Quadratic functions

$$f : \mathbb{R} \to \mathbb{R} \text{ such that } f(x) = ax^2 + bx + c, \quad a, b, c \in \mathbb{R},\ a \neq 0$$

- Graphically, this is the equation of a parabola.
- The parabola is convex if $a > 0$
- The parabola is concave if $a < 0$
- the vertex of the parabola is the point with coordinates $V = \left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$, with $\Delta = b^2 - 4ac$

Position of a parabola with respect to the x-axis

Let $\Delta = b^2 - 4ac$. Suppose that $a > 0$

1. If $\Delta > 0$ The parabola intercepts the x-axis at two points, which are the solutions of
$$ax^2 + bx + c = 0$$
2. If $\Delta = 0$ The parabola intercepts the x-axis at one point, which is the unique solution of
$$ax^2 + bx + c = 0$$
3. If $\Delta < 0$ The parabola stays always above the x-axis: the equation $ax^2 + bx + c = 0$ does not have any solution

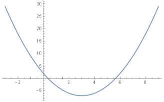


Figure: $a > 0, \quad \Delta > 0$

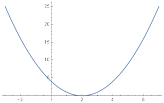


Figure: $a > 0, \quad \Delta = 0$

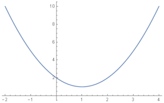


Figure: $a > 0, \quad \Delta < 0$

Position of a parabola with respect to the x-axis

Let $\Delta = b^2 - 4ac$. Suppose that $a < 0$

1. If $\Delta > 0$ The parabola intercepts the x-axis at two points, which are the solutions of
$$ax^2 + bx + c = 0$$
2. If $\Delta = 0$ The parabola intercepts the x-axis at one point, which is the unique solution of
$$ax^2 + bx + c = 0$$
3. If $\Delta < 0$ The parabola stays always below the x-axis: the equation $ax^2 + bx + c = 0$ does not have any solution

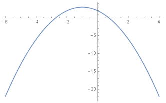


Figure: $a < 0, \quad \Delta > 0$

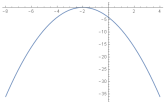


Figure: $a < 0, \quad \Delta = 0$

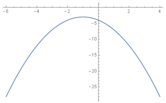


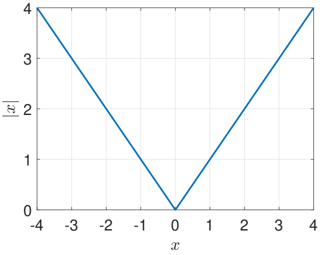
Figure: $a < 0, \quad \Delta < 0$

The function “Absolute Value”

Definition

The function “absolute value” of x, is given by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



- $D = \mathbb{R}$, $R_f = [0, +\infty)$

Power functions

For all $n \in \mathbb{N}$ we define the function:

$$f(x) = x^n$$

which is nothing but the multiplication of x by itself n times

- This function is defined for all $x \in \mathbb{R}$, $D = \mathbb{R}$
- If n is even, the range is $R_f = [0, +\infty)$
- If n is odd, the range is $R_f = \mathbb{R}$

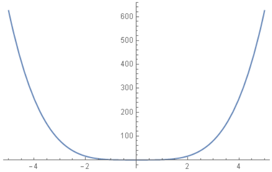


Figure: $f(x) = x^4$

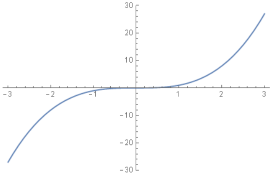


Figure: $f(x) = x^3$

A few characteristics of Power function with exponent $n \in \mathbb{N}$

If n is even, the function is not globally invertible. However if we consider only

$$f(x) : [0, +\infty) \to [0, +\infty)$$

the function is invertible and

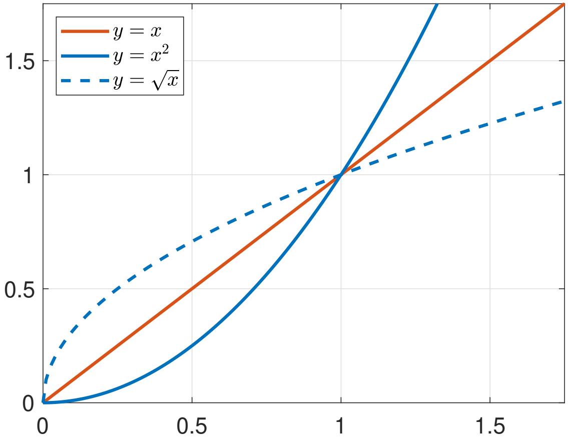
$$f^{-1}(y) = y^{\frac{1}{n}} = \sqrt[n]{y}$$

If n is odd, the function is globally invertible and

$$f^{-1}(y) = y^{\frac{1}{n}} = \sqrt[n]{y}$$

The inverse function: graphical representation

The graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ over the line $y = x$.



