Mathematics 2

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Week 4 - Rank, Linear Dependence and Linear Systems

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Outline

Suppose you are given vectors $\mathbf{v} \in \mathbb{R}^k$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in \mathbb{R}^k$. Can you tell me if there are coefficients $c_1, c_2, \ldots, c_n$, such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n \in \mathbb{R}^k?$$

Example (A simple case)

Any vector $\mathbf{v} \in \mathbb{R}^k$ is a linear combination of standard vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k \in \mathbb{R}^k$,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_k\mathbf{e}_k \in \mathbb{R}^k$$

In the general case the answer to this question amounts to solve a system!

Definition (Linear dependence)

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^k$ *is called linearly dependent if there exits constants* $c_1, c_2, \ldots, c_n$ *NOT ALL EQUAL TO ZERO such that*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{o}_k.$$

In this case we say that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^k$ *are linearly dependent*

Definition (Linear independence)

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^k$ *is called linearly independent if and only if the only linear combination such that*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{o}_k$$

is the trivial linear combination, that is $c_1, c_2, \ldots, c_n$ *are ALL equal to zero. In this case we also say that vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^k$ *are linearly independent*

Remark

- *There are at most* k *linearly independent vectors in* $\mathbb{R}^k$
- *Any set of vectors in* $\mathbb{R}^k$ *which includes the zero vector is linearly dependent*

Rank

Definition (Rank)

Let $A \in \mathcal{M}(m \times n)$*. We define the Rank of* A, $r(A)$, *as the maximum number of linearly independent columns.*

Equivalent definition:

Definition (Rank)

Let $A \in \mathcal{M}(m \times n)$*. We define the Rank of* A, $r(A)$, *as the maximum number of linearly independent rows.*

Remark

According to the definition we have that $r(A) \leqslant \min\{m, n\}$

How to compute the rank

Theorem

Let $A \in \mathcal{M}(m \times n)$. *The rank of* A *is an integer* $r \leqslant \min\{n, m\}$ *such that the following two conditions hold:*

1. *There is a non-zero minor of order* r;
2. *All minors of order* $r+1$ *are zero or there is no minor of order* $r+1$.

How to compute the rank, a shortcut

Theorem

Let $A \in \mathcal{M}(m \times n)$*. The rank of* A *is an integer* $r \leqslant \min\{n, m\}$ *such that the following two conditions hold:*

1. *There is a non-zero minor of order* r*;*
2. *All minors of order* $r + 1$ *that are obtained by adding to the submatrix of dimension* $r \times r$ *used above, are zero or there is no minor of order* $r + 1$*.*

Linear systems

Definition (Linear Equation)

A linear Equation in the variables $x_1, x_2, \ldots, x_n$ *is an equation that can be written as*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$a_1, \ldots, a_n \in \mathbb{R}$ *are called the coefficients and* $b \in \mathbb{R}$ *is the constant term of the equation.*

Example

- $3x_1 - 7x_2 + x_3 = 9$ *is a linear equation*
- $8x_1 - 2x_3 = x_2 + 1$ *is a linear equation*
- $8x_1 + 2x_2^2 = 5$ *is NOT a linear equation*
- $2x_1 - x_1x_2 = 0$ *is NOT a linear equation*

Linear systems

Definition (Linear System)

A system of linear equations is a set of k *linear equations in the same* n *variables* $x_1, \dots, x_n$ *that can be written as*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &= b_k. \end{aligned}$$

Does this expression remind you anything?

We say that the above set of equations is a linear system of k equations in n unknowns

Linear systems

Definition

A solution of the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &= b_k. \end{aligned}$$

is a set of n *values* $(s_1, s_2, \ldots, s_n)$ *such that every equality in the system, holds if we replace* x_i *with* s_i*, for every* $i = 1, \ldots, n$.

System of two linear equations in two variables

A linear equation in two variables has the form

$$ax + by = c$$

When at least one between a and b is not zero, this is the equation of a line in the xy-plane, say $\mathcal{L}$

Consider a system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

System of two linear equations in two variables

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Consider a system

$$\begin{aligned} a_1x + b_1y &= c_1 \quad \text{equation of line } \mathcal{L}_1 \\ a_2x + b_2y &= c_2 \quad \text{equation of line } \mathcal{L}_2 \end{aligned}$$

A system of two linear equations in two variables consists of a pair of equations which describe lines in the xy-plane

Geometric interpretation

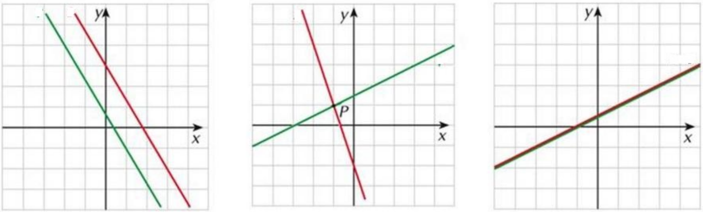
Definition

A solution of a system of two equations in two variables is a pair (s_1, s_2) *which represents the coordinate of a point lying on both* $\mathcal{L}_1$ *and* $\mathcal{L}_2$.

