Mathematics 2

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Week 1 - Integrals

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Outline

Indefinite Integral

Suppose you are given a function $g : \mathcal{D} \to \mathbb{R}$ and you want to compute a function F such that

$$F'(x) = g(x), \qquad \forall x \in \mathcal{D}$$

The function F is called an *anti-derivative* of g.

In fact, to compute F we will need to reverse the process of computing a derivative

Indefinite Integral

Remark (The antiderivative is not unique)

Suppose you are given the function

$$g(x) = 2x.$$

An anti-derivative of $g(x)$ *is*

$$F(x) = x^2$$

Another antiderivative of $g(x)$ *is*

$$F(x) = x^2 + 5$$

In fact the derivative of any constant is null.

Theorem

Let $g : D \subseteq \mathbb{R} \to \mathbb{R}$ *be a function.*
If $g(x)$ *has an anti-derivative* $F(x)$*, then it has infinitely many anti-derivatives that are given by*

$$F(x) + c$$

Proof. By hypothesis, $F'(x) = g(x)$. We want to show that the derivative of $F(x) + c$ is also $g(x)$. In fact we get that:

$$D[F(x) + c] = F'(x) + 0 = g(x).$$

This concludes the proof.

Definition (Indefinite integral)

Let $g : D \subseteq \mathbb{R} \to \mathbb{R}$ *be a function. The indefinite integral of* g *is the class of all antiderivatives of* g.

We indicate the integral of g *with the symbol*

$$\int g(x)dx.$$

and the function $g(x)$ *is called the integrand.*

Elementary Indefinite Integral

(i) Let $a \neq -1$.

$$\int x^a \, dx = \frac{x^{a+1}}{a+1} + c, \qquad \text{for all } c \in \mathbb{R}$$

Proof. We compute the derivative of $\frac{x^{a+1}}{a+1} + c$ and we would like to get x^a.

$$D\left[\frac{x^{a+1}}{a+1} + c\right] = \frac{1}{a+1} + D\left[x^{a+1}\right] + 0 = \frac{(a+1)x^a}{a+1} = x^a$$

Elementary Indefinite Integral

(ii) If $a = -1$, then $x^a = \frac{1}{x}$ which is well defined for all $x \neq 0$.

$$\int \frac{1}{x}\, dx = \lg|x| + c, \qquad \text{for all } c \in \mathbb{R},\ x \neq 0$$

Be careful! You need the ABSOLUTE VALUE of x, since the function $\frac{1}{x}$ is defined for all $x \neq 0$, BUT $\lg x$ is only defined for $x > 0$.

Elementary Indefinite Integral: Examples

1. $$\int x \, dx = \frac{x^2}{2} + c, \qquad \text{for all } c \in \mathbb{R}$$

2. $$\int x^2 \, dx = \frac{x^3}{3} + c, \qquad \text{for all } c \in \mathbb{R}$$

3. $$\int \frac{1}{x^4} \, dx = \int x^{-4} \, dx = \frac{x^{-4+1}}{-4+1} + c = -\frac{1}{3x^3} + c, \quad \text{for all } c \in \mathbb{R}$$

4. $$\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{2x\sqrt{x}}{3} + c, \quad \text{for all } c \in \mathbb{R}$$

Elementary Indefinite Integral

(iii) $\displaystyle\int e^x \, dx = e^x + c, \qquad \text{for all } c \in \mathbb{R}.$

(iv) $\displaystyle\int e^{ax} \, dx = \frac{e^{ax}}{a} + c, \qquad \text{for all } c \in \mathbb{R}.$

Proof.

$$D\left[\frac{e^{ax}}{a} + c\right] = \frac{1}{a} D[e^{ax}] + 0 = \frac{a e^{ax}}{a} = e^{ax}.$$

(v) Finally, since $a^x = e^{x \lg a}$, for all $a > 0$, we get that

$$\int a^x \, dx = \int e^{x \lg a} \, dx = \frac{1}{\lg a} e^{x \lg a} + c = \frac{1}{\lg a} a^x + c, \qquad \text{for all } c \in \mathbb{R}.$$

