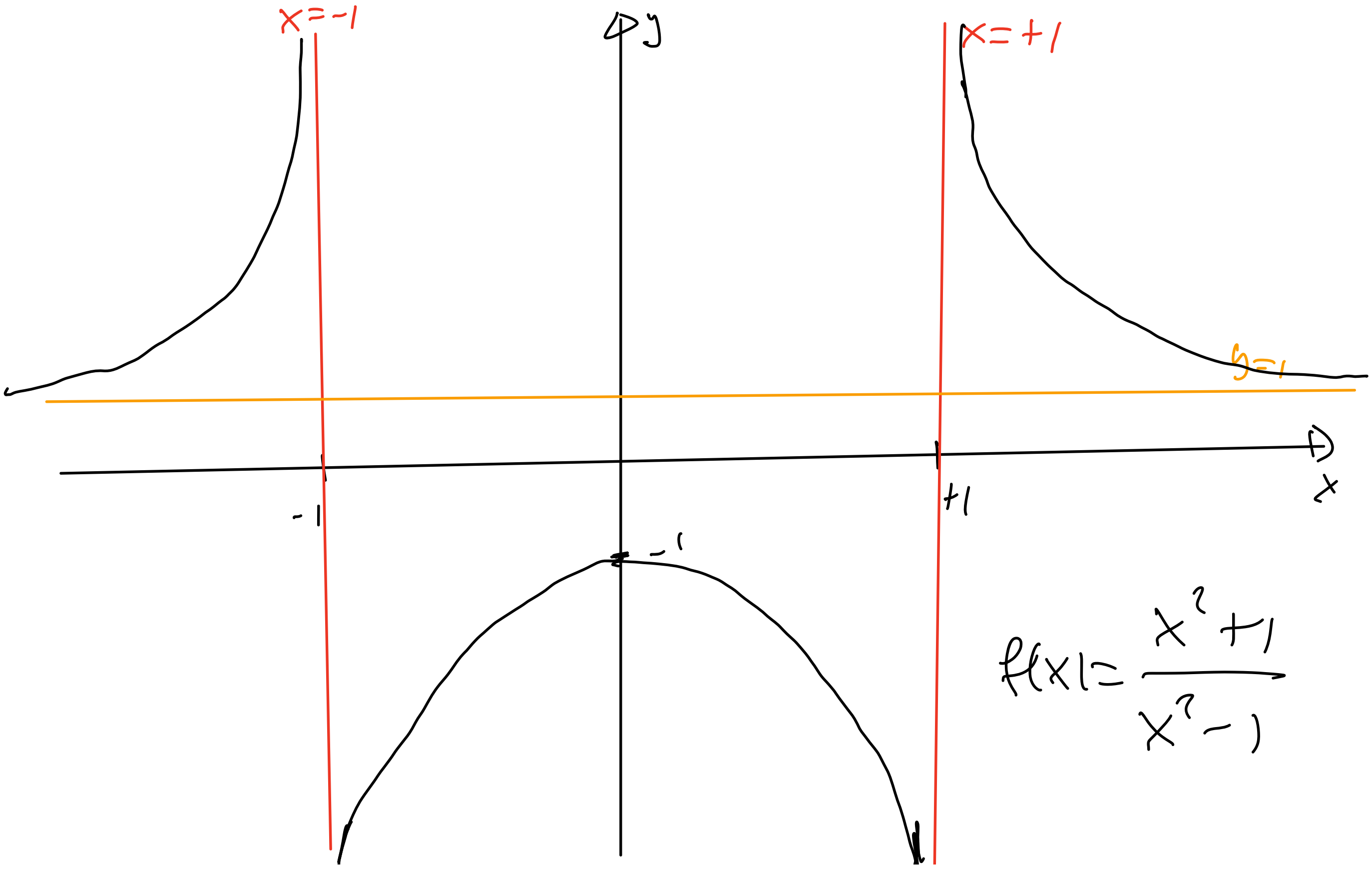
$\left(1+\frac{1}{x^2}\right) \to 1$, $\left(1-\frac{1}{x^2}\right) \to 1$

$$\lim_{x \to -\infty} \frac{x^2+1}{x^2-1} = \frac{\infty}{\infty} = 1$$

$x=0$

$$f(0) = \frac{1}{-1} = -1$$

y = 1 IS AN HORIZONTAL ASYMPTOTES.



$$f(x) = \frac{x^2+1}{x^2-1}$$

Thm: LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$ AND LET $x_0 \in \mathbb{R}$

$x_0 = \pm\infty$

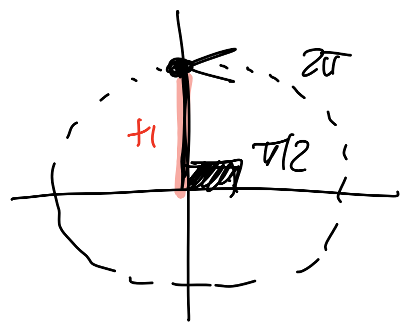
THEN

$$\boxed{\lim_{x \to x_0} f(x) = L} \iff \forall (x_n)_{n \in \mathbb{N}} \text{ SEQUENCE}$$

SUCH THAT $x_n \to x_0$

THEN $f(x_n) \to L$

$$\lim_{x \to +\infty} \sin(x) \quad \nexists$$



$\circ \; x_n = \frac{\pi}{2} + 2n\pi \to +\infty$

$\sin(x_n) = +1 \quad \forall n$

$n = 0 \Rightarrow x_0 = \frac{\pi}{2} \Rightarrow \sin(x_0) = 1$

$n = 1 \Rightarrow x_1 = \frac{\pi}{2} + 2\pi \Rightarrow \sin(x_1) = +1$

$\vdots$

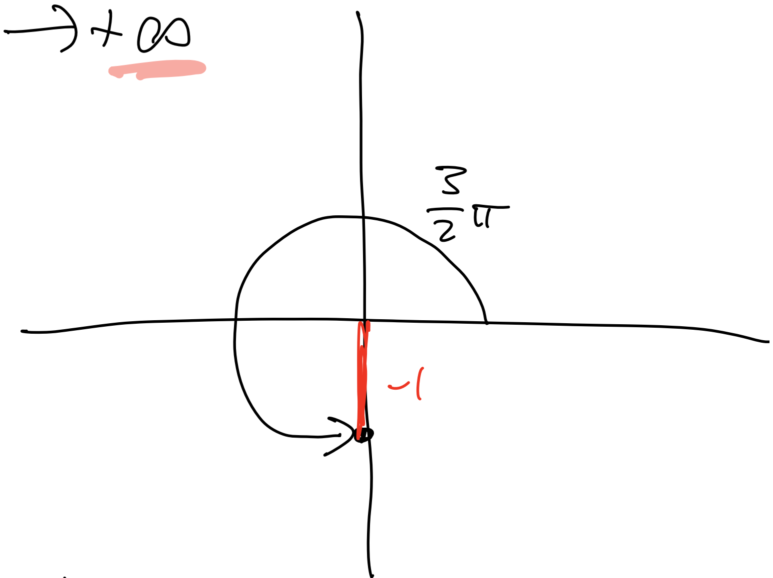
$x_n = \frac{\pi}{2} + 2\pi n \Rightarrow \sin(x_n) = +1$

