Mathematics 2, AY 2023-2024
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Additional Exercises n.1

Ex 1. Show that the function $F(x)$ is an antiderivative of the function $g(x)$:

1. $F(x) = e^{x^2+\sin(x^2+1)} - 1, \qquad g(x) = 2xe^{x^2+\sin(x^2+1)}(\cos(x^2+1)+1);$
2. $F(x) = \sin(x^2+3x) - 5, \qquad g(x) = (2x+3)\cos(x^2+3x);$
3. $F(x) = \dfrac{2x}{x^2+3x}, \qquad g(x) = \dfrac{2}{x^2+3x} - \dfrac{2x(2x+3)}{(x^2+3x)^2};$
4. $F(x) = \dfrac{\sqrt{x^2+2x}}{x}, \qquad g(x) = -\dfrac{1}{x\sqrt{x(x+2)}};$
5. $F(x) = \sqrt{e^{(x^2+2x)} - 3x}, \qquad g(x) = \dfrac{e^{x^2+2x}(2x+2) - 3}{2\sqrt{e^{x^2+2x} - 3x}}.$

SOLUTION.
We only need to show that $F'(x) = g(x)$.

Ex 2. Show that the functions $F(x)$ and $G(x)$ below

$$F(x) = \sin^2(x) + 7, \quad G(x) = -\frac{1}{2}\cos(2x) - 11$$

are antiderivatives of the same function $g(x)$.

SOLUTION.
We easily see that

$$F'(x) = 2\sin(x)\cos(x),$$
$$G'(x) = -\frac{1}{2}2(-\sin(2x)) = 2\sin(x)\cos(x)$$

Ex 3. Compute the following elementary indefinite integrals

1. $\displaystyle\int \frac{x^2}{\sqrt{x}}\,dx = \frac{2}{5}x^{\frac{5}{2}} + C$
2. $\displaystyle\int \sin x \cos x\,dx = \frac{\sin^2(x)}{2} + C$
3. $\displaystyle\int \frac{1}{x}\sqrt{\log x}\,dx = \frac{2}{3}\ln^{\frac{3}{2}}(x) + C$
4. $\displaystyle\int \tan x\,dx = -\ln|\cos(x)| + C$

5. $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln\left|x^2+1\right| + C$
6. $\int x^2 \sin x^3\,dx = -\frac{1}{3}\cos\left(x^3\right) + C$
7. $\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\arcsin\left(x^2\right) + C$
8. $\int \frac{1}{1+9x^2}\,dx = \frac{1}{3}\arctan\left(3x\right) + C$
9. $\int \frac{1}{\cos^2(2x)}\,dx = \frac{1}{2}\tan\left(2x\right) + C$
10. $\int \frac{e^x}{\cos^2(e^x)}\,dx = \tan\left(e^x\right) + C$
11. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + C$
12. $\int \frac{3x^2+4x+1}{\sqrt{x}}\,dx = \frac{6}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{\frac{3}{2}} + 2\sqrt{x} + C$
13. $\int \frac{1}{\sqrt{4-x^2}}\,dx = \arcsin\left(\frac{1}{2}x\right) + C$
14. $\int \frac{2x+3}{2x+1}\,dx = \ln|2x+1| + x + C$
15. $\int \frac{1}{3x^2+5}\,dx = \frac{1}{\sqrt{15}}\arctan\left(\sqrt{\frac{3}{5}}x\right) + C$
16. $\int xe^{-(x^2+1)}\,dx = -\frac{1}{2}e^{-x^2-1} + C$
17. $\int \frac{x}{\cos^2(x^2)}\,dx = \frac{1}{2}\tan\left(x^2\right) + C$
18. $\int \frac{5-3x}{\sqrt{4-3x^2}}\,dx = \frac{5}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right) + \sqrt{4-3x^2} + C$
19. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \arcsin\left(e^x\right) + C$
20. $\int x\ 7^{x^2}\,dx = \frac{1}{2\ln(7)}\cdot\ 7^{x^2} + C$
21. $\int \frac{3x^2+2x}{1+(x^3+x^2)^2}\,dx = \arctan\left(x^3+x^2\right) + C$
22. $\int (2x+1)\cos(x^2+x+1)\,dx = \sin\left(x^2+x+1\right) + C$
23. $\int \cos(x)\sqrt{\sin(x)}\,dx = \frac{2}{3}\sin^{\frac{3}{2}}(x) + C$
24. $\int \frac{\cos(\log(x))}{x}\,dx = \sin\left(\ln\left(x\right)\right) + C$
25. $\int \frac{x}{\sqrt{x^2+1}\cos^2(\sqrt{x^2+1})}\,dx = \tan(\sqrt{x^2+1}) + C$