The inverse function: graphical representation

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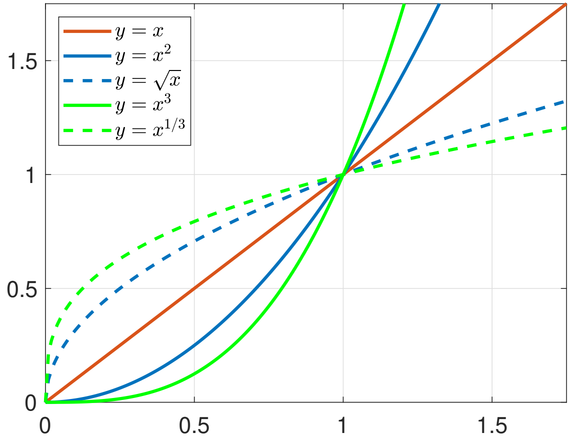

The inverse function: graphical representation

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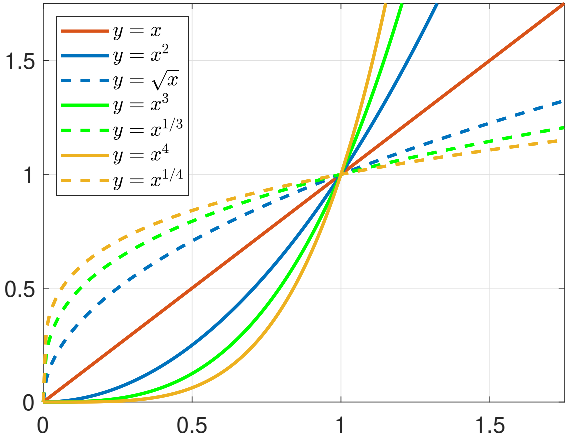


Power functions

Consider the function:

$$f(x) = x^r, \quad r \in \mathbb{R}$$

This is a power function with real exponent (which generalizes the case of a power function with natural exponent)
A few examples

1. $f(x) = x^{-1} = \dfrac{1}{x}$
2. $f(x) = x^{\frac{1}{2}} = \sqrt{x}$
3. $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$
4. $f(x) = x^{1.3}$

Notice that an extra care must be applied in computing the domain power functions with real exponent. In particular they are well defined when $x > 0$, but they may be undefined for $x = 0$ or $x < 0$. For instance function 1 is not defined when $x = 0$, functions 3 and 4 are not defined when $x < 0$.

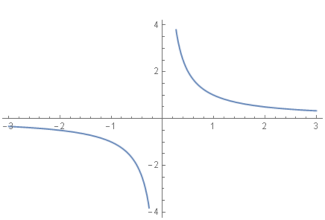


Figure: $f(x) = \frac{1}{x}$

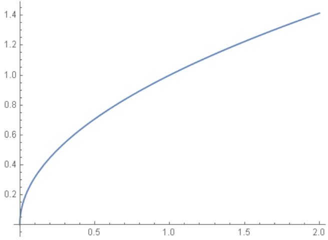


Figure: $f(x) = \sqrt{x}$

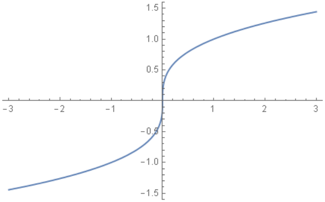


Figure: $f(x) = \sqrt[3]{x}$

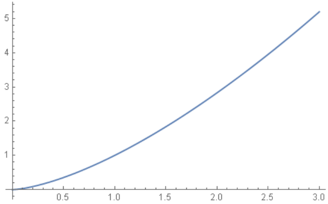


Figure: $f(x) = x^{1.3}$

The exponential function

$$f(x) = a^x, \qquad a > 0$$

Main characteristics:

- $D = \mathbb{R}$
- $R_f = (0, +\infty)$ meaning that $a^x > 0$ for all $x \in \mathbb{R}$
- $f(0) = a^0 = 1$
- if $a > 0$ the function is monotonic strictly increasing
- if $0 < a < 1$ the function is monotonic strictly decreasing
- if $a = 1$ we get the flat line

The exponential function

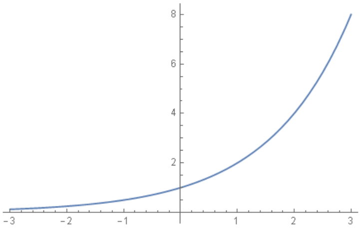


Figure: $f(x) = a^x$, $a > 1$

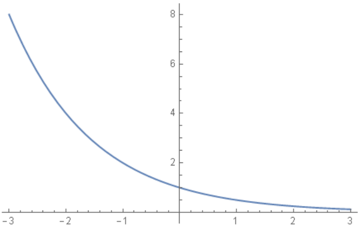


Figure: $f(x) = a^x$, $0 < a < 1$

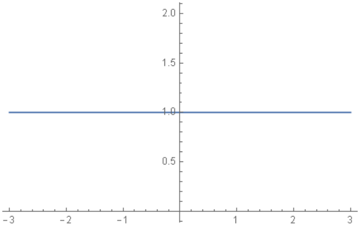


Figure: $f(x) = 1^x$

The logarithmic function

$$f(x) = \log_a(x), \qquad a > 0,\ a \neq 1$$

This is the inverse of the exponential function.