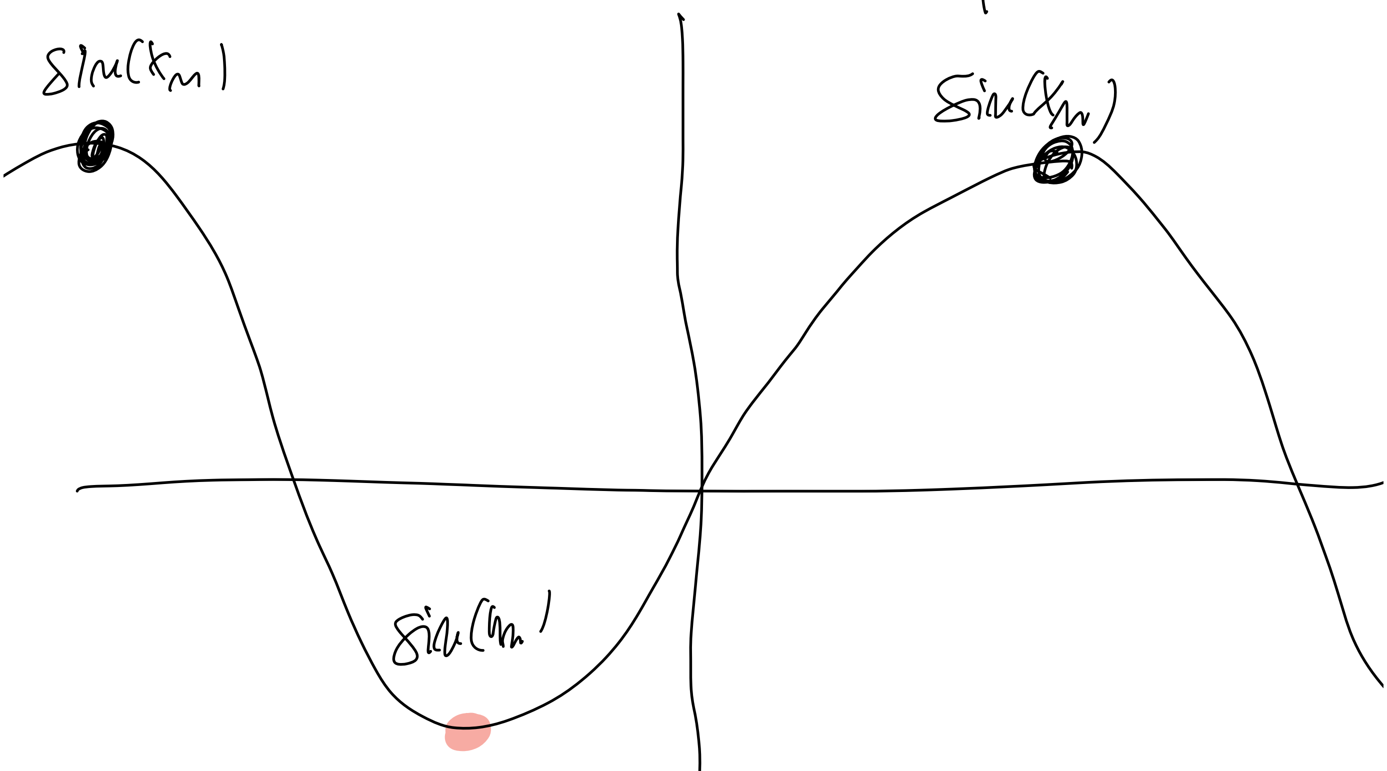
$\sin(x_n) = +1 \longrightarrow +1$

- $y_n = \frac{3}{2}\pi + 2\pi n \longrightarrow +\infty$

$\sin(y_n) = -1 \quad \forall n$

$\sin(y_n) = -1 \longrightarrow -1$





$f(x) = \sin\left(\frac{1}{x}\right) \qquad \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \nexists$

$\frac{1}{n}$

...

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \longrightarrow 0$$

$$a = \frac{\pi}{2} \qquad b = 2\pi$$

$$= \frac{1}{a + bn} \longrightarrow 0$$

$$\sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = +1 \longrightarrow +1$$

$$y_n = \frac{1}{\frac{3}{2}\pi + 2\pi n} \longrightarrow 0 \Rightarrow \sin\left(\frac{1}{y_n}\right) = \sin\left(\frac{3}{2}\pi + 2\pi n\right) = -1 \downarrow -1$$

$$\nexists \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

$$f(x) = x \log(x) \qquad D = (0, +\infty)$$

$$\cancel{\lim_{x \to 0^-} x \log(x)} =$$

ORDER OF INFINITY

$$t = \frac{1}{x} \longrightarrow +\infty$$

$$\lim_{x \to 0^+} x \log(x) = 0 \times (-\infty)$$

$$= \lim_{t \to +\infty} \frac{1}{t} \log\left(\frac{1}{t}\right) =$$

$$= \lim_{t \to +\infty} \frac{\log\left(\frac{1}{t}\right)}{t} = \lim_{t \to +\infty} \frac{\log\left(t^{-1}\right)}{t}$$

$D = (0, +\infty)$

$$= \lim_{t \to +\infty} \frac{-\log(t)}{t} = 0$$

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{\log(x^x)} = \lim_{x \to 0^+} e^{x \cdot \log(x)} = e^0 = 1$$

CONTINUITY

INTUITIVE DEFINITION: A FUNCTION IS CONTINUOUS IF ITS GRAPH HAS NO HOLES OR JUMPS.

DEF: LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$ BE A FUNCTION AND LET $\boxed{x_0 \in D}$. WE SAY THAT f IS CONTINUOUS IN x_0 IF

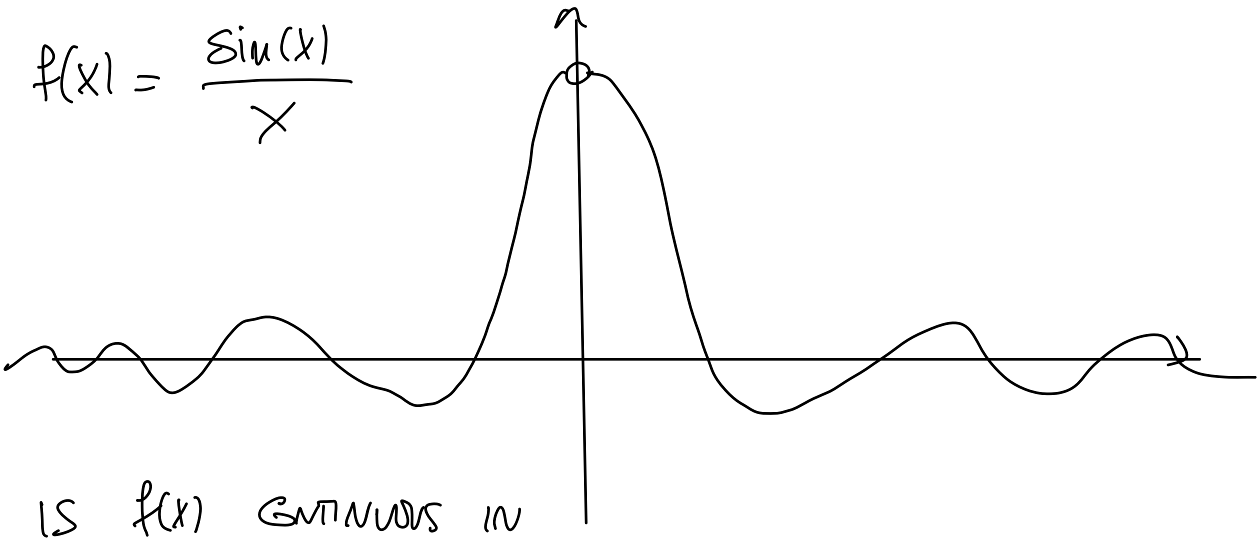
$$\boxed{\lim_{x \to x_0} f(x) = f(x_0).}$$

1) $x_0 \in D$

2) $\lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x)$

3) $L = f(x_0)$

$$f(x) = \frac{\sin(x)}{x}$$



IS $f(x)$ CONTINUOUS IN $x_0 = 0$? $0 \notin D$ SO NO!