26. $\int \frac{\sin(x)\cos(x)(3\cos(x)+2)}{\sin^2(x)-\cos^3(x)+2}\,dx = \ln|\sin^2(x)-\cos^3(x)+2)| + C$

27. $\int \frac{1}{\sin(x)\cos(x)}\,dx = \ln|\tan(x)| + C$

28. $\int \frac{5^x}{1+5^{2x}}\,dx = \frac{\arctan(5^x)}{\ln(5)} + C$

29. $\int 3\sin(x)\cos^3(x)\,dx = -\frac{3}{4}\cos^4(x) + C$

30. $\int (x^3+2x)(x^4+4x^2)^7\,dx = \frac{1}{32}(x^4+4x^2)^8 + C$

Ex 4. Use a suitable substitution to compute the following integrals

1. $\int \sqrt{1+4x}\,dx = \frac{1}{6}(1+4x)^{\frac{3}{2}} + C$

2. $\int \sin\left(\frac{3x-1}{2}\right)\,dx = -\frac{2}{3}\cos\left(\frac{1}{2}(3x-1)\right) + C$

3. $\int \frac{1}{e^x+1}\,dx = -\ln|e^x+1| + x + C$

4. $\int \frac{\sin(1+\sqrt{x})}{\sqrt{x}}\,dx = -2\cos\left(1+\sqrt{x}\right) + C$

5. $\int \frac{x}{\sqrt{x+1}}\,dx = 2\left(\frac{1}{3}(x+1)^{\frac{3}{2}} - \sqrt{x+1}\right) + C$

6. $\int \frac{e^x}{\sqrt{e^x+1}}\,dx = 2\sqrt{e^x+1} + C$

7. $\int \frac{x}{\sqrt{2x+1}}\,dx = \frac{1}{3}(x-1)\sqrt{2x+1} + C$

8. $\int \frac{\cos(\ln(x))}{x}\,dx = \sin(\ln(x)) + C$

9. $\int \frac{x}{\sqrt{x^2+1}\cos^2(\sqrt{x^2+1})}\,dx = \tan\left(\sqrt{x^2+1}\right) + C$

10. $\int \frac{-\sqrt{x-1}}{x+5}\,dx = -2\sqrt{6}\left(-\arctan\left(\sqrt{\frac{1}{6}(x-1)}\right) + \sqrt{\frac{1}{6}(x-1)}\right) + C$

11. $\int \frac{7\sqrt{x}+1}{\sqrt{x}(7\sqrt{x}-1)}\,dx = 2\sqrt{x} + \frac{4}{7}\ln\left|7\sqrt{x}-1\right| + C$

Ex 5. Use integration by parts to solve the following integrals

1. $\int xe^{-2x}\,dx = \frac{1}{4}\left(-2e^{-2x}x - e^{-2x}\right) + C$

2. $\int x^2\log x\,dx = \frac{1}{3}x^3\ln(x) - \frac{x^3}{9} + C$

3. $\int (x+2)\cos x\,dx = \sin(x)(x+2) + \cos(x) + C$

4. $\int x^2 \arcsin x \; dx = \frac{1}{3}x^3 \arcsin(x) - \frac{1}{3}\left(\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} - \sqrt{1-x^2}\right) + C$
5. $\int \arcsin x \; dx = x \arcsin(x) + \sqrt{1-x^2} + C$
6. $\int \arctan x \; dx = x \arctan(x) - \frac{1}{2}\ln\left|x^2+1\right| + C$
7. $\int (x^2 - 2x + 5)e^{-x} \; dx = -e^{-x}x^2 - 5e^{-x} + C$
8. $\int (x^3 + 2x)\log x \; dx = \frac{1}{4}x^4 \ln(x) - \frac{x^4}{16} + x^2 \ln(x) - \frac{x^2}{2} + C$
9. $\int e^x \sin(x) \; dx = -\frac{e^x \cos(x)}{2} + \frac{e^x \sin(x)}{2} + C$
10. $\int e^x \cos(x) \; dx = \frac{e^x \sin(x)}{2} + \frac{e^x \cos(x)}{2} + C$
11. $\int 2xe^{-x} \; dx = 2\left(-e^{-x}x - e^{-x}\right) + C$
12. $\int x \ln^2(5x) \; dx = \frac{1}{2}x^2 \ln^2(5x) - \frac{1}{2}x^2 \ln(5x) + \frac{x^2}{4} + C$