- $D = (0, +\infty)$,
- $R_f = \mathbb{R}$
- $f(1) = \log_a(1) = 0$ (this is a consequence of the fact that $a^0 = 1$)
- if $a > 0$ the function is monotonic strictly increasing
- if $0 < a < 1$ the function is monotonic strictly decreasing

The logarithmic function

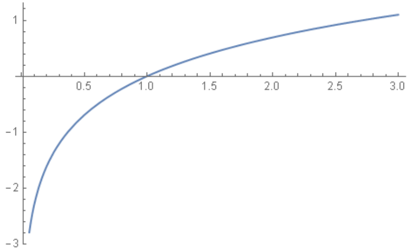


Figure: $f(x) = \log_a(x)$, $a > 1$

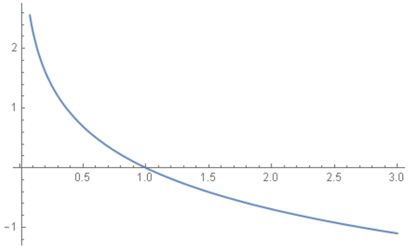
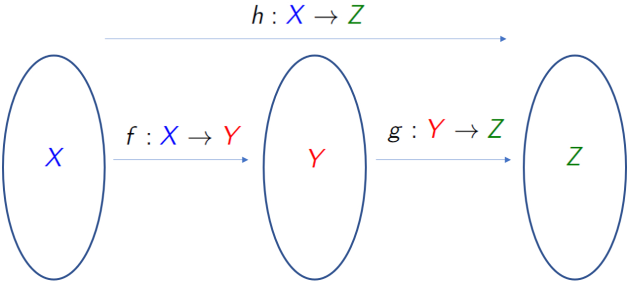


Figure: $f(x) = \log_a(x)$, $0 < a < 1$

The composite function: the intuition

Intuitively, the composition of two functions, $f : X \to Y$ and $g : Y \to Z$, is a function $h : X \to Z$ such that applying h to $x \in X$ produces the same results as applying first f to $x \in X$ and then applying g to $f(x) \in Y$.



The composite function: the definition

Definition

Consider a function $f : X \to Y$ and another function $g : Y \to Z$. The composite function, denoted by $g \circ f$, is defined as:

$$g \circ f : X \to Z$$

$$(g \circ f)(x) = g(f(x)), \quad \forall x \in X$$

Important: The order of composition matters. That is, in general,

$$g(f(x)) \neq f(g(x))$$

The composite function: examples

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$, $\quad g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 - 1$

  We set $y = f(x) = x + 1$

  $$(g \circ f)(x) = g(f(x)) = g(y) = y^2 - 1 = (x + 1)^2 - 1 = x^2 + 2x$$

- $f : \mathbb{R} \to (0, +\infty)$, $f(x) = x^2 + 1$, $\quad g : (0, +\infty) \to (0, +\infty)$, $g(y) = 1/y$

  We set $y = f(x) = x^2 + 1$. Thus:

  $$(g \circ f)(x) = g(f(x)) = g(y) = \frac{1}{y} = \frac{1}{x^2 + 1}$$

Intuitive plots of functions

Let $g : D \to \mathbb{R}$ be a function. For instance $g(x) = x^3 + 6x^2 - 15$.
How can we plot few composite function starting form the plot of $g(x)$?

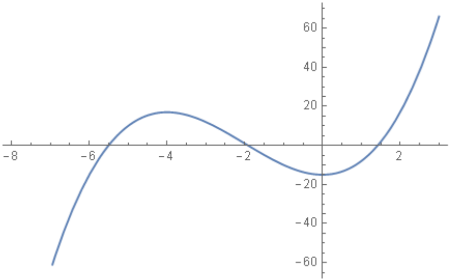


Figure: $g(x) = x^3 + 6x^2 - 15$

We want to derive the plot of a few composite functions, from the plot of $g(x)$.

Intuitive plots of functions

To plot $-g(x)$ we invert the graph of $g(x)$ along the x-axis: that is the negative part becomes positive and the positive part becomes negative.

