Mathematics 2, AA 2023-2024
Professor Katia Colaneri
Additional Exercises n.2

Ex 1. Compute the following definite integrals (the result is written in red). Amended solutions/exercises in blue!

1. $\displaystyle\int_0^1 x^2\ dx = \frac{1}{3}$
2. $\displaystyle\int_1^3 \frac{2x^2-1}{x}\ dx = 8 - \ln(3)$
3. $\displaystyle\int_{-1}^1 \frac{x^3}{2-x}\ dx = 8\ln(3) - \frac{26}{3}$
4. $\displaystyle\int_e^{e^2} \frac{1}{x\ln(x)}\ dx = \ln(2)$
5. $\displaystyle\int_1^3 \frac{1}{\sqrt{x}(1+x)}\ dx = \frac{\pi}{6}$
6. $\displaystyle\int_0^\pi x\ \sin(x)\ dx = \pi$
7. $\displaystyle\int_{-\pi/2}^{\pi/2} e^x\ \cos(x)\ dx = \frac{e^{\pi/2}+e^{-\pi/2}}{2}$
8. $\displaystyle\int_0^{\ln(2)} \frac{e^{x+2}}{e^x+1}\ dx = e^2\ln(3/2)$
9. $\displaystyle\int_0^1 \ln(1+x^2)\ dx = \ln(2) - 2 - \frac{\pi}{2}$
10. $\displaystyle\int_0^{\sqrt{2}} x\sqrt{1+x^2}\ dx = \frac{3\sqrt{3}-1}{3}$
11. $\displaystyle\int_{-2}^1 \ln(x+3)\ dx = 4\ln(4) - 3$
12. $\displaystyle\int_{-1}^0 \frac{1+x}{3-2x-x^2}\ dx = \ln\left(\sqrt{43}\right)$
13. $\displaystyle\int_1^{e^3} \frac{1}{x(9+\ln^2(x))}\ dx = \frac{\pi}{12}$
14. $\displaystyle\int_1^4 \frac{\sqrt{x}}{1+x\sqrt{x}}\ dx = \frac{2}{3}\left(2\ln(3) - \ln(2)\right)$
15. $\displaystyle\int_1^2 x\ln(x^2+x)\ dx = -1 - \frac{\ln(3)}{2} + \ln(36)$
16. $\displaystyle\int_0^{\frac{\pi}{2}} x^2\cos(x)\ dx = \frac{\pi^2}{4} - 2$
17. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos(x)}{1+4\sin^2(x)}\ dx = \frac{1}{2}\left(\arctan(2) - \arctan\left(\sqrt{2}\right)\right)$
18. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{2\cos^2(x)+5}{\cos^2(x)}\ dx = \frac{\pi}{6} + 5(\sqrt{3}-1)$
19. $\displaystyle\int_1^2 \frac{2x}{1-3x}\ dx = -\frac{2}{3}\left(1 + \frac{1}{3}\ln\left(\frac{2}{5}\right)\right)$
20. $\displaystyle\int_{-\frac{\pi}{2}}^{\pi} \sin(2x)\ dx = -1$

Ex. 2 Compute the derivatives of the following functions.

1. $\displaystyle F(x) = \int_0^{x\sqrt{x}} \frac{1}{\sqrt{1+t^4}}\ dt, \qquad F'(x) = \frac{3}{2}\frac{\sqrt{x}}{\sqrt{1+x^6}}$
2. $\displaystyle F(x) = \int_x^1 \frac{t^2-1}{1+t^4}\ dt, \qquad F'(x) = -\frac{x^2-1}{1+x^4}$
3. $\displaystyle F(x) = \int_{2x+1}^{e^x} \sin(t)\cos(t)\ dt, \qquad F'(x) = -2\sin(2x+1)\cos(2x+1) + e^x\sin(e^x)\cos(e^x)$
4. $\displaystyle F(x) = \int_{x^2-1}^{x^3} \ln(t)\ \arctan(t)\ dt, \qquad F'(x) = -2x\ln(2x+1)\arctan(x^2-1) + 3x^2\ln(x^3)\arctan(x^3)$

Ex. 3 Find all local maxima and minima of the following functions.

1. $F(x) = \int_0^x \frac{1}{\sqrt{1+t^4}}\,dt$, no local maxima nor local minima
2. $F(x) = \int_x^1 \frac{t^2-1}{1+t^4}\,dt$, $x = 1$ local max, $x = -1$ local min
3. $F(x) = \int_0^{x^2+2x} \frac{1}{t^4+2t^2+1}\,dt$, $x = -1$ local min
4. $F(x) = \int_{e^x}^1 (t-1)\ln(t)\,dt$, no local maxima nor local minima

Ex. 4 Compute the following limits.