Elementary Indefinite Integral:Examples

1. $\displaystyle\int e^{5x}\,dx = \frac{1}{5}e^{5x} + c, \qquad \text{for all } c \in \mathbb{R}.$
2. $\displaystyle\int e^{\frac{1}{5}x}\,dx = 5e^{\frac{1}{5}x} + c, \qquad \text{for all } c \in \mathbb{R}.$
3. $\displaystyle\int 7^{x}\,dx = \frac{1}{\lg 7}7^{x} + c, \qquad \text{for all } c \in \mathbb{R}.$

Elementary Indefinite Integral

(vi) $\displaystyle\int \cos x \, dx = \sin x + c, \qquad \text{for all } c \in \mathbb{R}.$

(vii) $\displaystyle\int \sin x \, dx = -\cos x + c, \qquad \text{for all } c \in \mathbb{R}.$

(viii) $\displaystyle\int \frac{1}{\cos^2 x} \, dx = \tan x + c, \qquad \text{for all } c \in \mathbb{R}.$

Proof.

$$D[\tan(x) + c] = \frac{1}{\cos^2 x} + 0$$

(ix) $\displaystyle\int \frac{1}{\sqrt{1+x^2}} \, dx = \arctan(x) + c, \qquad \text{for all } c \in \mathbb{R}.$

(x) $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x) + c, \qquad \text{for all } c \in \mathbb{R}.$

Elementary Indefinite Integral: the chain rule

By the chain rule we have that $D[f(g(x))] = f'(g(x)) \cdot f'(x)$. By reversing this rule we can compute the following elementary integrals:

1. For all $a \neq -1$

$$\int \big(g(x)\big)^a g'(x)\, dx = \frac{\big(g(x)\big)^{a+1}}{a+1} + c, \qquad \text{for all } c \in \mathbb{R}.$$

Proof. Using the chain rule:

$$D\left[\frac{\big(g(x)\big)^{a+1}}{a+1} + c\right] = \frac{1}{a+1} D\left[\big(g(x)\big)^{a+1}\right]$$
$$= \frac{1}{a+1}(a+1)\big(g(x)\big)^a g'(x) = \big(g(x)\big)^a g'(x).$$

Elementary Indefinite Integral: the chain rule

2. $\displaystyle\int e^{g(x)} g'(x)\, dx = e^{g(x)} + c, \qquad$ for all $c \in \mathbb{R}$.

Proof. Using the chain rule:

$$D\left[e^{g(x)} + c\right] = D\left[e^{g(x)}\right] + 0 = e^{g(x)} g'(x)$$

Elementary Indefinite Integral: the chain rule

3. $\displaystyle\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + c,$ for all $c \in \mathbb{R}$ and $f(x) \neq 0$.

Proof.

CASE 1: $g(x) > 0$. Then $\ln|g(x)| = \ln(g(x))$. By the chain rule,

$$D[\ln(g(x)) + c] = D[\ln(g(x))] + 0 = \frac{g'(x)}{g(x)}.$$

CASE 2: $g(x) < 0$. Then $\ln|g(x)| = \ln(-g(x))$. By the chain rule,

$$D[\ln(-g(x)) + c] = D[\ln(-g(x))] + 0 = \frac{-g'(x)}{-g(x)} = \frac{g'(x)}{g(x)}.$$

Elementary Indefinite Integral:Examples

- $\displaystyle\int \frac{\lg x}{x}\, dx = \frac{1}{2}\lg(x) + c,$ for all $c \in \mathbb{R}$, and $x > 0$.
- $\displaystyle\int 3x^2 e^{x^3}\, dx = e^{x^3} + c,$ for all $c \in \mathbb{R}$
- $\displaystyle\int \frac{2x+1}{x^2+x} dx = \lg|x^2 + x| + c,$ for all $c \in \mathbb{R}$, and $x \neq 0, -1$

Other Elementary integrals

4. $$\int g'(x)\sin(g(x))\;dx = -\cos(g(x)) + c,\;\; \text{for all } c \in \mathbb{R}$$

5. $$\int g'(x)\cos(g(x))\;dx = \sin(g(x)) + c,\;\; \text{for all } c \in \mathbb{R}$$

6. $$\int \frac{g'(x)}{\cos^2(g(x))}\;dx = \tan(g(x)) + c,\;\; \text{for all } c \in \mathbb{R}$$

7. $$\int \frac{g'(x)}{\sqrt{1+[g(x)]^2}}\;dx = \arctan(g(x)) + c,\;\; \text{for all } c \in \mathbb{R}$$