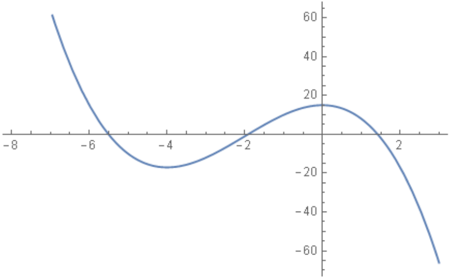


Figure: $-g(x) = -(x^3 + 6x^2 - 15)$

Intuitive plots of functions

To plot $|g(x)|$ we recall the definition of the absolute value:

$$|g(x)| = \begin{cases} g(x) & \text{if } g(x) \geq 0 \\ -g(x) & \text{if } g(x) < 0 \end{cases}$$

Then to plot $|g(x)|$ it is enough to overturn the negative part of the function above the x-axis

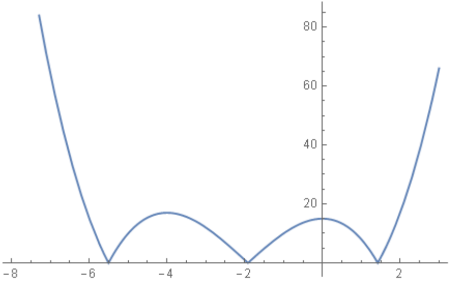


Figure: $|g(x)| = |x^3 + 6x^2 - 15|$

Intuitive plots of functions

To plot $g(x) + c$ we shift the plot of $g(x)$ up by the quantity c, if c is positive and down if c is negative.

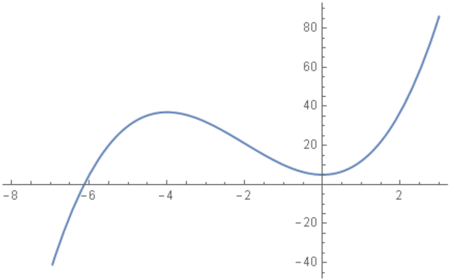


Figure: $g(x) + 20 = x^3 + 6x^2 - 15 + 20$

Intuitive plots of functions

To plot $g(x + a)$ we shift the plot of $g(x)$ on the left by the quantity c, if c is positive and on the right if c is negative.

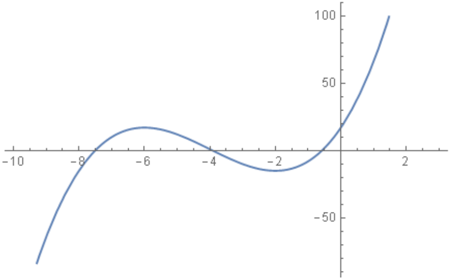


Figure: $g(x+2) = (x+2)^3 + 6(x+2)^2 - 15$

Sequences: the intuition

Intuitively, a sequence is a function which associates to each **natural** number $n \in \mathbb{N}$ a **real** number $s_n \in \mathbb{R}$.

Examples

- $s_n = \frac{1}{n} \quad \Rightarrow \quad s_1 = 1, \quad s_2 = \frac{1}{2}, \quad s_3 = \frac{1}{3}, \quad s_4 = \frac{1}{4}, \quad \dots$
- $s_n = \sqrt{n} \quad \Rightarrow \quad s_1 = 1, \quad s_2 = \sqrt{2}, \quad s_3 = \sqrt{3}, \quad s_4 = 2, \quad \dots$
- $s_n = \frac{n}{n+1} \quad \Rightarrow \quad s_1 = \frac{1}{2}, \quad s_2 = \frac{2}{3}, \quad s_3 = \frac{3}{4}, \quad s_4 = \frac{4}{5}, \quad \dots$
- $s_n = (-1)^n \quad \Rightarrow \quad s_1 = -1, \quad s_2 = 1, \quad s_3 = -1, \quad s_4 = 1, \quad \dots$

Sequences: the definition

Definition

A sequence is any function $s : \mathbb{N} \to \mathbb{R}$. A sequence is denoted by $(s_n)_{n\in\mathbb{N}}$, whereas we denote by s_n the n-th element of the sequence.

Limit of sequences

What happens when n is "very large"?

Examples

$$s_n = \frac{1}{n}$$

n	1	2	3	4	5	...
$\frac{1}{n}$	1	$\frac{1}{2} = 0.5$	$\frac{1}{3} = 0.3333$	$\frac{1}{4} = 0.25$	$\frac{1}{5} = 0.2$	...

$$s_n = \frac{n}{n+1}$$

n	1	2	3	4	5	...
$\frac{n}{n+1}$	$\frac{1}{2} = 0.5$	$\frac{2}{3} = 0.6667$	$\frac{3}{4} = 0.75$	$\frac{4}{5} = 0.8$	$\frac{5}{6} = 0.8333$	...

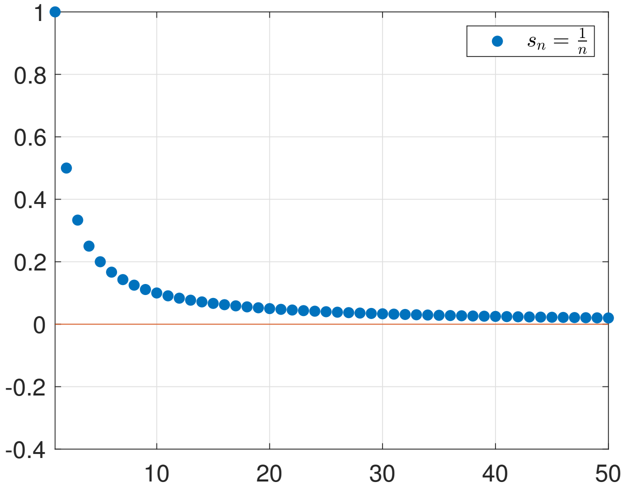
$$s_n = (-1)^n$$

n	1	2	3	4	5	...
$(-1)^n$	-1	1	-1	1	-1	...