DEF: LET $I \subseteq D$. WE SAY THAT f IS CONTINUOUS IN I IF f IS CONTINUOUS IN ALL $x \in I$

- x^b $b>0$
- a^x $0<a<1$ OR $a>1$
- $\log_a(x)$
- $\sin(x)$, $\cos(x)$, ALL THE TRIGONOMETRIC FUNCTIONS

ARE CONTINUOUS IN THEIR DOMAINS

IS $\log(x)$ CONTINUOUS IN $x_0 = 0$? NO!

LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$ AND LET $x_0 \in \mathbb{R}$,

IF $\exists \lim_{x \to x_0^-} f(x) = L_1$

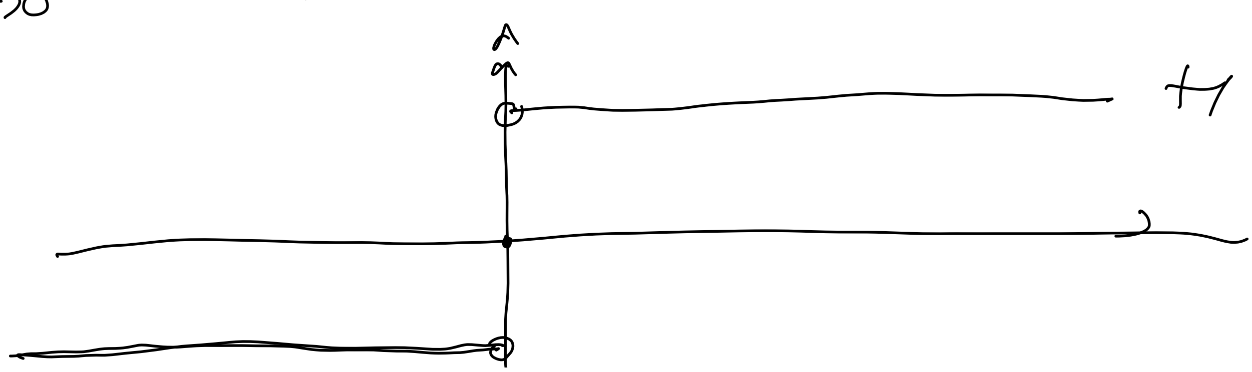
$\exists \lim_{x \to x_0^+} f(x) = L_2$

BUT $L_1 \neq L_2 \Rightarrow$ WE SAY THAT f HAS A JUMP DISCONTINUITY IN x_0

$$f(x) = \frac{|x|}{x} \qquad D = \mathbb{R} \setminus \{0\}$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{+x}{x} = \underline{1}$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \underline{-1}$$



- 2

$x_0 \in D$ IF

$$\boxed{\lim_{x \to x_0} f(x) = L} \neq f(x_0)$$

WE SAY THAT f HAS A REMOVABLE DISCONTINUITY IN x_0.

BECAUSE IN THIS CASE THE FUNCTION

$$g(x) = \begin{cases} f(x) & x \neq x_0 \\ L & x = x_0 \end{cases}$$

IS CONTINUOUS IN x_0

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} f(x) = L = g(x_0)$$

IF $x_0 \notin D$ BUT STILL $\boxed{\lim_{x \to x_0} f(x) = L}$

THEN WE SAY THAT f CAN BE EXTENDED TO A CONTINUOUS FUNCTION IN x_0.

$$f(x) = \frac{\sin(x)}{x} \qquad D = \mathbb{R} \setminus \{0\}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

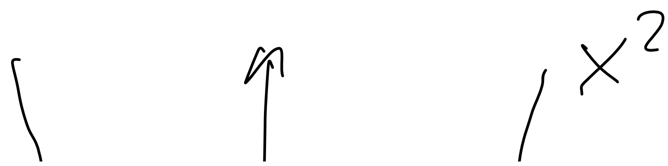
f IS NOT DEFINED IN $x_0 = 0$ HOWEVER

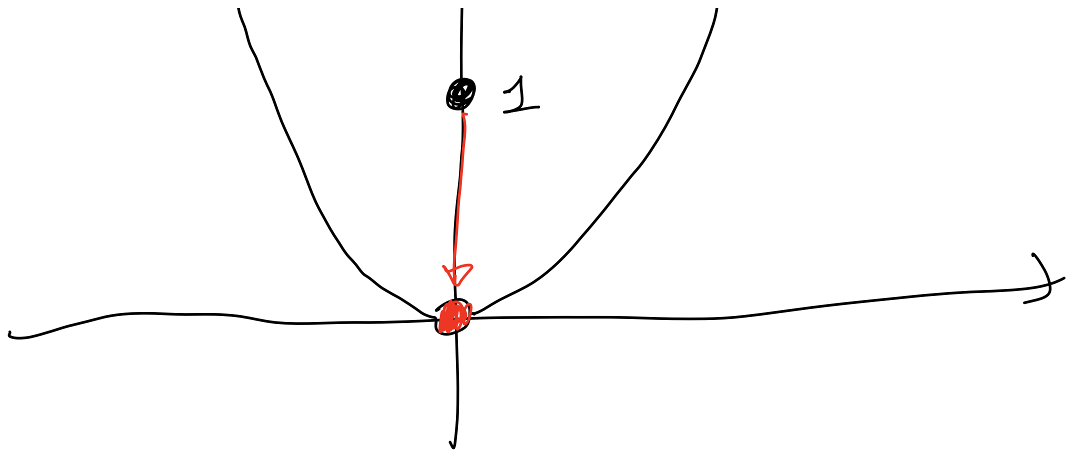
$$g(x) = \begin{cases} \dfrac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

COINCIDES WITH $f(x)$ FOR ALL $x \in D$

AND MOREOVER IS CONTINUOUS IN $x_0 = 0$

$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$$





$$\lim_{x\to 0} f(x) = 0 \neq f(0) = 1$$

$$g(x) = \begin{cases} x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

THIS IS CONTINUOUS.

3) IF EITHER

$$\lim_{x\to x_0^+} f(x)$$

OR

$$\lim_{x\to x_0^-} f(x)$$

OR BOTH ARE $\pm\infty$ OR

DO NOT EXIST WE SAY THAT

x_0 IS AN ESSENTIAL DISCONTINUITY

$$\boxed{f(x) = \sin\left(\frac{1}{x}\right)}$$

$$\nexists \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

SO $x_0 = 0$ IS AN ESSENTIAL DISCONTINUITY

$$f(x) = x \cdot \sin\left(\frac{1}{x}\right) \qquad D = \mathbb{R} \setminus \{0\}$$

$$\lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right) \qquad \boxed{-1 \leq \sin \leq +1}$$

$$0 \leq \left| x \cdot \sin\left(\frac{1}{x}\right) \right| \leq |x|$$

$$|x|\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \cdot |x|$$

if $x \to 0$ $|x| \to 0$

$$\Rightarrow \left|x \cdot \sin\left(\frac{1}{x}\right)\right| \longrightarrow 0$$