8. $$\int \frac{g'(x)}{\sqrt{1-[g(x)]^2}}\;dx = \arcsin(g(x)) + c,\;\; \text{for all } c \in \mathbb{R}$$

Properties of integrals: Linearity

- $\displaystyle\int kf(x)\ dx = k\int f(x)\ dx$
- $\displaystyle\int [f(x)+g(x)]\ dx = \int f(x)\ dx + \int g(x)\ dx$
- $\displaystyle\int k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)\ dx =$

  $\displaystyle k_1\int f_1(x)\ dx + k_2\int f_2(x)\ dx + \cdots + k_n\int f_n(x)\ dx$

Properties of integrals: examples

- $\displaystyle\int 3x^4 + 5x^2 + 2 \, dx = \frac{3}{5}x^5 + \frac{5}{3}x^3 + 2x + c, \quad \text{for all } c \in \mathbb{R}$
- $\displaystyle\int \frac{2}{x} + \frac{4}{\sqrt[3]{x}} + 6e^{2x} \, dx = 2 \lg |x| + \frac{3}{2}\sqrt[3]{x^2} + 3e^{2x} + c,$

  for all $c \in \mathbb{R}$ and $x \neq 0$

Integration by substitution

We want to compute:

$$\int g(x)\ dx$$

5 step procedure!

1. Change of variable: Define $x = h(t)$ for a suitable invertible function h
2. Differentiate: By differentiation we get $dx = h'(t)dt$
3. Substitute: Substitute x and dx $\int g(h(t))h'(t)dt$
4. Solve the integral with variable t: (perhaps simpler!)

$$\int g(h(t))\ h'(t)\ dt = F(t) + c$$

5. Come back to x: $\int g(x)\ dx = F(h^{-1}(x)) + c$

Examples

Solve the following integrals:

1. $$\int \frac{\lg \sqrt{x}}{x} dx$$
2. $$\int \frac{1}{x + \sqrt{x}} dx$$
3. $$\int \frac{1}{e^x + e^{-x}} dx$$

Integration by parts

Recall the product rule:

$$D[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Suppose you want to compute

$$\int f'(x)g(x)\ dx$$

By the product rule we get

$$\int f'(x)g(x)\ dx = f(x)g(x) - \int f(x)g'(x)\ dx$$

This formula is called Integration by parts rule

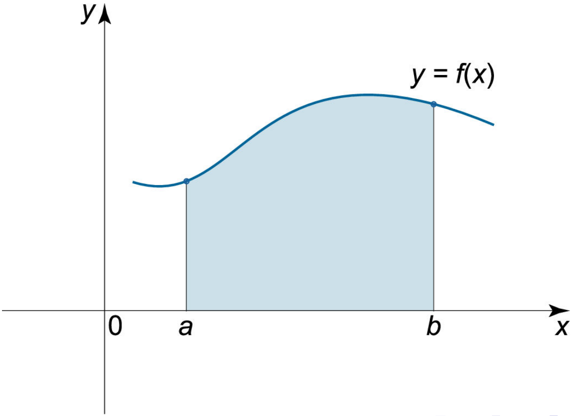
Examples

Compute the following integrals:

1. $$\int x \sin x \; dx$$
2. $$\int x e^{2x} \; dx$$
3. $$\int \lg x \; dx$$

Area under a curve

Suppose you want to compute the area bounded by the graph of the positive and continuous function $f(x)$ and the x-axis for $a \leqslant x \leqslant b$:



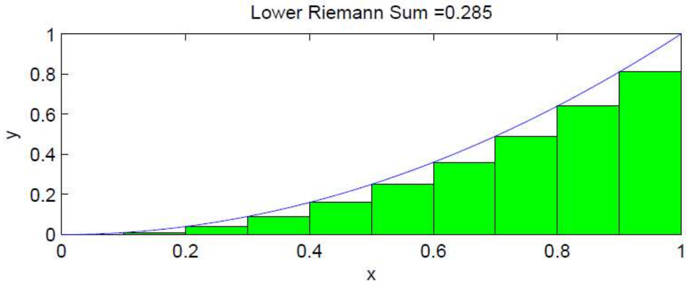
Area under a curve

- Divide the interval $[a, b]$ into n sub-intervals, by choosing points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

