Mathematics 2

Prof. Katia Colaneri

Week 2 - Integrals, Linear Algebra

University of Rome Tor Vergata

13 – 17 November, 2023

Outline

Properties of Riemann Integral

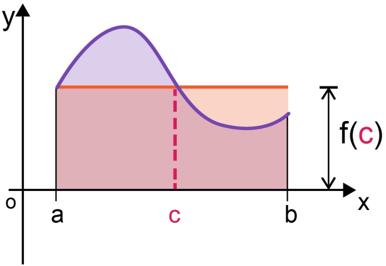
1. The Riemann Integral is defined for every continuous function (positive and negative).
2. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$ if $a > b$.
3. $\int_a^a f(x)\ dx = 0$.
4. $\int_a^b [\alpha f(x) + \beta g(x)]\ dx = \alpha \int_a^b f(x)\ dx + \beta \int_a^b g(x)\ dx, \quad \alpha, \beta \in \mathbb{R}$.
5. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$.
6. If $f(x) \leqslant g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\ dx \leqslant \int_a^b g(x)\ dx$.

Theorem (The Mean Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then there exists a point* $c \in [a, b]$ *such that*

$$\int_a^b f(x)dx = f(c)(b - a)$$

Meaning: If $f(x) > 0$ in the interval $[a, b]$, the area under the graph of $f(x)$ is equivalent to the area of a rectangle with width $(b - a)$ and height $f(c)$.



Let $f : [a, b] \to \mathbb{R}$ be a continuous function. We define the integral function $F : [a, b] \to \mathbb{R}$ as

$$F(x) = \int_a^x f(t)\, dt.$$

Theorem (The Fundamental Theorem of Calculus)

The function F s differentiable for all $x \in [a, b]$ and satisfies

1. $F'(x) = f(x)$
2. $F(a) = 0$

The importance of the *Fundamental Theorem of Calculus*

1. F is an antiderivative of f
2. For any other antiderivative G of f we have that

$$F(x) = G(x) - G(a)$$

In particular

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

Example

Compute

$$\int_{-2}^{1} x^2 + 1 \, dx$$

An antiderivative is $G(x) = \frac{x^3}{3} + x$. Then

$$G(1) - G(-2) = \left(\frac{1}{3} + 1\right) - \left(\frac{-8}{3} - 2\right) = \frac{9}{3} + 3 = 6$$

In short we write

$$\int_{-2}^{1} x^2 + 1 \, dx = \left[\frac{x^3}{3} + x\right]_{-2}^{1} = \left(\frac{1}{3} + 1\right) - \left(\frac{-8}{3} - 2\right) = 6$$

Example

$$\int_0^2 e^{x+2}\,dx = \left[e^{x+2}\right]_0^2 = e^4 - e^2$$

Example (IMPORTANT EXAMPLE!)

Compute the derivative of the function

$$F(x) = \int_1^x (1+4t)dt$$

By the Fundamental Theorem of Calculus:

$$F'(x) = f(x) = 1 + 4x$$

Example (IMPORTANT EXAMPLE!)

Compute the derivative of the function

$$F(x) = \int_1^{x^2+x} (1+4t)dt$$

We cannot apply the Fundamental Theorem of Calculus directly!

Call $g(x) = x^2 + x$ and $h(u) = \displaystyle\int_1^u (1+4t)\,dt$. Then

$$F(x) = h(g(x))$$

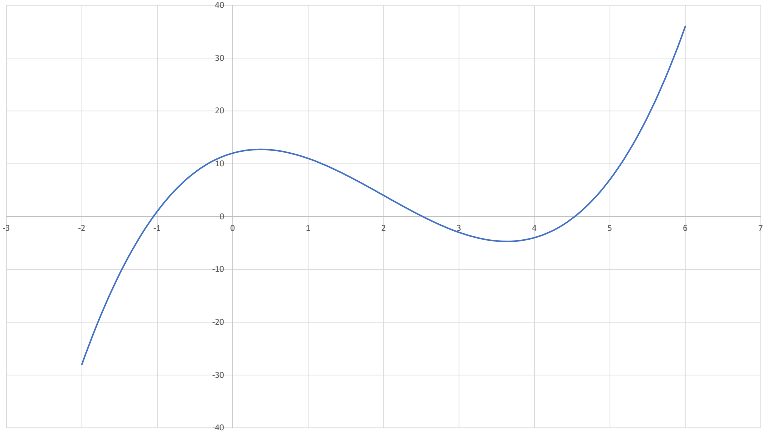
To compute $F'(x)$ we apply the differentiation rule

$$F'(x) = h'(g(x)) \cdot g'(x) = (1 + 4g(x))g'(x) = (1 + 4(x^2 + x)) \cdot (2x + 1)$$