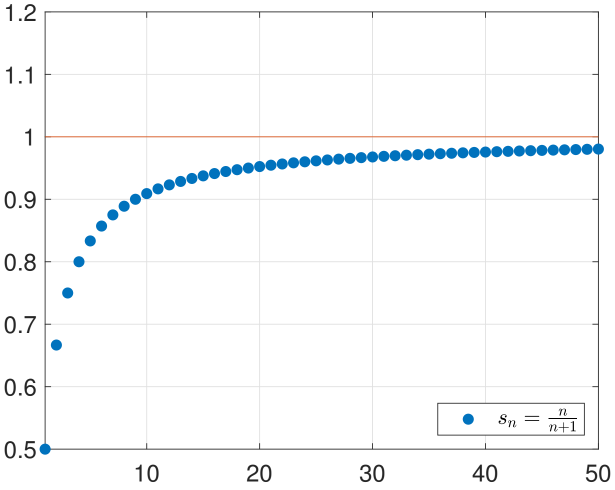
Limit of sequences

- The elements of $s_n = \frac{1}{n}$ approach 0 as n grows
- The elements of $s_n = \frac{n}{n+1}$ approach 1 as n grows
- The elements of $s_n = (-1)^n$ swing between 1 and -1

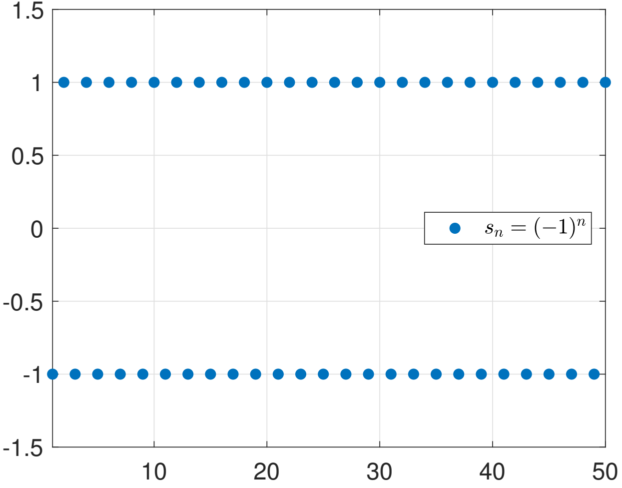
Limit of sequences, cont'd



Limit of sequences, cont'd



Limit of sequences, cont'd



Limit of sequences: the definition

Definition (Convergent sequence)

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. We say that $(s_n)_{n\in\mathbb{N}}$ converges and we write

$$\lim_{n\to\infty} s_n = \ell$$

where $\ell \in \mathbb{R}$ is a finite real number, if:

$$\forall \epsilon > 0 \quad \exists\, n^* \in \mathbb{N} \;:\; \forall n > n^* \;\Rightarrow\; |s_n - \ell| < \epsilon$$

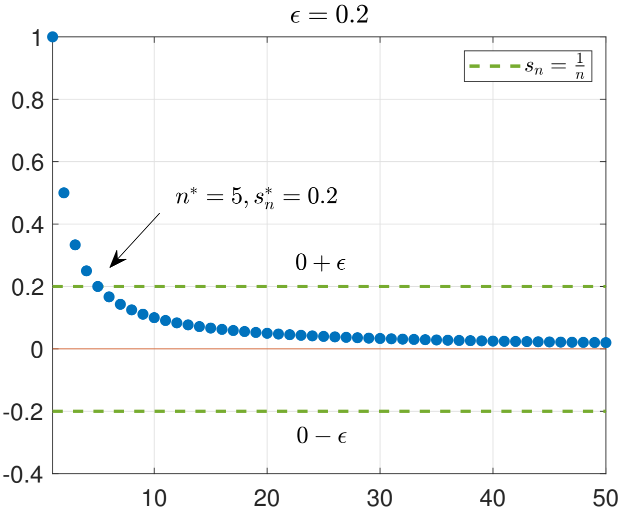
Sometimes we also use the notation $s_n \to \ell$ to indicate $\lim_{n\to\infty} s_n = \ell$.

