Mathematics 2

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Week 3 - Matrices

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Outline

1. Matrix product
   - Matrix-vector product
   - Matrix product
2. Determinant and Invertibility

The matrix-vector product

Let $A \in \mathcal{M}(k \times n)$ and $\mathbf{u} \in \mathbb{R}^n$. Then $\mathbf{w} = A \cdot \mathbf{u} \in \mathbb{R}^k$ given by

$$\mathbf{w} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_n\mathbf{a}_n$$

where $\mathbf{a}_1, \ldots \mathbf{a}_n$ are the columns of the matrix A and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$.

A special case: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$, then $\mathbf{u}^\top \cdot \mathbf{v}$ is a real number given by

$$\mathbf{u}^\top \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_kv_k$$

The (square) identity matrix

Definition

The identity matrix $I_n \in \mathcal{M}(n \times n)$ *is a square matrix with entries equal to* 1 *on the diagonal and* 0 *otherwise,*

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

The (square) identity matrix

Definition

The identity matrix $\mathrm{I}_n \in \mathcal{M}(n \times n)$ *is a square matrix whose columns are the standard vectors in* $\mathbb{R}^n$.

Property: The identity matrix is a multiplicative identity, that is for any vector $\mathbf{v} \in \mathbb{R}^n$ we have that

$$\mathrm{I}_n \cdot \mathbf{v} = \mathbf{v}.$$

Proof. @ the blackboard

Properties of the matrix-vector product

Theorem

Let $A, B \in \mathcal{M}(k \times n)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$. *Then*

1. $A \cdot (\mathbf{u} + \mathbf{v}) = A \cdot \mathbf{u} + A \cdot \mathbf{v}$
2. $A \cdot (\lambda \mathbf{u}) = \lambda A \cdot \mathbf{u}$
3. $(A + B) \cdot \mathbf{u} = A \cdot \mathbf{u} + B \cdot \mathbf{u}$
4. $A \cdot \mathbf{e}_j = \mathbf{a}_j$ *where* $\mathbf{a}_j$ *is the j-th column of* A
5. $A \cdot \mathbf{o}_n = \mathbf{o}_k$, *where* $\mathbf{o}_n$ *is the zero vector in* $\mathbb{R}^n$ *and* $\mathbf{o}_k$ *is the zero vector in* $\mathbb{R}^k$
6. $O_{k \times n} \cdot \mathbf{v} = \mathbf{o}_k$ *where* $O_{k \times n}$ *is the null* $k \times n$*-matrix*
7. $I_n \cdot \mathbf{v} = \mathbf{v}$, *where* I_n *is the* $n \times n$*-identity matrix*

Matrix multiplication

Let $A \in \mathcal{M}(k \times n)$ and $B \in \mathcal{M}(n \times m)$, then $P = A \cdot B$ is a $k \times m$ matrix whose columns are given by

$$\mathbf{p}_1 = A \cdot \mathbf{b}_1, \quad \mathbf{p}_2 = A \cdot \mathbf{b}_2, \ldots, \mathbf{p}_m = A \cdot \mathbf{b}_m$$

Notice that this is just a repetition of the matrix-vector product.

Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 5 \end{pmatrix} \in \mathcal{M}(3 \times 2), \qquad B = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \in \mathcal{M}(2 \times 2)$$

Then $P = AB \in \mathcal{M}(3 \times 2)$ with columns

$$\mathbf{p}_1 = A\mathbf{b}_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}$$

$$\mathbf{p}_2 = A\mathbf{b}_2 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

That is

$$P = \begin{pmatrix} 3 & 1 \\ 5 & 1 \\ 10 & 0 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \in \mathcal{M}(2 \times 3), \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 2 \end{pmatrix} \in \mathcal{M}(3 \times 2)$$

Then $P = AB \in \mathcal{M}(2 \times 2)$ with columns

$$\mathbf{p}_1 = A\mathbf{b}_1 = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix}$$

p_{11} = 1st row of A times 1st column of $B \Rightarrow (2 \ -1 \ \ 3) \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

p_{21} = 2nd row of A times 1st column of $B \Rightarrow -5$

$$\mathbf{p}_2 = A\mathbf{b}_2 = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix}$$