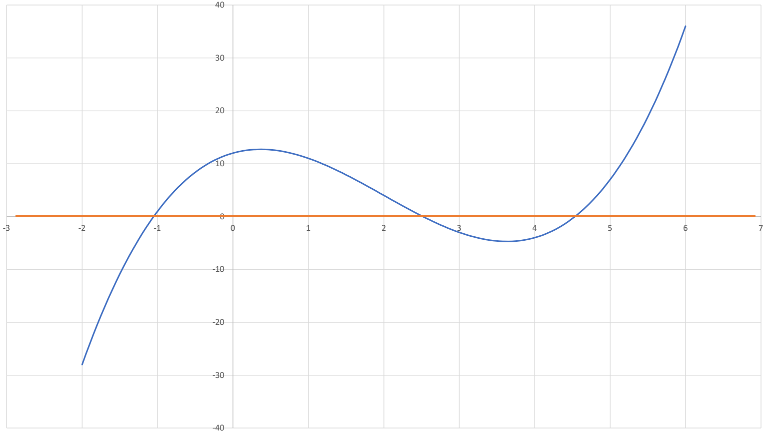
Again on the area under a curve

- Suppose that the function f is always positive on the interval $[a, b]$. Then the quantity
$$A = \int_a^b f(x)\, dx$$
indicates the area between the graph of the function f and the x-axis.
- If the function f is always negative on the interval $[a, b]$, then the area between the graph of the function f and the x-axis is
$$A = -\int_a^b f(x)\, dx.$$
- If f changes its sign we need to detect all points where f vanishes and divide the interval $[a, b]$ so that, when f is positive, we keep the sign $+$, when f is negative we put a $-$ in front of the integral