1. $\lim_{x\to 0} \frac{\int_0^x \sin(t)\,dt}{x^2} = 1/2$
2. $\lim_{x\to 0} \frac{\int_0^x e^t - 1\,dt}{x^2} = 1/2$
3. $\lim_{x\to 0} \frac{\int_0^x \ln(t+1)\,dt}{x^2} = 1/2$
4. $\lim_{x\to 1^+} \frac{\int_0^x t^2 - 1\,dt}{x^2-2x+1} = -\infty$, $\quad \lim_{x\to 1^-} \frac{\int_0^x t^2 - 1\,dt}{x^2-2x+1} = -\infty$
5. $\lim_{x\to 2^+} \frac{\int_2^x \frac{t^2-2t}{t+1}\,dt}{(x-2)^2} = \frac{1}{3}$, $\quad \lim_{x\to 2^-} \frac{\int_2^x \frac{t^2-2t}{t+1}\,dt}{(x-2)^2} = \frac{1}{3}$
6. $\lim_{x\to 5^+} \frac{\int_5^x \ln(t-4)\,dt}{(x-5)^3} = +\infty$

Ex. 5 Determine the equation of the tangent line to the function $F(x)$ at the given point x_0.

1. $F(x) = \int_0^x \frac{1}{1+t^4}\,dt$ at $x_0 = 1$, $\quad y = \frac{1}{2}x + q$, $\quad q = \int_0^1 \frac{1}{1+t^4}\,dt$
2. $F(x) = \int_{-\pi}^x \sin(t)\,dt$ at $x_0 = \pi$, $\quad y = 0$
3. $F(x) = \int_{-1}^x t^3\,dt$ at $x_0 = 0$, $\quad y = -\frac{1}{4}$

Ex. 6 For the following function

a. Provide the sketch of the graph of the function
b. Evaluate the area between the curve $y = f(x)$, the axis $y = 0$ in the interval $I = [a, b]$

1. $f(x) = \frac{x^2}{x+1}$ and $I = [0, 2]$, $\quad A = \ln(3)$
2. $f(x) = \frac{x^2-x}{x^2+1}$ and $I = [-3, 3]$, $\quad A = 4 - 2\arctan(3) + \ln(2) + \frac{\pi}{2}$
3. $f(x) = -1 + \frac{1}{2}\sin(x)$ and $I = [0, \pi]$, $\quad A = \pi - 1$
4. $f(x) = (1-x)e^x$ and $I = [0, 2]$, $\quad A = 2(e-1)$
5. $f(x) = 1 + \ln^2(x)$ and $I = [1, e]$, $\quad A = 2e - 3$

Ex. 7 Compute the following improper integrals.

1. $\displaystyle\int_1^{+\infty} \frac{4+x}{x^2}\,dx$ does not converge

2. $\displaystyle\int_0^{+\infty} \frac{1}{(x+2)\sqrt{x+2}}\,dx = \sqrt{2}$

3. $\displaystyle\int_0^{1} \frac{1}{x^2}e^{-\frac{1}{x}}\,dx = e^{-1}$

4. $\displaystyle\int_0^{+\infty} xe^{-2x} = \frac{1}{4}$

Ex. 8 Compute the mean value of the function $f(x)$ in the given interval I.

1. $f(x) = x^3 - x$ and $I = [0, 2]$, 1
2. $f(x) = \dfrac{2-x}{x+2}$ and $I = [-1, 4]$, $4/5\ln(6) - 1$
3. $f(x) = \sin(x)$ and $I = \left[0, \frac{\pi}{2}\right]$, $2/\pi$
4. $f(x) = \dfrac{x}{1+x^2}$ and $I = [0, 1]$, $\ln(2)/2$
5. $f(x) = \ln(x+1)$ and $I = [0, 2]$, $3\ln(3)/2 - 1$

Ex. 9 Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 7 & -1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 2 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Compute

$$A + B = \begin{pmatrix} 1 & 3 & -1 & 5 \\ 2 & \frac{15}{2} & -2 & -\frac{1}{2} \\ 1 & 0 & \frac{3}{2} & 5 \end{pmatrix}, \quad A - B = \begin{pmatrix} 1 & 1 & -1 & 3 \\ -2 & \frac{13}{2} & 0 & \frac{1}{2} \\ -1 & 0 & \frac{1}{2} & 5 \end{pmatrix}$$

$$B - A = \begin{pmatrix} -1 & -1 & 1 & -3 \\ 2 & -\frac{13}{2} & 0 & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} & -5 \end{pmatrix}, \quad 3A = \begin{pmatrix} 3 & 6 & -3 & 12 \\ 0 & 21 & -3 & 0 \\ 0 & 0 & 3 & 15 \end{pmatrix}$$

$$2B = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 4 & 1 & -2 & -1 \\ 2 & 0 & 1 & 0 \end{pmatrix} \quad 2B - 3A = \begin{pmatrix} -3 & -4 & 3 & -10 \\ 4 & -20 & 1 & -1 \\ 2 & 0 & -2 & -15 \end{pmatrix}$$

$$A^\top = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ -1 & -1 & 1 \\ 4 & 0 & 5 \end{pmatrix} \quad B^\top = \begin{pmatrix} 0 & 2 & 1 \\ 1 & \frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 \end{pmatrix} \quad (A+B)^\top = \begin{pmatrix} 1 & 2 & 1 \\ 3 & \frac{15}{2} & 0 \\ -1 & -2 & \frac{3}{2} \\ 5 & -\frac{1}{2} & 5 \end{pmatrix}.$$