Mathematics 2, AA 2023-2024
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Additional Exercises n.3

Ex. 1 Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 7 & -1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 2 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Compute

$$A + B = \begin{pmatrix} 1 & 3 & -1 & 5 \\ 2 & \frac{15}{2} & -2 & -\frac{1}{2} \\ 1 & 0 & \frac{3}{2} & 5 \end{pmatrix}, \quad A - B = \begin{pmatrix} 1 & 1 & -1 & 3 \\ -2 & \frac{13}{2} & 0 & \frac{1}{2} \\ -1 & 0 & \frac{1}{2} & 5 \end{pmatrix}$$

$$B - A = \begin{pmatrix} -1 & -1 & 1 & -3 \\ 2 & -\frac{13}{2} & 0 & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} & -5 \end{pmatrix}, \quad 3A = \begin{pmatrix} 3 & 6 & -3 & 12 \\ 0 & 21 & -3 & 0 \\ 0 & 0 & 3 & 15 \end{pmatrix}$$

$$2B = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 4 & 1 & -2 & -1 \\ 2 & 0 & 1 & 0 \end{pmatrix} \quad 2B - 3A = \begin{pmatrix} -3 & -4 & 3 & -10 \\ 4 & -20 & 1 & -1 \\ 2 & 0 & -2 & -15 \end{pmatrix}$$

$$A^\top = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ -1 & -1 & 1 \\ 4 & 0 & 5 \end{pmatrix} \quad B^\top = \begin{pmatrix} 0 & 2 & 1 \\ 1 & \frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 \end{pmatrix} \quad (A + B)^\top = \begin{pmatrix} 1 & 2 & 1 \\ 3 & \frac{15}{2} & 0 \\ -1 & -2 & \frac{3}{2} \\ 5 & -\frac{1}{2} & 5 \end{pmatrix}.$$

Ex. 2 Compute the following matrix-vector products

1. $\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & -2 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{2} \\ -\frac{7}{2} \end{pmatrix}$
2. $\begin{pmatrix} 3 & 1 \\ 4 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ 5 \end{pmatrix}$
3. $\begin{pmatrix} 1 & -1 & 0 \\ 3 & 5 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 20 \end{pmatrix}$

Ex. 3 Compute, if possible, the matrix products $A \cdot B$ and $B \cdot A$:

1. $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 0 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 2 & -1 \\ 0 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad BA = \begin{pmatrix} -1 & 0 & 5 \\ 3 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -3 & -2 \\ 5 & -1 & 7 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & -3 & -2 \\ 15 & -3 & 21 \end{pmatrix}, \quad BA \text{ not computable}$$

3. $A = \begin{pmatrix} \frac{1}{2} & 1 & -1 \\ 2 & -1 & 0 \\ 1 & 0 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{pmatrix}$

$$AB = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & 2 \\ -5 & -3 & 4 \\ -7 & 2 & 5 \end{pmatrix}, \quad BA = \begin{pmatrix} -\frac{1}{2} & 0 & -5 \\ \frac{7}{2} & 2 & -3 \\ -2 & 3 & 1 \end{pmatrix}$$

4. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 1 & 5 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 2 & 0 \\ -1 & -5 & 1 \\ 3 & -4 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & -2 & 2 \\ 7 & -19 & 7 \end{pmatrix}, \quad BA \text{ not computable}$$

Ex 4. Given vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

compute the following linear combinations

$$2\mathbf{v_1} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, \quad 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \begin{pmatrix} 4 \\ -10 \\ 7 \end{pmatrix}, \quad 5\mathbf{v}_1 - \mathbf{v}_3 = \begin{pmatrix} 6 \\ -8 \\ 3 \end{pmatrix}, \quad \frac{1}{2}\mathbf{v}_2 + \frac{3}{2}\mathbf{v}_3 = \begin{pmatrix} -3/2 \\ 7/2 \\ 9/2 \end{pmatrix}$$

Ex 5. Compute the following matrix-vector product

1. $\begin{pmatrix} 3 & -2 & 1 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 14 \end{pmatrix}$

2. $\begin{pmatrix} 2 & -3 \\ -4 & 5 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 10 \end{pmatrix}$

3. $\begin{pmatrix} 2 & -3 & 4 \\ -4 & 5 & -2 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

4. $\left(\begin{pmatrix} 3 & 0 \\ -2 & 4 \end{pmatrix}^\top + \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix}^\top \right) \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}$

5. $\left(\begin{pmatrix} 3 & 0 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix} \right) \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 26 \\ 9 \end{pmatrix}$

6. $\begin{pmatrix} -\frac{1}{2} & 1 & \frac{1}{3} \\ 0 & -2 & \frac{2}{3} \\ \frac{1}{2} & 1 & 0 \\ 0 & -3 & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -2 \\ \frac{1}{2} \\ -3 \end{pmatrix}$