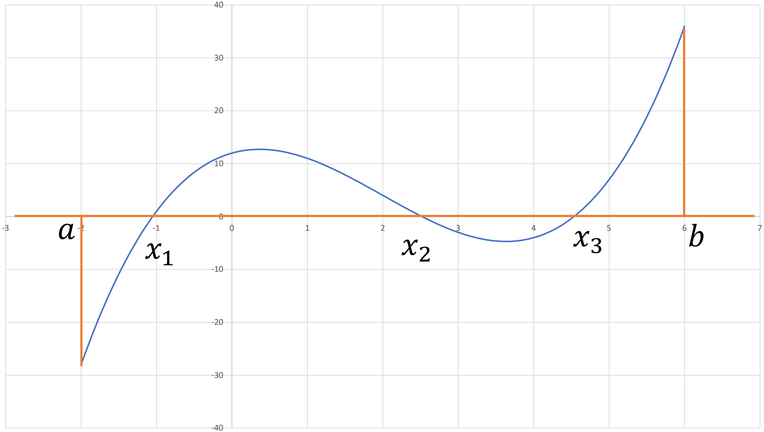
$$f(x) = x^3 - 6x^2 + 4x + 12$$



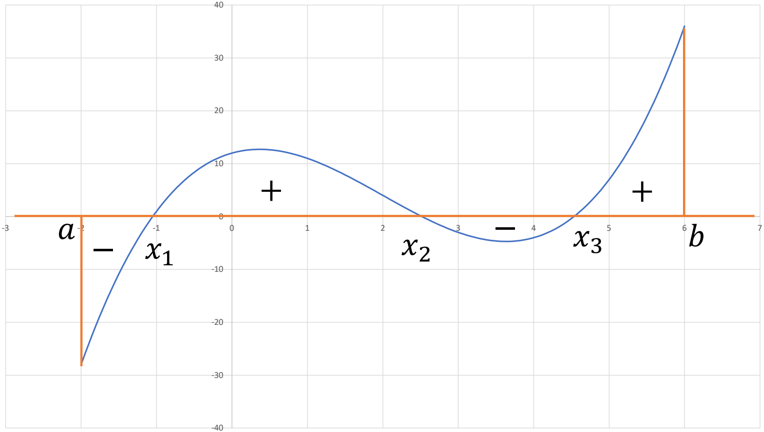
$$f(x) = x^3 - 6x^2 + 4x + 12$$



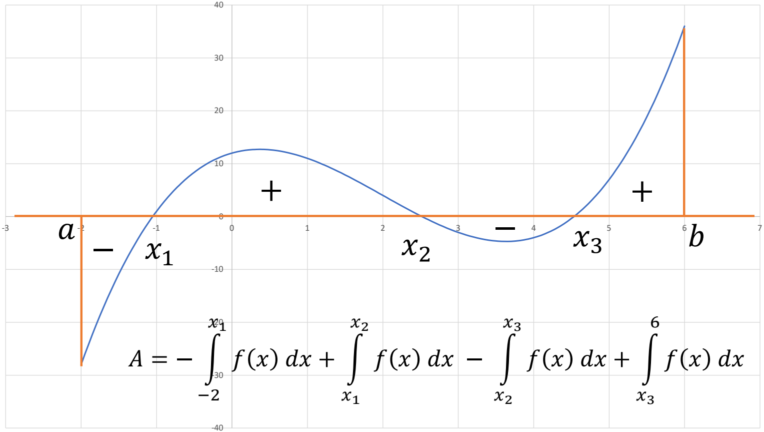
$$f(x) = x^3 - 6x^2 + 4x + 12$$



$$f(x) = x^3 - 6x^2 + 4x + 12$$



$$f(x) = x^3 - 6x^2 + 4x + 12$$



$$A = -\int_{-2}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx - \int_{x_2}^{x_3} f(x)\,dx + \int_{x_3}^{6} f(x)\,dx$$

Improper Integrals

In statistics and in economics it is common to consider definite integrals over an infinite interval, that is either $a = -\infty$, or $b = +\infty$ or both, or over intervals where the function is not well defined.

An improper integral is the limit, **if it exists**, of a definite integral when an endpoint of the integration interval

- goes to ∞
- is a point of discontinuity of the integrand $f(x)$

We write, for instance

$$\int_a^{\infty} f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx$$

Example

For any $\lambda > 0$*, show that the following improper integral satisfies*

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1$$

The function $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ is very well known in statistics and it is called the *density of the exponential distribution*.

Exercises

Prove that the following integrals converge and compute their values:

1. For $c > 0$

$$\int_{-\infty}^{\infty} x e^{-cx^2} \, dx$$

2. For $\alpha > 1$

$$\int_{1}^{\infty} \frac{1}{x^{\alpha}} \, dx$$

Introduction

Suppose a company owns two bookstores. Each of them sells books, newspapers, and magazines. Assume that sales (in thousand dollars) of the two bookstores in June and July are summarised by the following tables:

	June	
Store	A	B
Books	45	64
Newspapers	6	8
magazines	11	7

	July	
Store	A	B
Books	42	68
Newspapers	8	11
magazines	21	19

The two rectangular arrays of real numbers (scalars) are called Matrices

Introduction

Why do we need matrices?

- Organization of (numerical) data, and perform useful operations/significant analysis on them.
- Any regular problem can be approximated by a linear problem (to wit, by a linear system).
- A number of efficient algorithms for the resolution of linear systems is available

Introduction

A matrix is a rectangular array of numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix}$$

Introduction

The size of a matrix is indicated by the number of rows and the number of its columns.

A matrix with k rows and n columns is called a $k \times n$ ("k by n") matrix.

We also use the notation $A \in \mathcal{M}(k \times n)$ to indicate that A ia a $k \times n$ matrix.

The number in row i and column j is called the (i, j)-th entry and it is denoted by a_{ij}.

We get a square matrix if $k = n$

Examples

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 3 & 3 \end{pmatrix}$$

Submatrix

Sometimes we are interested only in part of the information collected in a matrix. For instance, in the bookstore example the matrix M is given by

	June	
Store	A	B
Books	45	64
Newspapers	6	8
magazines	11	7

Submatrix

Sometimes we are interested only in part of the information collected in a matrix. For instance, in the bookstore example the matrix M is given by

	June	
Store	A	B
Books	45	64
Newspapers	6	8
magazines	11	7

We only need information about books and newspapers. The matrix

$$E = \begin{pmatrix} 45 & 64 \\ 6 & 8 \end{pmatrix}$$

is called a *a submatrix* of M.