$$\Rightarrow \lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$$

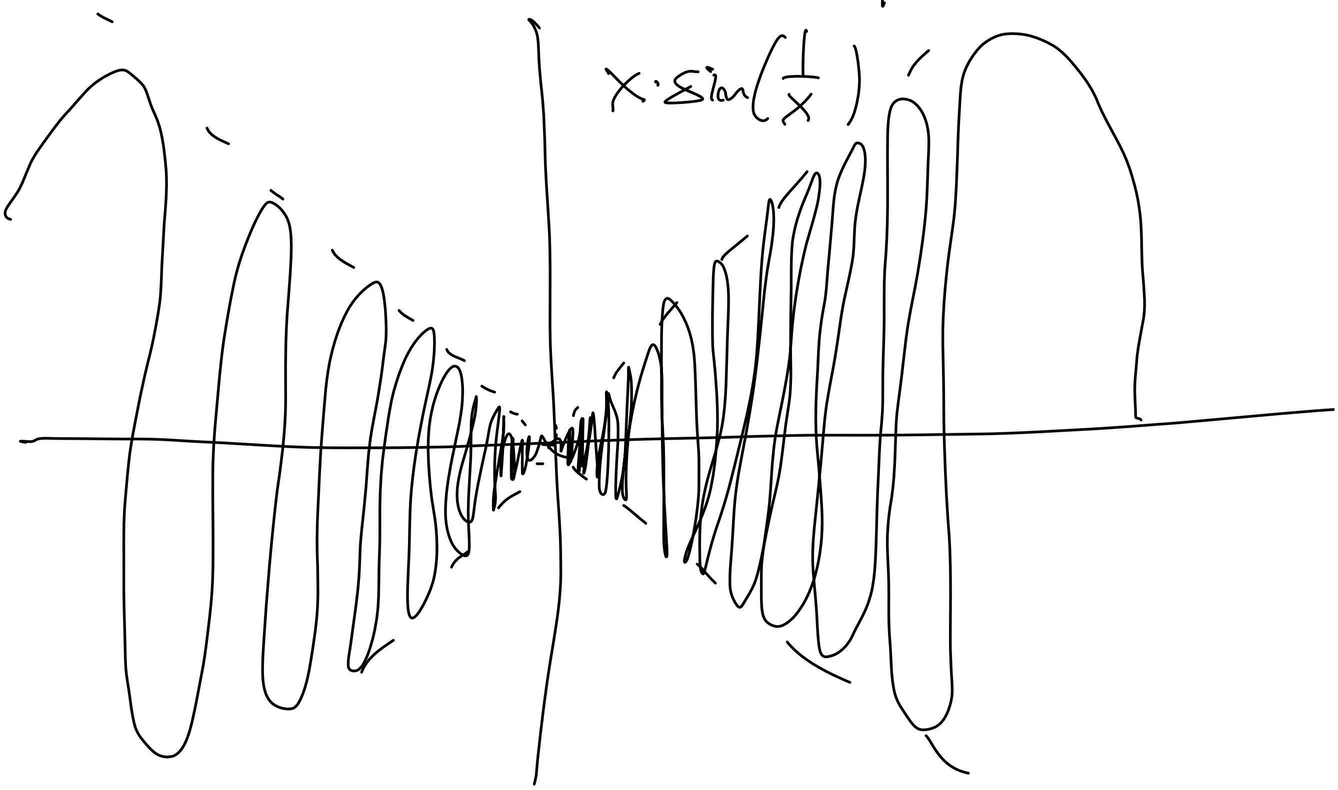
$$|x| \cdot \left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \cdot |x|$$

$$0 \leq \underbrace{\left|x \cdot \sin\left(\frac{1}{x}\right)\right|}_{\to 0} \leq |x| \longrightarrow 0$$

$$\Rightarrow \quad x \cdot \sin\left(\frac{1}{x}\right) \longrightarrow 0$$

$$g(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

g IS CONTINUOUS EVERYWHERE



$f(x) = \frac{1}{x}$ WHICH TYPE OF DISCONTINUITY HAS THE FUNCTION IN $x_0 = 0$?

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

ESSENTIAL

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

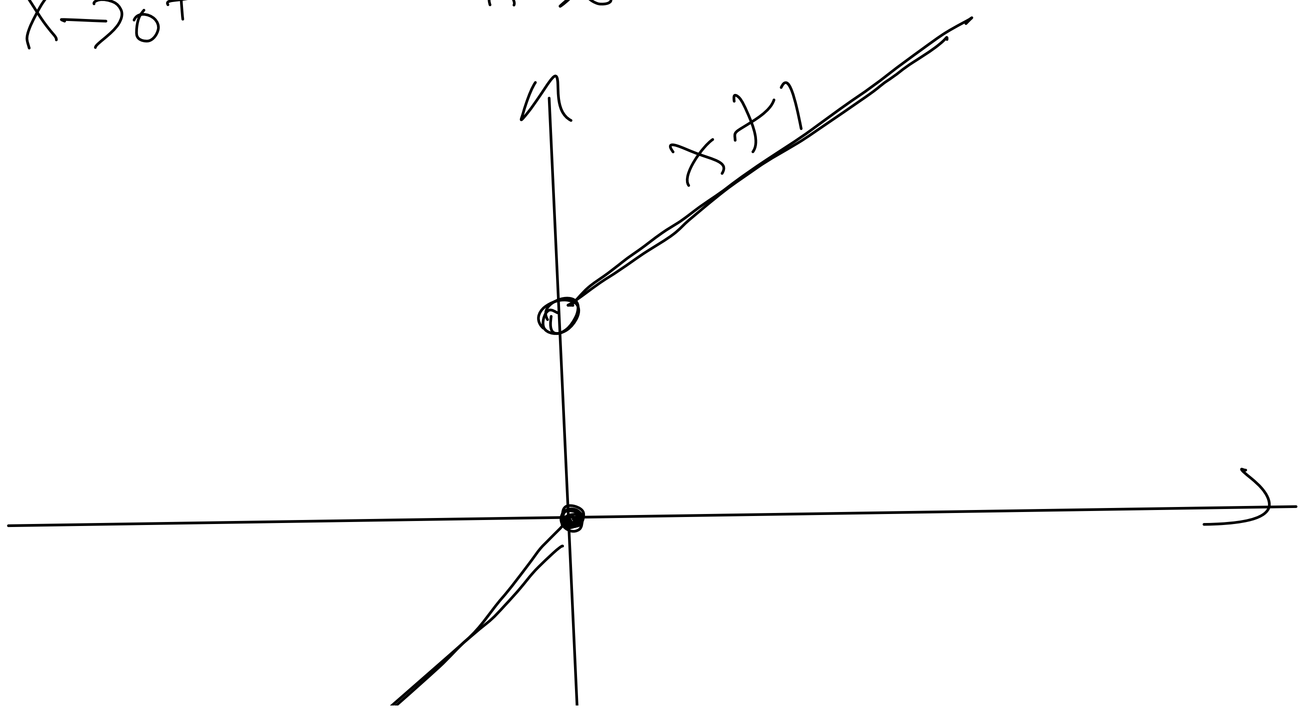
$$f(x) = \begin{cases} x & x \leq 0 \\ x+1 & x > 0 \end{cases}$$

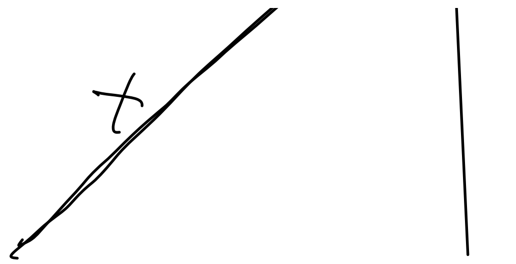
IS f CONTINUOUS IN $x_0 = 0$?

IF NOT, WHICH TYPE OF DISCONTINUITY?