Remark The above definition says that,

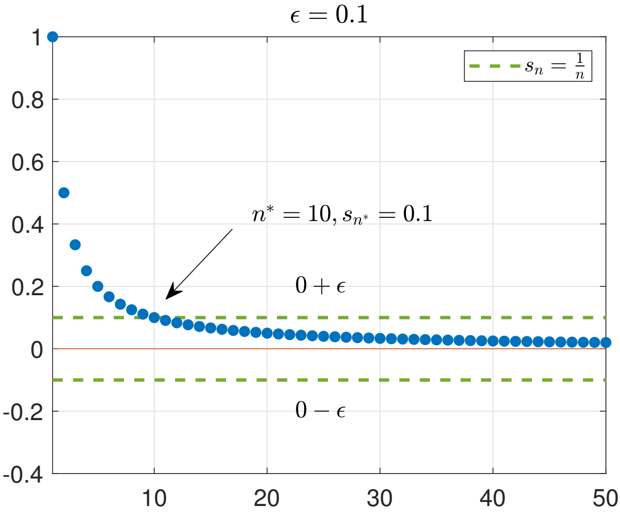
- for all length ϵ (as small as we like)
- we can find a natural number n^*
- such that for all indices n that are larger than n^*
- the distance between s_n and the limit ℓ is smaller than ϵ

That means, s_n approaches ℓ, when n is very large.

Limit of sequences, cont'd



Limit of sequences, cont'd



If we choose a smaller ϵ, the number n^* for which the definition is true gets larger!

Limit of sequences: exercises

Prove that:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Solution

We have to find an n^* such that, for $n > n^*$, we have

$$\left|\frac{1}{n} - 0\right| < \epsilon$$

Let's solve the above inequality:

$$\left|\frac{1}{n} - 0\right| < \epsilon \;\Leftrightarrow\; \frac{1}{n} < \epsilon \Leftrightarrow\; n > \frac{1}{\epsilon}$$

We fix $n^* = \left[\frac{1}{\epsilon}\right]$. Then for all $n > n^*$ we get that $\left|\frac{1}{n} - 0\right| < \epsilon$, and hence the definition is verified.

For example, if $\epsilon = 0.2$, then $n^* = 5$; if $\epsilon = 0.1$, then $n^* = 10$; if $\epsilon = 0.014$, then $n^* = 72$, etc.

Limit of sequences: exercises, cont'd

Prove that:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

Solution

We have to find an n^* such that, for $n > n^*$, we have

$$\left|\frac{n}{n+1} - 1\right| < \epsilon$$

Let's solve the above inequality:

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-n-1}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \epsilon$$

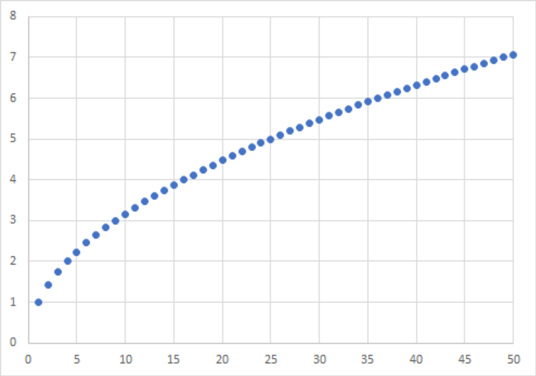
which is solved for $n > \frac{1-\epsilon}{\epsilon}$. We can set $n^* = \left[\frac{1-\epsilon}{\epsilon}\right]$. Then, for all $n > n^*$ it holds that $\left|\frac{n}{n+1} - 1\right| < \epsilon$, and hence the definition is verified.

Limit of sequences: cont'd

Consider the following sequence:

$$s_n = \sqrt{n}$$

Does it have a limit?



Limit of sequences, cont'd

Definition

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. We say that $(s_n)_{n\in\mathbb{N}}$ diverges to $+\infty$, and we write

$$\lim_{n\to+\infty} s_n = +\infty$$

if:

$$\forall M > 0, \quad \exists\, n^* \ :\ \forall n > n^* \ \Rightarrow\ s_n > M$$

Definition

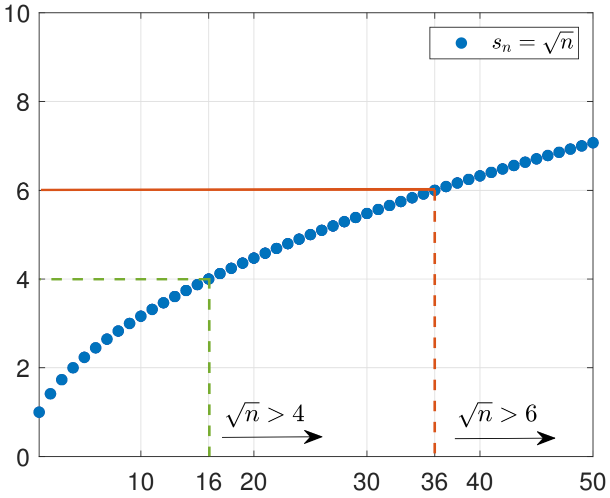
Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. We say that $(s_n)_{n\in\mathbb{N}}$ diverges to $-\infty$, and we write

$$\lim_{n\to+\infty} s_n = -\infty$$

if:

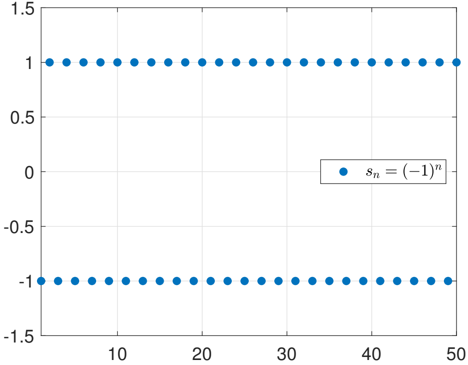
$$\forall M > 0, \quad \exists\, n^* \ :\ \forall n > n^* \ \Rightarrow\ s_n < -M$$

Limit of sequences, cont'd



Sequences: the limit may not exist

The limit of a sequence may not exist.
For instance consider the sequence $(s_n)_{n\in\mathbb{N}}$, with $s_n = (-1)^n$



When n is large this sequence does not approach any specific value.