Submatrix

Definition

A submatrix of a matrix A is any matrix obtained from A by removing entire rows and/or columns

Example

Given

$$A = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Then

$$E = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 3 & 5 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

are all submatrices of A

Operations with matrices

As mathematics objects, one can do operations with matrices. Under suitable assumptions we can

- Sum/Subtract matrices
- Multiply a matrix by a number
- Multiply matrices
- Invert a matrix

Sum/Subtraction

Let $A, B \in \mathcal{M}(k \times n)$ be two matrices with the same size, i.e. the same number of rows and columns.

- We define $C = A + B$.
  C is a $k \times n$ matrix with entries $c_{ij} = a_{ij} + b_{ij}$.
- We define $D = A - B$.
  D is a $k \times n$ matrix with entries $d_{ij} = a_{ij} - b_{ij}$.

Sum: Example

Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 6 & 9 \end{pmatrix}$$

Then $C = A + B$ is given by

$$C = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 7 & 10 \end{pmatrix}$$

and $D = A - B$ is given by

$$D = \begin{pmatrix} 1 & 2 & 4 \\ -4 & -5 & -8 \end{pmatrix}$$

The null matrix (or zero matrix)

The null matrix $O_{k\times n} \in \mathcal{M}(k \times n)$ is a matrix whose entries are 0.

$$O = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

This is an addition identity or neutral matrix, that is $A + O_{k\times n} = A$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix}$$

Scalar multiplication

Let $\lambda \in \mathbb{R}$ be a scalar (i.e. a number) and let $A \in \mathcal{M}(k \times n)$ be a matrix. We can define the matrix $L = \lambda A$.

L is a $(k \times n)$ matrix with entries $\ell_{ij} = \lambda a_{ij}$

Example

Let

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 3 & 3 \end{pmatrix}$$

Then $L = 2A$ is given by

$$L = \begin{pmatrix} 4 & 4 \\ -4 & 2 \\ 6 & 6 \end{pmatrix}$$

Properties of matrix addition and scalar multiplication

Theorem

Let $A, B, C \in \mathcal{M}(k \times n)$ *and* $\alpha, \beta \in \mathbb{R}$. *We have the following properties*

1. $A + B = B + A$ *(commutative law of matrix addition)*
2. $(A + B) + C = A + (B + C)$ *(associative law of matrix addition)*
3. $A + O_{k \times n} = A$ *(O is the neutral matrix for addition)*
4. $A + (-A) = O_{k \times n}$
5. $(\alpha + \beta)A = \alpha A + \beta A$
6. $\alpha(A + B) = \alpha A + \alpha B$

Homework: Prove the theorem.

Matrix Transposes

The transpose of a $k \times n$ matrix A is an $n \times k$ matrix denoted by $A^\top$ where one inverts rows and columns.

Example

Let

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 3 & 3 \end{pmatrix}$$

Then $A^\top$ is given by

$$A^\top = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Properties of the transpose

Definition

Transpose of a matrix Let $A \in \mathcal{M}(k \times n)$, the transpose of A is a matrix $A^\top \in \mathcal{M}(n \times k)$ with entries $a^\top_{ij} = a_{ji}$.

Theorem

Let $A, B \in \mathcal{M}(k \times n)$ and $\lambda \in \mathbb{R}$. We have the following properties

1. $(A + B)^\top = A^\top + B^\top$
2. $(\lambda A)^\top = \lambda A^\top$
3. $(A^\top)^\top = A$

Homework: Prove the theorem.

Vectors

A matrix with exactly one column is called a vector.

Entries of a vector are called components.

The set of all column vectors with k components is $\mathbb{R}^k$

Example

$$\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$$

Sometimes we use the notation $\mathbf{u} = (1, -2, 3)^\top$, for saving space.

For any $\mathbf{u} \in \mathbb{R}^k$ we write $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}$, or $\mathbf{u} = (u_1, u_2, \ldots, u_k)^\top$.