$p_{12} =$ 1st row of A times 2nd column of $B \Rightarrow 5$

$p_{22} =$ 2nd row of A times 2nd column of $B \Rightarrow 0$

Finally,

$$P = \begin{pmatrix} 11 & 5 \\ -5 & 0 \end{pmatrix}$$

Remark

An important property! The matrix product is NOT commutative

Theorem

Properties of the matrix product Let $A, B \in \mathcal{M}(k \times n)$, $C \in \mathcal{M}(n \times m)$ *and* $P, Q \in \mathcal{M}(m \times l)$. *We get that:*

1. $\lambda(AC) = (\lambda A)C = A(\lambda C)$, *for all* $\lambda \in \mathbb{R}$
2. $(AC)P = A(CP)$
3. $(A + B)C = AC + BC$
4. $C(P + Q) = CP + CQ$
5. $I_k A = A = A I_n$
6. $A O_{n\times m} = O_{k\times m}$
7. $(AC)^\top = C^\top A^\top$

By now we know how to *multiply* matrices. Can we divide matrices?

For real numbers, we know that $\frac{a}{b} = ab^{-1}$. Similarly we want to define the operation of multiplication by the *inverse* of matrix.

Definition

Let $A \in \mathcal{M}(n \times n)$. *We say that* A *is* ***invertible*** *if there exists a matrix* $B \in \mathcal{M}(n \times n)$ *such that*

$$AB = BA = I_n.$$

B *is called the inverse of* A

In the sequel we will denote the inverse of A by A^{-1}.

The inverse of a 2×2 matrix

Theorem

Let $A \in \mathcal{M}(2 \times 2)$ *given by*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - cb \neq 0$. *Then*

$$A^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Remark

A 2×2 *matrix is invertible if and only if* $ad - cb \neq 0$ *(we will see later why and where this condition comes from).*

What about higher dimension matrices? We need more mathematical instruments

Theorem

Let $A, P \in \mathcal{M}(n \times n)$ *and* ***invertible****. Then*

1. $(A^{-1})^{-1} = A$
2. $(A^{\top})^{-1} = (A^{-}1)^{\top}$
3. AP *is invertible and* $(AP)^{-1} = P^{-1}A^{-1}$

When is a matrix invertible? (conditions will be provided later in the lecture)

Determinant

- $n = 1$ A 1×1 matrix is a real number $a \in \mathbb{R}$. We define $\det(a) = a$.
- $n = 2$ Let
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
we define $\det(A) = ad - cb$
- $n \geqslant 3$ For higher dimension the determinant has a more complicated definition which requires additional definitions

Determinant

For any square matrix $A \in \mathcal{M}(n \times n)$ we denote by A_{ij} the matrix obtained from A by deleting the row i and the column j

Notice that we can express the determinant of a 2×2 matrix in terms of these *reduced* matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$\det(A) = a \det(A_{11}) - b \det(A_{12})$

Definition (Determinant)

For any $A \in \mathcal{M}(n \times n)$ *the determinant of* A *is given by*

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+1} a_{i2} \det(A_{i2}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in})$$

Define $c_{ij} = (-1)^{i+j} \det(A_{ij})$. This coefficient is called cofactor. Then we can rewrite the determinant as:

$$\det(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

This equation is called the cofactor expansion.

Definition (Determinant (equivalent definition))

For any $A \in \mathcal{M}(n \times n)$ *the determinant of* A *is given by*

$$\det(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj}$$

Theorem

The determinant of a ***diagonal, lower triangular or upper triangular*** *matrix is the product of its diagonal elements*

Proof. @ the blackboard

Theorem (Properties of determinants)

Let $A, B \in \mathcal{M}(n \times n)$. *Then the following properties hold:*

1. A *is invertible if and only if* $\det A \neq 0$
2. $\det(AB) = \det(A)\det(B)$
3. $\det(A^\top) = \det(A)$
4. *If* A *is invertible then* $\det(A^{-1}) = \dfrac{1}{\det(A)}$
5. $\det(\lambda A) = \lambda^n \det(A)$

Remark: $\det(A + B) \neq \det(A) + \det(B)$

Inverse of a matrix

Definition

Let $A \in \mathcal{M}(n \times n)$. We define the cofactor matrix of A, $\mathrm{Cof}(A)$, the matrix of all cofactors of A, that is the matrix with entries c_{ij}.

Theorem

Let $A \in \mathcal{M}(n \times n)$ be invertible. Then

$$A^{-1} = \frac{1}{\det(A)}(\mathrm{Cof}(A))^{\top}$$