Sequences: the limit may not exist

Definition

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence, and let $(k_n)_{n\in\mathbb{N}}$ be a sequence

$$k_n : \mathbb{N} \to \mathbb{N}$$

with $k_n < k_{n+1}$. The sequence $(s_{k_n})_{n\in\mathbb{N}}$ is called a subsequence of $(s_n)_{n\in\mathbb{N}}$.

Example. Let $s_n = \frac{1}{n}$.
The even subsequence is:

$$s_2 = \frac{1}{2}, \quad s_4 = \frac{1}{4}, \quad s_6 = \frac{1}{6}, \quad s_8 = \frac{1}{8}, \ldots$$

and it is denotes as $(s_{2n})_{n\in\mathbb{N}}$.
The odd subsequence is:

$$s_3 = \frac{1}{3}, \quad s_5 = \frac{1}{5}, \quad s_7 = \frac{1}{7}, \quad s_9 = \frac{1}{9}, \ldots$$

and it is denotes as $(s_{2n+1})_{n\in\mathbb{N}}$.

Sequences: the limit may not exist

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. We have that

$$\lim_{n\to\infty} s_n = \ell$$

if and only if ***for all subsequences*** *$(s_{k_n})_{n\in\mathbb{N}}$ it holds that*

$$\lim_{n\to\infty} s_{k_n} = \ell$$

Simplified formulation which is used in exercises:

Corollary

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If

$$\lim_{n\to\infty} s_{2n} = \ell_1 \quad \textit{and} \quad \lim_{n\to\infty} s_{2n+1} = \ell_2$$

with $\ell_1 \neq \ell 2$ then the sequence $(s_n)_{n\in\mathbb{N}}$ does not have a limit.

Sequences: the limit may not exist, cont'd

If the limit exists, it does **not** change when considering "sub-sequences".

If we find two subsequences that converge to different limits, then the original sequence does not converge.

Notice that, if the even and the odd subsequences converge to the same limit, this does not tell us anything about the sequence $(s_n)_{n\in\mathbb{N}}$, which may converge or not.

Example

Consider the following sequence:

$$s_n = (-1)^n$$

Then:

$$s_{2n} = (-1)^{2n} = 1 \to 1$$
$$s_{2n+1} = (-1)^{2n+1} = -1 \to -1$$

Since the two sub-sequences converge to different limits, the limit of $s_n = (-1)^n$ does not exist.

What about $\lim_{n\to\infty} \cos(n\pi)$?

Sequences: some useful theorems

Theorem (Uniqueness of the limit)

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If the sequence converges then the limit is unique.

This theorem says that it is impossible that a sequence converges to two different limits.

Sequences: some useful theorems, cont'd

Theorem (Absolute value theorem for sequences)

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If $|s_n| \to 0$, then $s_n \to 0$.

Important: This theorem holds only if the limit is zero!

Exercise

Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

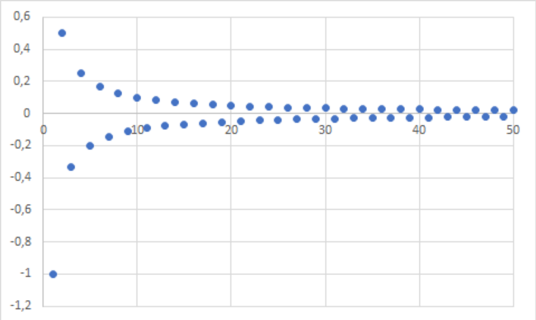
Let's compute first $\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right|$. Observe that:

$$\left|\frac{(-1)^n}{n}\right| = \frac{|(-1)^n|}{|n|} = \frac{1}{n}$$

But we know that $\frac{1}{n} \to 0$. Thus, by the absolute value theorem, we conclude that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Sequences: some useful theorems, cont'd

$$s_n = \frac{(-1)^n}{n}$$



We easily see from the plot that the sequence converges to zero.