Three possible situations:

1. $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel: No solution
2. $\mathcal{L}_1$ and $\mathcal{L}_2$ intercept at one point: Unique solution
3. $\mathcal{L}_1$ and $\mathcal{L}_2$ are the same line: Infinitely many solutions

Geometric interpretation



In case 1 we say that the system is inconsistent

In case 2 and 3 we say that the system is consistent

These definitions extend to any dimension!

Systems of equations in matrix form

Consider the system

$$3x_1 + 4x_2 = 5$$

$$7x_1 - 2x_2 = 2$$

Define

$$A = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

We can re-write the system using matrix-vector multiplication form:

$$A\mathbf{x} = \mathbf{b}$$

Systems of equations in matrix form

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix}$$

Then the system can be written in the equivalent form

$$A\mathbf{x} = \mathbf{b}$$

Consistency

Definition

A linear system is consistent if it has at least one solution.

A linear system is inconsistent if it has no solution.

Homogeneous system

A linear system with all constant terms equal to zero, i.e.

$$A\mathbf{x} = \mathbf{o}_k,$$

is called homogeneous.

Theorem

A homogeneous system is always consistent.

Proof: It is enough to show that an homogeneous system always has the trivial solution. However it may also have other non-trivial solutions (we will see this later!)

Importance of Rank

Proposition

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^k$ *be vectors and let* $\mathbf{b} \in \mathbb{R}^k$ *be a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$*. Let* V *be the matrix with columns* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *and* $\widetilde{V}$ *be the matrix with columns* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{b}\}$*. Then*

$$\mathrm{rank}(V) = \mathrm{rank}(\widetilde{V})$$

.

Rouché-Capelli Thorem

Theorem (Rouché-Capelli)

Consider a system $A\mathbf{x} = \mathbf{b}$ *of* k *equations and* n *unknowns, with* $k \leqslant n$.
The system is consistent if and only if $\operatorname{rank}(A) = \operatorname{rank}(\widetilde{A})$.
If the system is consistent, let $r = \operatorname{rank}(A) = \operatorname{rank}(\widetilde{A})$. *Then it has*

a. *a unique solution if* $r = n$
b. ∞^{n-r} *solutions if* $r < n$

Interpretation of Rouché-Capelli Theorem

Consider the system $A\mathbf{x} = \mathbf{b}$ of k equations and n unknowns

1. A system is consistent if the vector $\mathbf{b}$ is LINEARLY DEPENDENT of the columns of A;
2. A system is inconsistent if the vector $\mathbf{b}$ is LINEARLY INDEPENDENT of the columns of A.

The square case

Consider a system

$$A\mathbf{x} = \mathbf{b}$$

where $A \in M(n \times n)$ with $\det(A) \neq 0$, $\mathbf{b} \in \mathbb{R}^n$.

- In this case the system is consistent, $r(A) = r(\tilde{A}) = n$, and it has a unique solution.
- To compute the solution we let $D_i \in M(n \times n)$ be the square matrix where we replace column i of A with the vector $\mathbf{b}$.

The square case

Then the solution of the system is $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ where

$$x_1 = \frac{|D_1|}{|A|}, \quad x_2 = \frac{|D_2|}{|A|}, \ldots, x_n = \frac{|D_n|}{|A|}.$$

The general case

Consider a system

$$A\mathbf{x} = \mathbf{b}$$

where $A \in M(k \times n)$ and rank $r < n$, $\mathbf{b} \in \mathbb{R}^k$.

If the system is consistent, $r(A) = r(\tilde{A}) = r$, and it has ∞^{n-r} solutions.

The general case

To compute the solution:

- Consider a nonzero minor of order r
- Consider only the equations of the system used for the calculation of this minor and remove all the others
- Keep on the left side of the equality the variables whose coefficients are used top compute the minor and move to the right all other variables. The variables on the left are called basic variables, and the variables on the right are called free variables. Free variables correspond to parameters, basic variables are considered as unknowns.
- Now we ended up with a square case, and we can apply Cramer's rule.