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1$$





$$f(x) = x \cdot \ln(|x|)$$

$$D = \mathbb{R} \setminus \{0\}$$

$$\lim_{x\to 0^+} x\ln(|x|) =$$

$$= \lim_{x\to 0^+} x\ln(x) = 0$$

$$\lim_{x\to 0^-} x\ln(|x|) = \lim_{x\to 0^-} x\ln(-x)$$

$-x = t$

$$= \lim_{t \to 0^+} -t \ln(t) = 0$$

$\boxed{\text{REMOVABLE}}$

$$g(x) = \begin{cases} x \ln(|x|) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

g IS CONTINUOUS IN $x_0 = 0$

ex: FIND FOR WHICH VALUES OF THE PARAMETER $\alpha \in \mathbb{R}$ THE FOLLOWING FUNCTION

$$f(x) = \begin{cases} \alpha \cdot \frac{\sin(x)}{x} & \text{IF } x > 0 \\ 2x^2+3 & \text{IF } x \leq 0 \end{cases}$$

IS CONTINUOUS.

$$\lim_{x \to 0^+} f(x) = \lim_{\substack{x \to 0 \\ x > 0}} f(x) = \alpha$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2x^2+3 = 3$$

THE FUNCTION IS CONTINUOUS IF AND ONLY IF $\alpha = 3$

ex: INVESTIGATE THE CONTINUITY OF

$$f(x) = \begin{cases} \frac{\sin^2(x) \cdot \cos\left(\frac{1}{x}\right)}{e^x - 1} & x < 0 \\ \ln(1+x) & x \geq 0 \end{cases}$$

$f(0) = \ln(1) = 0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \ln(1+x) = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin^2(x)\cos\left(\frac{1}{x}\right)}{e^x - 1} = \frac{0}{0}$$

$$= \lim_{x\to 0^-} \frac{\frac{\sin^2(x)}{x^2}\cdot x^2 \cos\left(\frac{1}{x}\right)}{e^x - 1} = \lim_{x\to 0^-} \frac{\left(\frac{\sin(x)}{x}\right)^2 x \cos\left(\frac{1}{x}\right)}{\left(\frac{e^x-1}{x}\right)} = 0$$

$\to 1$, $\to x\cdot x$, $\to 1$, $\downarrow \varnothing$, 1, φ

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{y\to 0} \frac{y}{\log(1+y)} = \lim_{y\to 0} \frac{1}{\log(1+y)^{\frac{1}{y}}} = \frac{1}{\log(e)} = 1$$

$\downarrow e$

$$\boxed{y = e^x - 1} \Rightarrow x = \log(1+y)$$

$$\parallel \quad \frac{0}{0}$$

$$y+1 = e^x \Rightarrow x = \log(1+y)$$

$$= \lim_{y\to 0} \frac{y}{\log(1+y)} = \lim_{y\to 0} \frac{1}{\frac{1}{y}\log(1+y)} =$$

$$= \lim \frac{1}{\dots} = \frac{1}{\dots} = 1$$

$$\lim_{y\to 0} \log(1+y)^{1/y} \quad \log(e) = 1$$

$$\lim_{y\to 0} (1+y)^{\frac{1}{y}} = \lim_{t\to +\infty} \left(1+\frac{1}{t}\right)^t = e$$

EX: LET $f(x) = e^{-\frac{1}{x^2}} = e^{-\frac{1}{x^2}}$

1) FIND THE DOMAIN

2) CLASSIFY (IF THERE WERE ANY) THE DISCONTINUITY POINT OF f.

$$D = (-\infty, 0) \cup (0, +\infty) = \mathbb{R} \setminus \{0\}$$

$$\lim_{x\to 0} e^{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{1}{e^{\frac{1}{x^2}}} = \frac{1}{e^{+\infty}} = \frac{1}{+\infty} = 0$$

REMOVABLE

$$g(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

g IS CONTINUOUS EVERYWHERE.

$$f(x) = e^{+\frac{1}{x^2}} \qquad D = \mathbb{R} \setminus \{0\}$$

$$\lim_{x \to 0} e^{\frac{1}{x^2}} = e^{+\frac{1}{0^+}} = e^{+\infty} = +\infty$$

ESSENTIAL

$$f(x) = \begin{cases} x^2 - 2x + 3\alpha - 4 & x \leq 0 \\ \dfrac{\sin(\alpha x)}{x} & x > 0 \end{cases}$$

- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x^2 - 2x + 3\alpha - 4\right) = 3\alpha - 4$
- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \alpha \left[\dfrac{\sin(\alpha x)}{\alpha x}\right]^{\to 1} = \alpha$

$3\alpha - 4 = \alpha$

$$\boxed{3\alpha - 4 = \alpha}$$

$$2\alpha - 4 = 0$$

$$\Rightarrow \boxed{\alpha = 2}$$

MAXIMUM AND MINIMUM OF A FUNCTION.

DEF: LET $f : D \subseteq \mathbb{R} \to \mathbb{R}$

LET $I \subseteq D$ BE A SUBSET OF D

REACHES MINIMUM

WE SAY THAT f ATTAINS A MAXIMUM IN I IF:

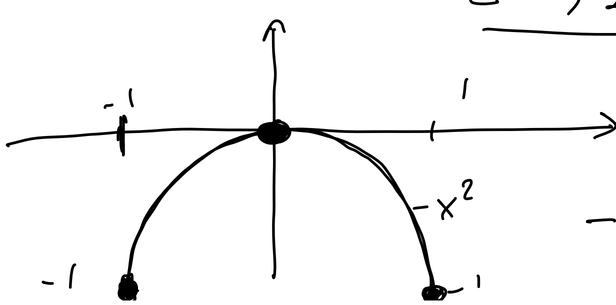
$x_m \in I$ $\quad f(x_m) \le f(x)$

$$\exists\, x_M \in I : \forall x \in I \Rightarrow f(x) \le f(x_M)$$

$f(x_M)$ IS CALLED THE MAXIMUM VALUE OF f IN I.

$f(x) = -x^2 \quad D = \mathbb{R} \quad I = [-1, 1]$

$M = 0$



$m = -1$

$-x^2 \le 0 \quad \forall x$

yes $X_n = 0$

$f(x) = \sqrt{x}$ $I = [0, +\infty)$ no

$m = 0$

$x = 0$

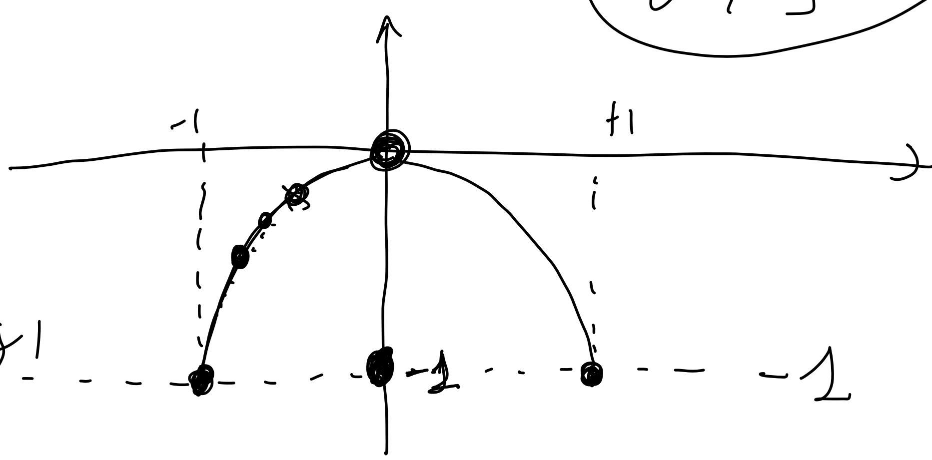
$$f(x) = \begin{cases} -x^2 & x \neq 0 \\ -1 & x = 0 \end{cases}$$