 Call the partition P
- Let $\Delta^{(i)} = x_i - x_{i-1}$ be the length of each sub-interval.
- Let m_i be the minimum value of the function f in the interval $[x_{i-1}, x_i]$ That is, there is $c_i \in [x_{i-1}, x_i]$ such that $m_i = f(c_i)$
- Draw, for each interval $[x_{i-1}, x_i]$ a rectangle with dimensions $\Delta^{(i)}, m_i$
- By summing up the area of each single rectangle we approximate from below the area under the curve
- That is

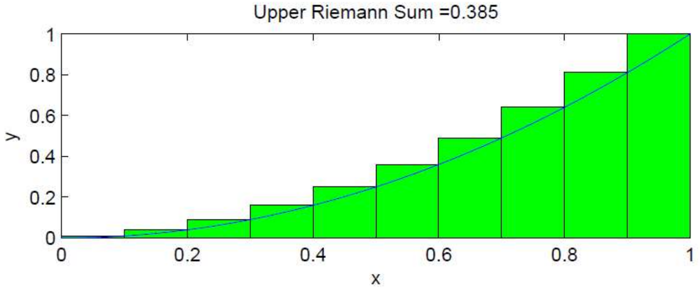
$$\sum_{i=0}^{n-1} m_i \Delta^{(i)} = \sum_{i=0}^{n-1} f(c_i)\Delta^{(i)} < A$$


Lower Riemann Sum =0.285
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x

Area under a curve

- Now choose the maximum value M_i assumed by the function f in the interval $[x_{i-1}, x_i]$ That is, there is $d_i \in [x_{i-1}, x_i]$ such that $M_i = f(d_i)$
- Draw, for every interval $[x_{i-1}, x_i]$ a rectangle with dimensions Δ^i and M_i
- By summing up the area of each single rectangle approximate from above the area under the curve
- That is

$$\sum_{i=0}^{n-1} M_i \Delta^{(i)} = \sum_{i=0}^{n-1} f(d_i) \Delta^{(i)} > A$$


Upper Riemann Sum =0.385
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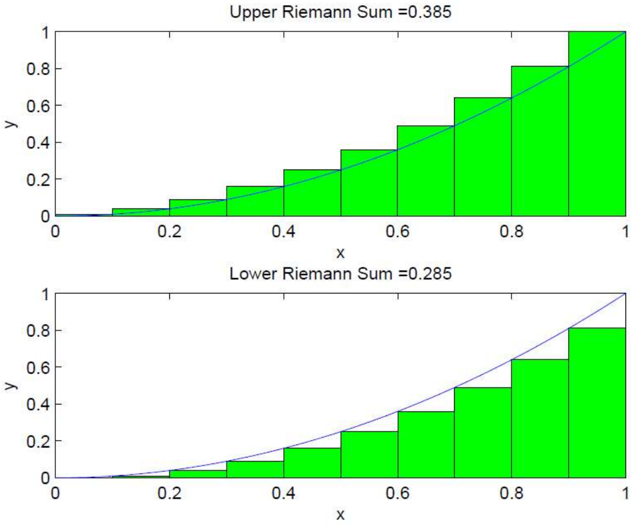
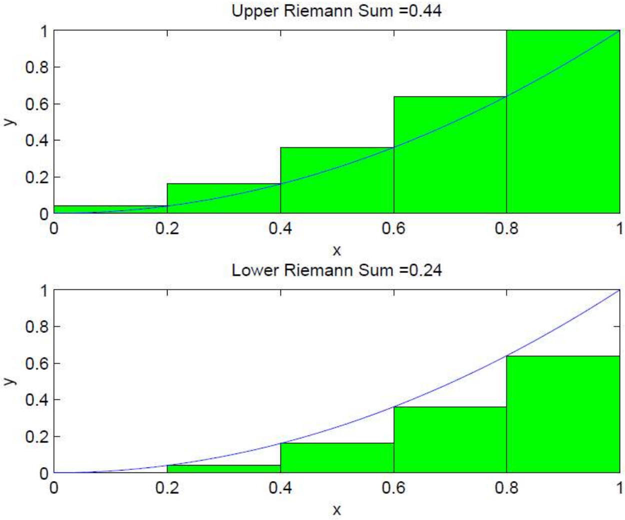
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x