Operations with vectors

Since vectors are just special matrices we can sum/subtract vectors of the same length, we can multiply vectors by a scalar and they inherit the same properties of matrices.

1. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$ we define $\mathbf{w} = \mathbf{u} + \mathbf{v} \in \mathbb{R}^k$ with components $w_i = u_i + v_i$.
2. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$ we define $\mathbf{z} = \mathbf{u} - \mathbf{v} \in \mathbb{R}^k$ with components $z_i = u_i - v_i$.
3. For any $\mathbf{u} \in \mathbb{R}^k$ and $\lambda \in \mathbb{R}$ we define $\lambda\mathbf{u} = (\lambda u_1, \dots, \lambda u_k)^\top \in \mathbb{R}^k$.
4. The neutral element in $\mathbb{R}^k$ is the vector $\mathbf{o} = (0, \dots, 0)^\top \in \mathbb{R}^k$

Corollary

For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^k$, *and* $\alpha, \beta \in \mathbb{R}$ *we have that*

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{o} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{o}$
5. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
6. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

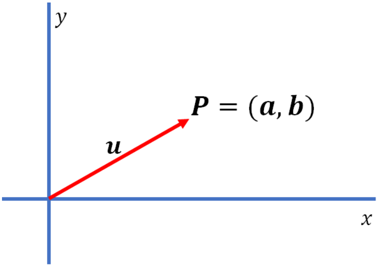
Geometry of vectors

For many applications it is convenient to represent vectors geometrically as a directed line segment or arrows.

Geometry of vectors

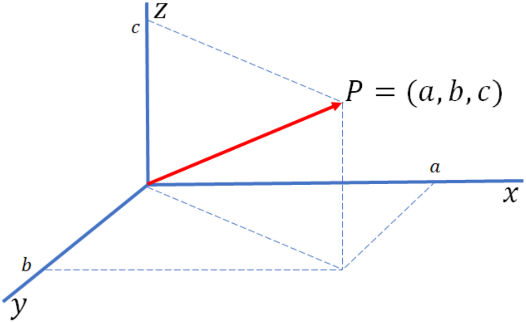
For many applications it is convenient to represent vectors geometrically as a directed line segment or arrows.

For example if $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ is a vector in $\mathbb{R}^2$, then we can represent the vector as a segment from the origin to the point (a, b).



Vectors in $\mathbb{R}^3$

Analogously, vectors in $\mathbb{R}^3$ can be represented as arrows in the three-dimensional space.



Linear combinations, matrix-vector products, special matrices

Definition

A ***linear combination*** *of vectors* $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p \in \mathbb{R}^k$ *is a vector of the form*

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p \in \mathbb{R}^k$$

for scalars $c_1, c_2, \dots, c_p$*, which are called the* ***coefficients*** *of the linear combination.*

We can always write any vector as a linear combination of vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Indeed $\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are called the standard vectors in $\mathbb{R}^2$.

Similarly $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are standard vectors in $\mathbb{R}^3$.

In general, we have the following definition

Definition

Vectors $\mathbf{e}_1, \dots \mathbf{e}_k \in \mathbb{R}^k$ *given by*

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad \mathbf{e}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

are called the standard vectors in $\mathbb{R}^k$.

Any vector $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} \in \mathbb{R}^k$ can be written as a linear combination of standard vectors $\mathbf{e}_1, \dots \mathbf{e}_k \in \mathbb{R}^k$.

Indeed,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \dots u_k\mathbf{e}_k$$

The matrix-vector product

Let $A \in \mathcal{M}(k \times n)$ and $\mathbf{u} \in \mathbb{R}^n$. Then $\mathbf{w} = A \cdot \mathbf{u} \in \mathbb{R}^k$ given by

$$\mathbf{w} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_n\mathbf{a}_n$$

where $\mathbf{a}_1, \ldots \mathbf{a}_n$ are the columns of the matrix A and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$.

A special case: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$, then $\mathbf{u}^\top \cdot \mathbf{v}$ is a real number given by

$$\mathbf{u}^\top \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_kv_k$$

The (square) identity matrix

Definition

The identity matrix $I_n \in \mathcal{M}(n \times n)$ *is a square matrix with entries equal to* 1 *on the diagonal and* 0 *otherwise,*

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

The (square) identity matrix

Definition

The *identity matrix* $I_n \in \mathcal{M}(n \times n)$ *is a square matrix whose columns are the standard vectors in* $\mathbb{R}^n$.