$[-1, 1]$

$\mathbb{R}$

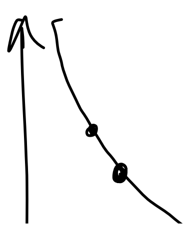
$m = -1$

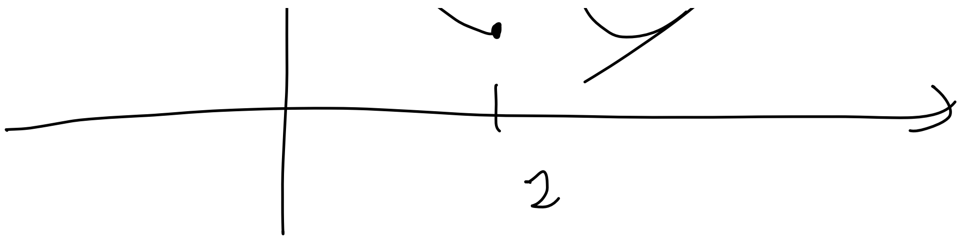
$x = -1, +1$



$f(x) = \frac{1}{x}$ $I = (0, 1]$

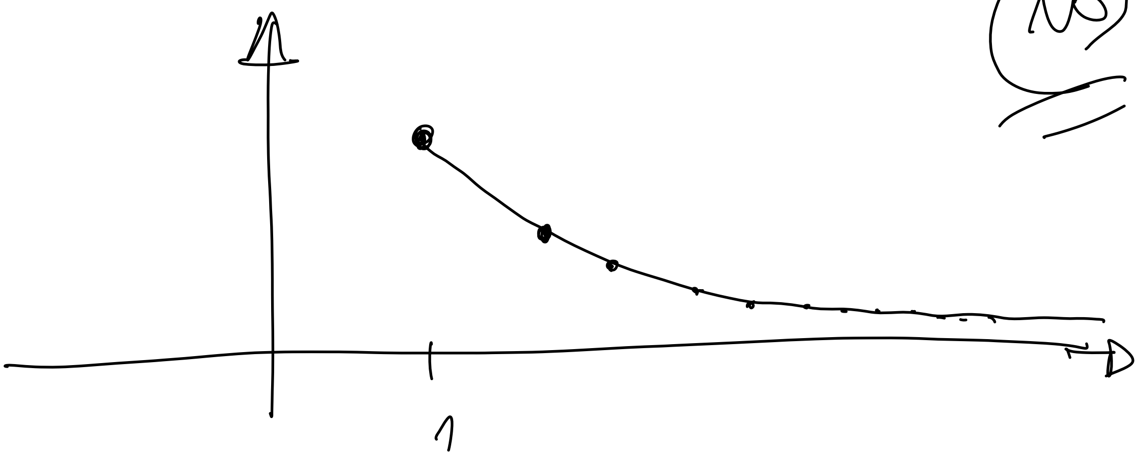
no





$f(x) = \frac{1}{x}$ $\quad I = [1, +\infty)$

NO !



WEIERSTRASS THEOREM.

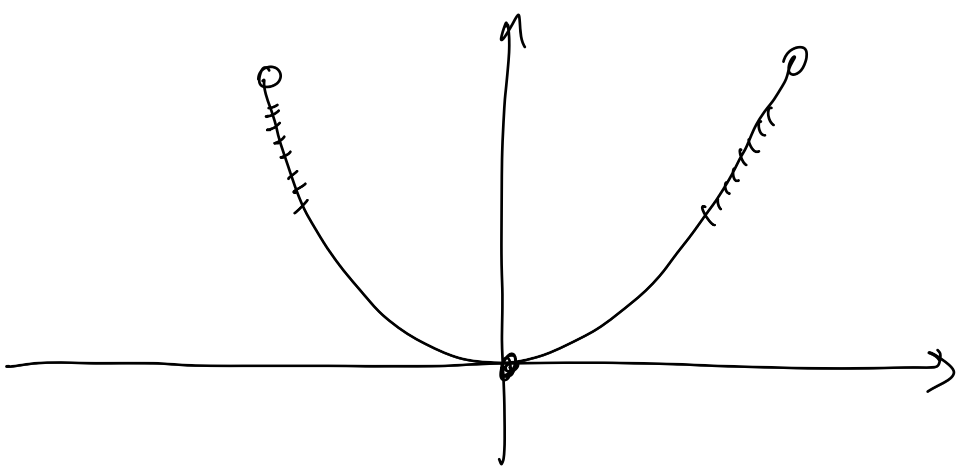
LET $f: D \subseteq \mathbb{R} \to \mathbb{R}$ BE A FUNCTION AND LET $[a, b] \subseteq D$ BE A CLOSED AND BOUNDED INTERVAL OF D. IF f IS CONTINUOUS ON $[a, b]$ THEN f ATTAINS A

MAXIMUM AND A MINIMUM
IN $[a,b]$

1) $[a,b]$ CLOSED AND BOUNDED

2) f MUST BE CONTINUOUS ON $[a,b]$

$f(x) = x^2 \qquad (-2,2)$



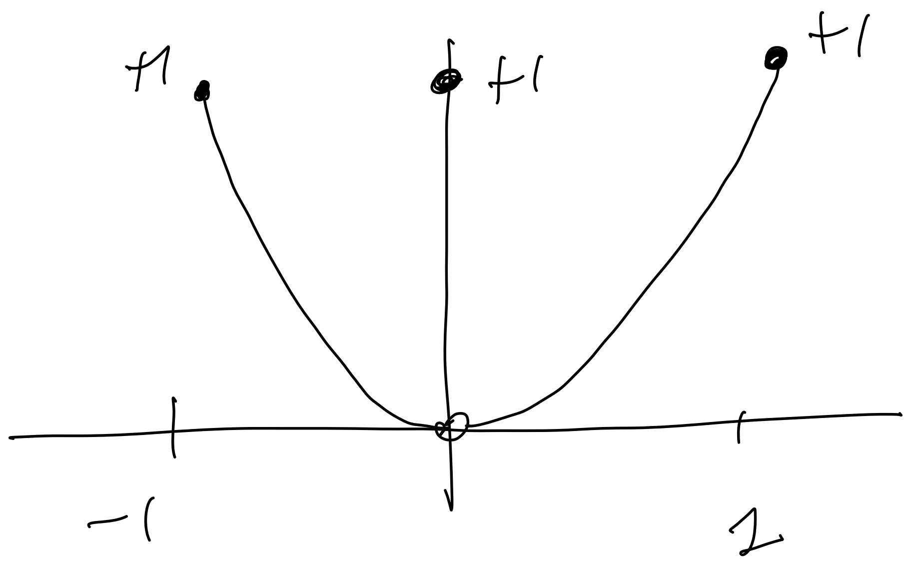
$f(x) = \frac{1}{x} \qquad [1,+\infty)$

$f(x) \le 1 \quad \forall x \in [1,+\infty)$

$f(1) = 1$

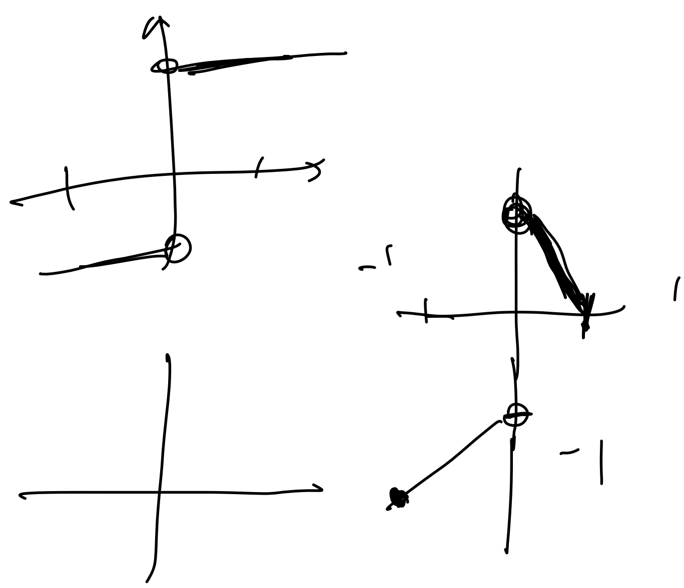
$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$I = [-1, 1]$



$(1-x)\ \frac{|x|}{x}$

$x\ -1$



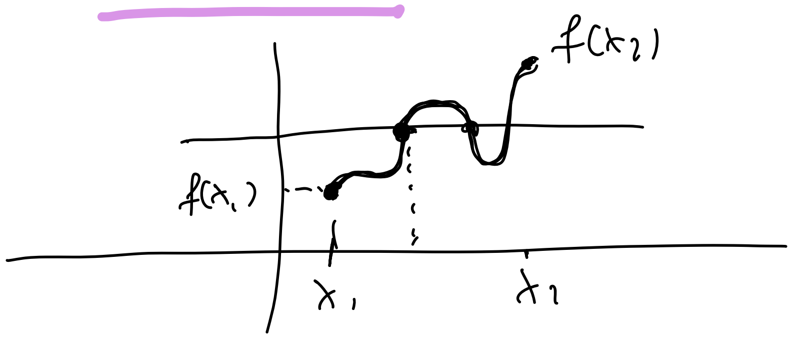
INTERMEDIATE VALUE THEOREM.