Upper Riemann Sum =0.385
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x
Lower Riemann Sum =0.285
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x

Area under a curve

- By making the partition of the interval $[a, b]$ finer and finer, the upper and lower approximations get better
- In the limit (that is when taking $\Delta^{(i)} \to 0$) the upper and the lower approximations coincide and they also coincide with the Area under the curve from a to b
- That is

$$A = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(c_i)\Delta^{(i)} = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(d_i)\Delta^{(i)} = \int_a^b f(x)\, dx$$


Upper Riemann Sum =0.44
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0.6
0.4
0.2
0
y
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0.2
0.4
0.6
0.8
1
x
Lower Riemann Sum =0.24
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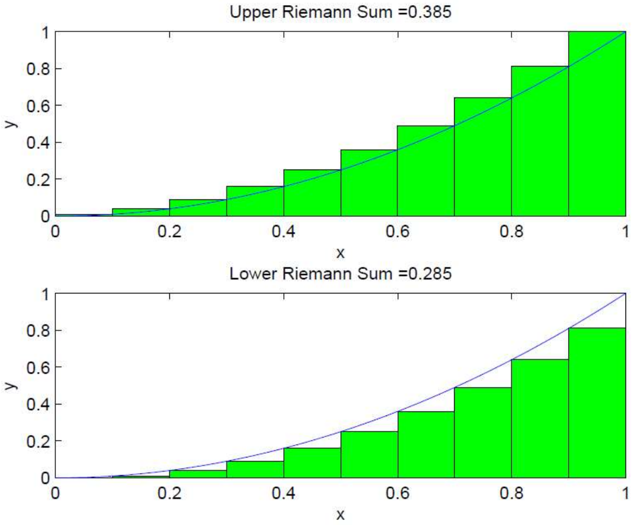
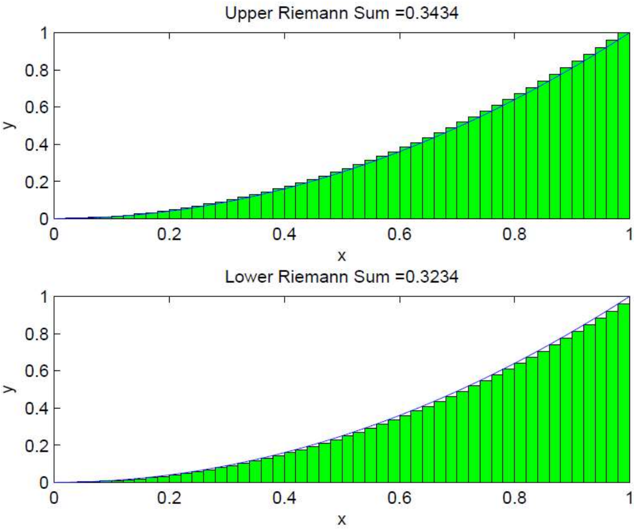
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x


Upper Riemann Sum =0.385
Lower Riemann Sum =0.285
y
x
0
0.2
0.4
0.6
0.8
1


Upper Riemann Sum =0.3434
y
x
0
0.2
0.4
0.6
0.8
1
Lower Riemann Sum =0.3234
y
x
0
0.2
0.4
0.6
0.8
1

Riemann Integral

- Let $P = [a = x_0, x_1, \ldots, x_n = b]$ be a partition of the interval $[a, b]$
- Define the *Upper Riemann sum of* f *with respect to the partition* P as

$$U(f, P) = \sum_{i=0}^{n-1} M_i \Delta^{(i)}$$

- Define the *Lower Riemann sum of* f *with respect to the partition* P as

$$L(f, P) = \sum_{i=0}^{n-1} m_i \Delta^{(i)}$$

Riemann Integral

Let Π the set of all possible partitions of the interval $[a, b]$ and define the **Upper Riemann sum** as

$$U(f) = inf_{P \in \Pi} U(f, P)$$

and the **Lower Riemann sum** as

$$L(f) = sup_{P \in \Pi} L(f, P)$$

Definition

A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if the Upper Riemann sum $U(f)$ and Lower Riemann sum $L(f)$ are equal.
The Riemann integral, denoted by

$$\int_a^b f(x)\, dx$$

is the value $U(f)$ (or $L(f)$).