Property: The identity matrix is a multiplicative identity, that is for any vector $\mathbf{v} \in \mathbb{R}^n$ we have that

$$I_n \cdot \mathbf{v} = \mathbf{v}.$$

Proof. @ the blackboard

Properties of the matrix-vector product

Theorem

Let $A, B \in \mathcal{M}(k \times n)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$. *Then*

1. $A \cdot (\mathbf{u} + \mathbf{v}) = A \cdot \mathbf{u} + A \cdot \mathbf{v}$
2. $A \cdot (\lambda \mathbf{u}) = \lambda A \cdot \mathbf{u}$
3. $(A + B) \cdot \mathbf{u} = A \cdot \mathbf{u} + B \cdot \mathbf{u}$
4. $A \cdot \mathbf{e}_j = \mathbf{a}_j$ *where* $\mathbf{a}_j$ *is the j-th column of* A
5. $A \cdot \mathbf{o}_n = \mathbf{o}_k$, *where* $\mathbf{o}_n$ *is the zero vector in* $\mathbb{R}^n$ *and* $\mathbf{o}_k$ *is the zero vector in* $\mathbb{R}^k$
6. $O_{k \times n} \cdot \mathbf{v} = \mathbf{o}_k$ *where* $O_{k \times n}$ *is the null* $k \times n$*-matrix*
7. $I_n \cdot \mathbf{v} = \mathbf{v}$, *where* I_n *is the* $n \times n$*-identity matrix*

Matrix multiplication

Let $A \in \mathcal{M}(k \times n)$ and $B \in \mathcal{M}(n \times m)$, then $P = A \cdot B$ is a $k \times m$ matrix whose columns are given by

$$\mathbf{p}_1 = A \cdot \mathbf{b}_1, \quad \mathbf{p}_2 = A \cdot \mathbf{b}_2, \ldots, \mathbf{p}_m = A \cdot \mathbf{b}_m$$

Notice that this is just a repetition of the matrix-vector product.

Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 5 \end{pmatrix} \in \mathcal{M}(3 \times 2), \qquad B = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \in \mathcal{M}(2 \times 2)$$

Then $P = AB \in \mathcal{M}(3 \times 2)$ with columns

$$\mathbf{p}_1 = A\mathbf{b}_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}$$

$$\mathbf{p}_2 = A\mathbf{b}_2 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

That is

$$P = \begin{pmatrix} 3 & 1 \\ 5 & 1 \\ 10 & 0 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \in \mathcal{M}(2 \times 3), \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 2 \end{pmatrix} \in \mathcal{M}(3 \times 2)$$

Then $P = AB \in \mathcal{M}(2 \times 2)$ with columns

$$\mathbf{p}_1 = A\mathbf{b}_1 = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix}$$

p_{11} = 1st row of A times 1st column of B $\Rightarrow (2 \;\; -1 \;\; 3) \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

p_{21} = 2nd row of A times 1st column of B $\Rightarrow -5$

$$\mathbf{p}_2 = A\mathbf{b}_2 = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix}$$

p_{12} = 1st row of A times 2nd column of B $\Rightarrow 5$

p_{22} = 2nd row of A times 2nd column of B $\Rightarrow 0$

Finally,

$$P = \begin{pmatrix} 11 & 5 \\ -5 & 0 \end{pmatrix}$$

Remark

An important property! The matrix product is NOT commutative

Theorem

Properties of the matrix product Let $A, B \in \mathcal{M}(k \times n)$, $C \in \mathcal{M}(n \times m)$ *and* $P, Q \in \mathcal{M}(m \times l)$. *We get that:*

1. $\lambda(AC) = (\lambda A)C = A(\lambda C)$, *for all* $\lambda \in \mathbb{R}$
2. $(AC)P = A(CP)$
3. $(A + B)C = AC + BC$
4. $C(P + Q) = CP + CQ$
5. $I_k A = A = A I_n$
6. $A O_{n \times m} = O_{k \times m}$
7. $(AC)^\top = C^\top A^\top$