Sequences: some useful theorems, cont'd

Theorem (The comparison theorem)

Let $(a_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ *be three sequences such that*

- $a_n \le s_n \le b_n$ *for every* n
- $\lim_{n\to+\infty} a_n = \ell$ *and* $\lim_{n\to+\infty} b_n = \ell$

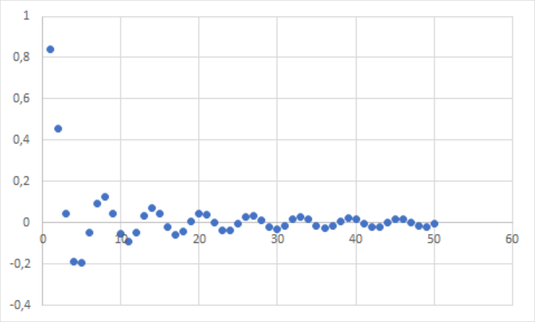
Then

$$\lim_{n\to+\infty} s_n = \ell$$

Sequences: some useful theorems, cont'd

Example

Consider the sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = \frac{\sin(n)}{n}$



Sequences: some useful theorems, cont'd

Example

Consider the sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = \frac{\sin(n)}{n}$.
Since

$$-1 \leq \sin(n) \leq 1,$$

then

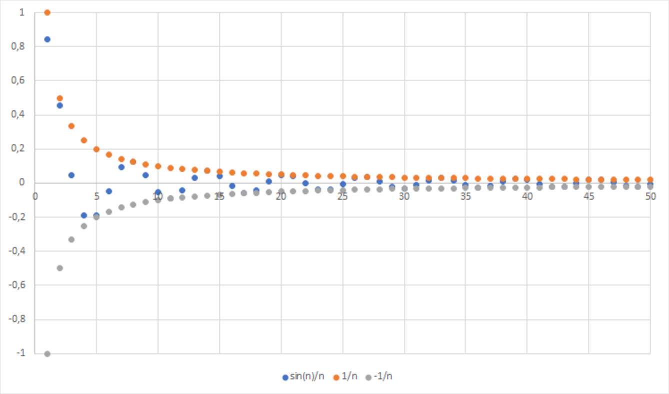
$$\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}, \quad \text{for every } n.$$

We call $a_n = \frac{-1}{n}$ and $b_n = \frac{1}{n}$ and observe that $\lim_{n\to+\infty} \frac{-1}{n} = 0$ and $\lim_{n\to+\infty} \frac{1}{n} = 0$. By the comparison theorem also

$$\lim_{n\to+\infty} \frac{\sin(n)}{n} = 0$$

Sequences: some useful theorems, cont'd

Example, cont'd



Sequences: some useful theorems, cont'd

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ be two sequences with $s_n \to \ell$ and $q_n \to \ell'$, $\ell, \ell' \in \mathbb{R}$ (finite numbers). Then:

- $s_n + q_n \to \ell + \ell'$
- $s_n - q_n \to \ell - \ell'$
- $s_n \cdot q_n \to \ell \cdot \ell'$.
- *if* $\ell' \neq 0$, *then* $\frac{s_n}{q_n} \to \frac{\ell}{\ell'}$.

Moreover,

- *If $s_n \to +\infty$ and $q_n \to +\infty$, then $s_n + q_n \to +\infty$ and $s_n \cdot q_n \to +\infty$.*
- *If $s_n \to -\infty$ and $q_n \to -\infty$, then $s_n + q_n \to -\infty$ and $s_n \cdot q_n \to +\infty$.*
- *If $s_n \to +\infty$ and $q_n \to -\infty$, then $s_n + q_n$ is **indetermined** and $s_n \cdot q_n \to -\infty$.*

"Practical" rules

We have formally proved that:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

This limit can also be computed using the following "practical" rule:

$$\text{For } \alpha \in \mathbb{R}, \quad \frac{\alpha}{\infty} = 0$$

Similarly, for $\alpha \in \mathbb{R}$, we have:

$$\alpha + \infty = +\infty$$

so that, for instance:

$$\lim_{n\to+\infty} \pi + n = +\infty$$

Can we always use these rules?

“Practical rules” and indeterminate forms

Let $\alpha \in \mathbb{R}$. Then:

- $\alpha + \infty = +\infty$, and $\alpha - \infty = -\infty$
- $(+\infty) + (+\infty) = +\infty$ and $(-\infty) + (-\infty) = -\infty$
- $(+\infty) \times (+\infty) = +\infty$ and $(-\infty) \times (-\infty) = +\infty$
- $(+\infty) \times (-\infty) = -\infty$ and $(-\infty) \times (+\infty) = -\infty$
- $\frac{\alpha}{+\infty} = 0$, $\frac{\alpha}{-\infty} = 0$, $0^{+\infty} = 0$.
- If $\alpha > 0$ then $\alpha \times (+\infty) = +\infty$ and $\alpha \times (-\infty) = -\infty$
- If $\alpha < 0$ then $\alpha \times (+\infty) = -\infty$ and $\alpha \times (-\infty) = +\infty$
- $(+\infty) - (+\infty)$ and $(-\infty) + (+\infty)$ are INDETERMINATE
- $0 \times (+\infty)$ and $0 \times (-\infty)$ are INDETERMINATE
- $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are INDETERMINATE
- ∞^0 and 0^0 and 1^∞ are INDETERMINATE