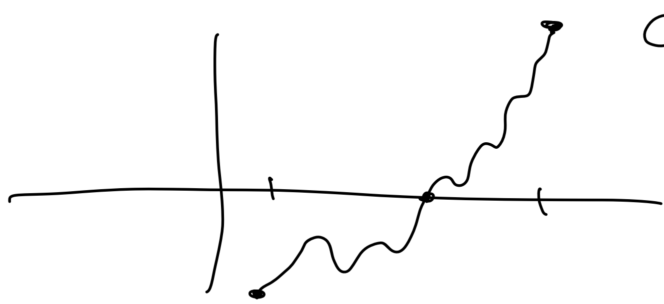
LET $f: [a,b] \to \mathbb{R}$ BE A CONTINUOUS FUNCTION FROM THE CLOSED AND BOUNDED INTERVAL $[a,b]$ TO $\mathbb{R}$.

SUPPOSE THAT $\exists\ x_1, x_2 \in [a,b]$ SUCH THAT

$$\boxed{f(x_1) \leq c \leq f(x_2)}$$

THEN $\exists\ x_0 \in [x_1, x_2]$ SUCH THAT $f(x_0) = c$





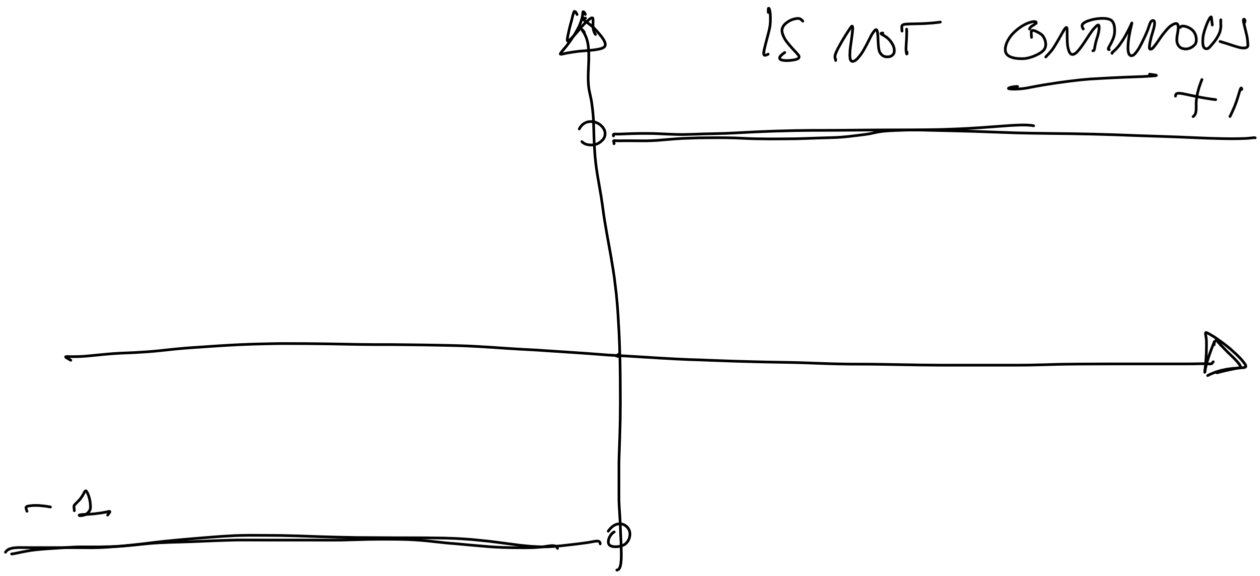
$c = 0$

$$f(x) = \frac{|x|}{x} \qquad f(-1) = -1 = \frac{|-1|}{-1} = \frac{+1}{-1} = -1$$

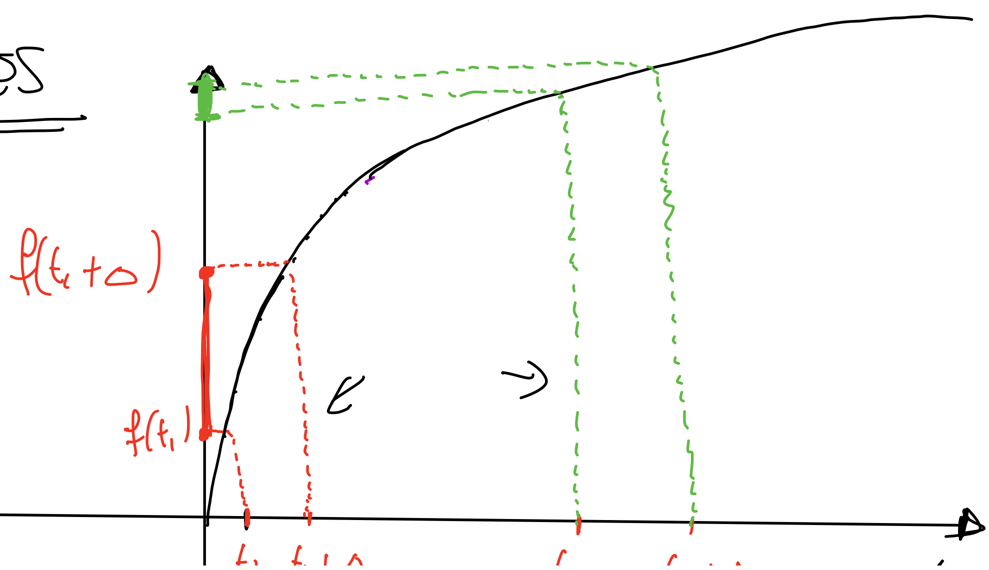
$$f(+1) = +1$$

$$f(-1) < 0 < f(+1)$$

$$\exists\, x_0 : f(x_0) = 0$$



DERIVATIVES



1) THE GDP GROWS WITH TIME

2) IN THE FIRST PART OF THE GRAPH THE GDP GROWS FASTER THAN IN THE OTHER PART

$$\frac{f(t_1+\Delta)-f(t_1)}{(\cancel{t_1}+\Delta)-\cancel{t_1}} = \frac{f(t_1+\Delta)-f(t_1)}{\Delta}$$

$$\exists \lim_{\Delta\to 0} \boxed{\frac{f(t_1+\Delta)-f(t_1)}{\Delta}} = f'(t_1)$$

INCREMENTAL RATIO

DEF: $f : D \subseteq \mathbb{R} \to \mathbb{R}$

LET $(a,b) \subseteq D$ $\quad (a,b)$ OPEN.

LET $x_0 \in (a,b)$.