Ex 5. Given matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ -3 & -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 4 & 1 \\ 3 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & -1 \\ 4 & 3 & -2 \end{pmatrix}$$

compute the following products:

$$AB = \begin{pmatrix} 16 & -4 \\ 6 & -9 \\ -5 & -2 \end{pmatrix}, \quad BC = \begin{pmatrix} -2 & -1 & 1 \\ 12 & 7 & -6 \\ -2 & -3 & 1 \end{pmatrix}, \quad ABC = = \begin{pmatrix} 16 & 4 & -8 \\ -24 & -21 & 12 \\ -18 & -11 & 9 \end{pmatrix}$$

$$CA = \begin{pmatrix} 7 & 5 & 10 \\ 16 & 9 & 24 \end{pmatrix}, \quad CAB = \begin{pmatrix} 43 & -15 \\ 92 & -39 \end{pmatrix}, \quad CB = \begin{pmatrix} -1 & 3 \\ 2 & 7 \end{pmatrix}$$

$$B^\top A = \begin{pmatrix} -2 & -12 & 13 \\ 8 & 3 & 4 \end{pmatrix}, \quad B^\top C^\top = \begin{pmatrix} -1 & 2 \\ 3 & 7 \end{pmatrix}$$

Ex 6. Compute, if possible, the following matrix products

1. $\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 & 0 \\ 3 & -1 & 2 & 1 \end{pmatrix} = = \begin{pmatrix} 8 & -1 & 3 & 2 \\ 12 & -4 & 8 & 4 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 1 & 0 & 1 \\ 3 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & 1 & 4 \\ -3 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 2 \\ 0 & 3 & 9 \end{pmatrix}$

3. $\begin{pmatrix} 2 & -2 \\ 4 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 8 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 14 & 6 & -4 \\ 35 & 12 & 20 \\ 13 & 3 & 22 \end{pmatrix}$

4. $\begin{pmatrix} 8 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 4 & 3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 30 & 3 \\ 6 & 18 \end{pmatrix}$

5. $\begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 1 \\ 7 & 3 & -7 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{pmatrix}$

Ex 7. Compute the determinant of the following matrices

1. $\begin{pmatrix} 2 & 2 \\ 1 & 5 \end{pmatrix}$ 8

2. $\begin{pmatrix} 2 & 0 \\ 3 & -2 \end{pmatrix}$ 6

3. $\begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 1 \\ -1 & 3 & 2 \end{pmatrix}$ 9

4. $\begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$ 11

5. $\begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$ 12

6. $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 1

7. $\begin{pmatrix} 5 & 2 & 6 & 2 \\ 0 & 1 & 0 & 0 \\ 4 & 2 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 1

8. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 2 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{pmatrix}$ 30

9. $\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 4 & -3 & 2 & -1 \\ 0 & 3 & 0 & -2 \end{pmatrix}$ -4

10. $\begin{pmatrix} 4 & 1 & 0 \\ 0 & 3 & -2 \\ 2 & 0 & 5 \end{pmatrix}$ 56

Ex 8. Compute, if it exists, the inverse of each of the following matrices

1. $\begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$. Not invertible

2. $\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$. $\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

3. $\begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 1 \\ -1 & 3 & 2 \end{pmatrix}$. $\begin{pmatrix} -\frac{1}{9} & \frac{4}{9} & -\frac{2}{9} \\ -\frac{5}{9} & \frac{2}{9} & -\frac{1}{9} \\ \frac{7}{9} & -\frac{1}{9} & \frac{5}{9} \end{pmatrix}$

4. $\begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$. $\begin{pmatrix} \frac{2}{11} & \frac{1}{11} & \frac{4}{11} \\ -\frac{3}{11} & \frac{4}{11} & -\frac{17}{11} \\ -\frac{5}{11} & \frac{3}{11} & -\frac{10}{11} \end{pmatrix}$

5. $\begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$. $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{6} & \frac{5}{12} & -\frac{7}{12} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{5}{6} \end{pmatrix}$

6. $\begin{pmatrix} 4 & 1 & 0 \\ 0 & 3 & -2 \\ 2 & 0 & 5 \end{pmatrix}$. $\begin{pmatrix} \frac{15}{56} & -\frac{5}{56} & -\frac{1}{28} \\ -\frac{1}{14} & \frac{5}{14} & \frac{1}{7} \\ -\frac{3}{28} & \frac{1}{28} & \frac{3}{14} \end{pmatrix}$

Ex 9. Determine for which values of the parameter t, the following matrices are invertible

1. $\begin{pmatrix} t & t-1 \\ -8 & t-6 \end{pmatrix}$, $t \neq 2,\ t \neq -4$

2. $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & t \\ 0 & t & -15 \end{pmatrix}$, $t \neq -5,\ t \neq 3$

3. $\begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 4 \\ 2 & 1 & t \end{pmatrix}$, $t \neq -14$

4. $\begin{pmatrix} -1 & t-1 & 1-t \\ -t-2 & 2t-3 & 4-t \\ -t-2 & t-1 & 2 \end{pmatrix}$, $t \neq -1,\ t \neq 2$