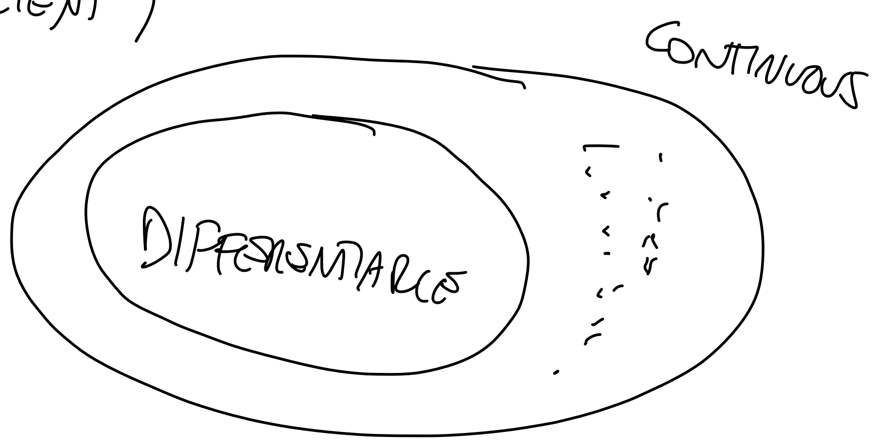
WE SAY THAT f IS DIFFERENTIABLE IN x_0

IF THE FOLLOWING LIMIT EXISTS AND IT IS FINITE

$$\lim_{h \to 0} \left[\frac{f(x_0+h) - f(x_0)}{h} \right] = f'(x_0)$$

<u>Thm</u>: IF f IS DIFFERENTIABLE IN x_0 THEN f IS CONTINUOUS IN x_0

(AKA... CONTINUITY IS A <u>NECESSARY</u> CONDITION FOR DIFFERENTIABILITY ALTHOUGH IT IS NOT SUFFICIENT)



LET f BE DIFFERENTIABLE IN x_0.

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{(x - x_0)} \cdot (x - x_0)$$

$$\lim_{x\to x_0}\left(f(x)-f(x_0)\right)=\lim_{x\to x_0}\underbrace{\left[\frac{f(x)-f(x_0)}{x-x_0}\right]}_{\to f'(x_0)=L}\cdot\underbrace{(x-x_0)}_{\to 0}$$

$$=0$$

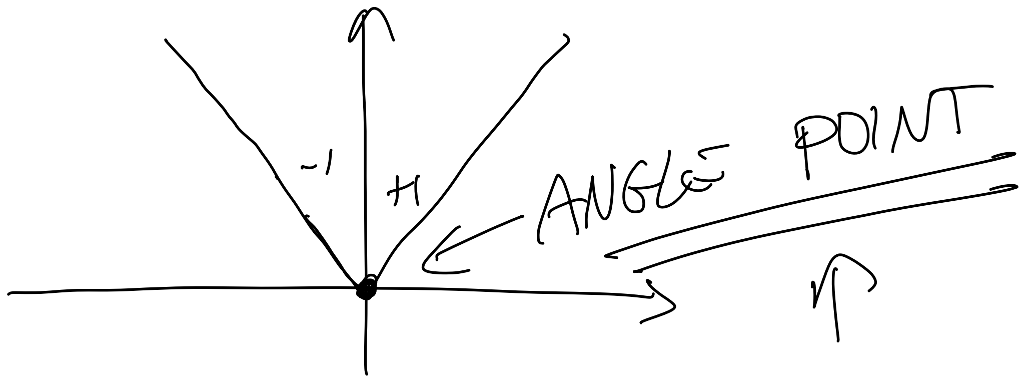
$$\lim_{x\to x_0} f(x)=f(x_0)$$

$f(x)=|x|$ $\quad$ $x_0=0$

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-|0|}{h}$$

$$=\cancel{\lim_{h\to 0}\frac{|h|}{h}}\ \nexists$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = -1 \qquad \lim_{h \to 0^+} \frac{|h|}{h} = +1$$



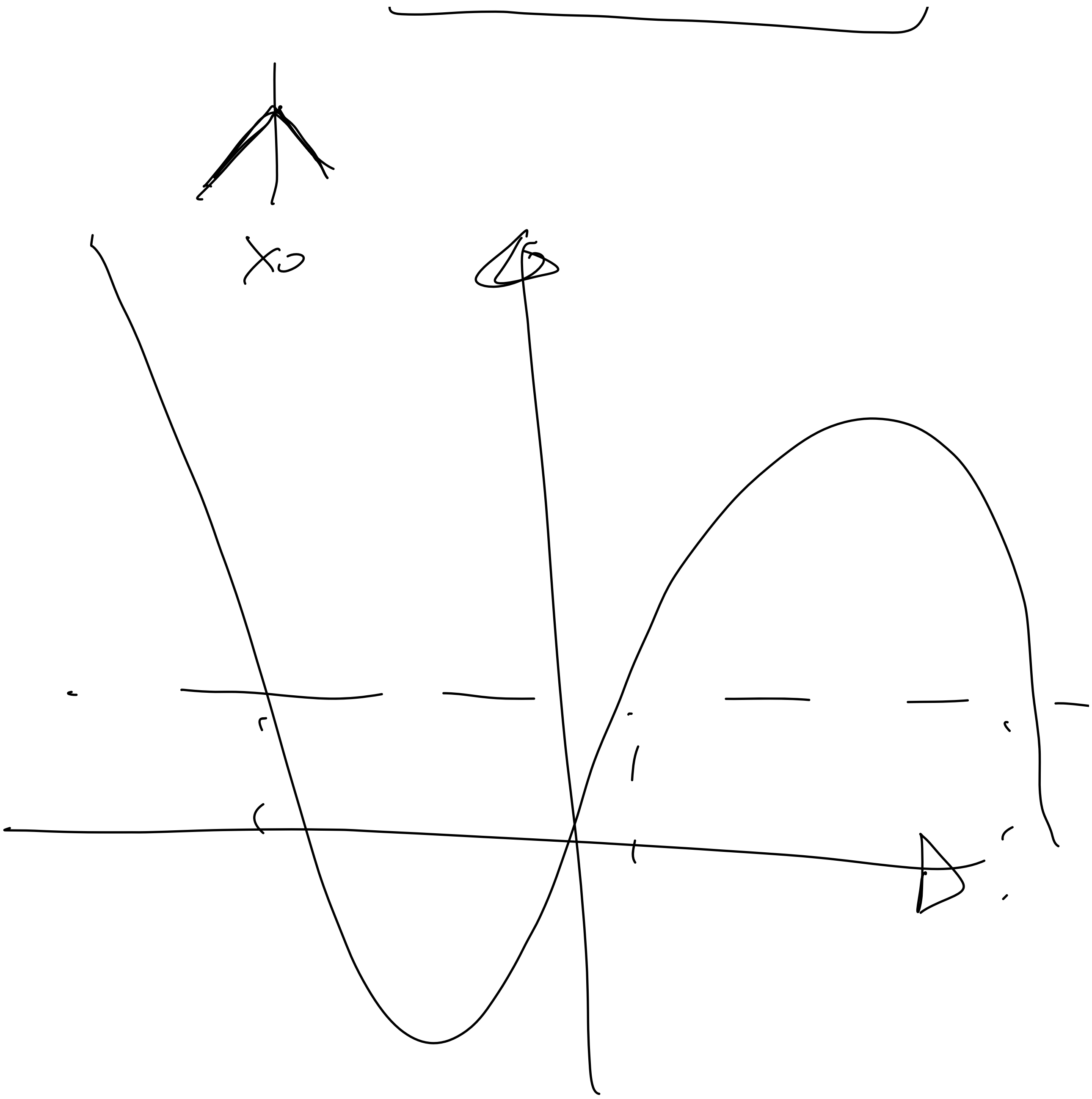
DEF: IF

$$\lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = L_1 \quad \text{FINITO}$$

$$\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = L_2 \quad \text{FINITO}$$

$L_1 \neq L_2 \Rightarrow x_0$ IS AN

ANGLE POINT


x0