Mathematics 2, AY 2023-2024
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Additional Exercises n.4

Ex 1. Decide if the following sets of vectors are linearly dependent or linearly independent. If they are linearly dependent write down the largest subset of linearly independent vectors.

1.
$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -4 \\ 1 \\ 3 \\ 1 \end{pmatrix}$$

2.
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix},$$

3.
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ -1 \\ 3 \\ -2 \\ 7 \end{pmatrix}$$

4.
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

5.
$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \\ 2 \\ -2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 4 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -3 \end{pmatrix}$$

6.
$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Ex 2. Find the rank of the following matrices.

1.
$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

2.
$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 3 & 5 \\ -1 & 1 & 0 & -1 \end{pmatrix}$$

3.
$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}$$

4.
$$\begin{pmatrix} -1 & 1 & 0 & 1 & 1 \\ 0 & -2 & 1 & -3 & 1 \\ 1 & -3 & 2 & -5 & 1 \\ -2 & 0 & 0 & 0 & 2 \end{pmatrix}$$

5.

$$\begin{pmatrix} -1 & -2 \\ 0 & \frac{5}{2} \\ -\frac{2}{3} & 3 \end{pmatrix}$$

6.

$$\begin{pmatrix} 5 & -3 & 2 \\ 10 & -6 & 4 \\ 1 & 4 & 3 \\ 4 & -7 & -1 \end{pmatrix}$$

7.

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 5 \\ 0 & 2 & -\frac{1}{2} & 4 & 3 \\ -1 & -3 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 6 \end{pmatrix}$$

8.

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ \frac{1}{2} & 3 & -1 & 2 \\ 4 & -3 & 5 & 7 \\ 5 & -4 & 7 & 8 \end{pmatrix}$$

Ex 3 Discuss the rank of the following matrices as k changes

1.

$$\begin{pmatrix} 1 & k & -1 \\ 2k & -2 & 1 \\ k+1 & 1 & 1 \end{pmatrix}$$

2.

$$\begin{pmatrix} 2 & -1 & k^2 \\ 1 & k & 0 \\ 0 & -1 & 1-k \end{pmatrix}$$

3.

$$\begin{pmatrix} k & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & k & -2 & 0 \end{pmatrix}$$

4.

$$\begin{pmatrix} 1 & 2 & -k & 6 \\ 0 & 2k & -\frac{1}{2} & 5 \\ -1 & -3 & 0 & 3 \end{pmatrix}$$

Ex 4. Use Rouché-Capelli Theorem to discuss whether the following systems are consistent or not and specify the number of solutions.

1.

$$\begin{cases} x_1 - x_2 + x_4 = 1 \\ x_3 - x_4 = -1 \\ x_1 - x_2 + 2x_3 - x_4 = 1 \end{cases}$$

2.

$$\begin{cases} x_1 + 3x_2 - x_3 = 4 \\ 2x_1 + x_2 + 3x_3 = 3 \\ x_1 + 2x_2 = 3 \end{cases}$$

3.

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ -2x_1 + 3x_3 = 3 \\ x_1 + 3x_2 = 0 \end{cases}$$

4.

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 0 \\ x_1 - 2x_2 = 1 \\ 3x_1 + x_2 + x_3 = 0 \end{cases}$$

Ex 4. Use Rouché-Capelli Theorem to discuss whether the following systems are consistent or not and specify the number of solutions as k changes

1.

$$\begin{cases} -x_1 + x_2 + x_3 = 2 \\ x_1 - x_2 = 1 \\ kx_1 - 2x_2 - 2x_3 = k \end{cases}$$

2.

$$\begin{cases} x_1 - x_2 - x_3 = 0 \\ 3x_1 + x_2 + 2x_3 = 0 \\ 4x_1 + kx_2 = k \end{cases}$$

3.

$$\begin{cases} x_1 - kx_2 = 1 \\ 2x_1 - x_2 + x_3 = k \\ x_1 + x_2 = 2 \end{cases}$$

4.

$$\begin{cases} 2x_1 + 3x_2 = 1 \\ -kx_1 + 2x_2 = -2 \\ 2x_1 - 2x_2 = k \end{cases}$$

5.

$$\begin{cases} x_1 + kx_2 = 1 \\ -x_1 + x_2 + x_3 = 2 \\ x_1 + kx_2 - 2kx_3 = 2 \end{cases}$$

6.

$$\begin{cases} x_1 - kx_2 + x_3 = 0 \\ x_1 + x_2 = 1 \\ x_1 + x_3 = 2k \end{cases}$$

7.
$$\begin{cases} x_1 + kx_2 = k \\ -x_1 - x_2 = 1 \\ x_1 + x_2 + kx_3 = -k - 1 \end{cases}$$

8.
$$\begin{cases} 2x_1 + (k-1)x_2 = 2k \\ -kx_1 - x_2 = 1 \\ kx_1 + x_2 = k + 2 \end{cases}$$

9.
$$\left\{ \begin{array}{ccc} x_1 + x_2 + kx_3 & = & 0 \\ x_1 + kx_2 & = & 0 \\ kx_1 & = & 0 \end{array} \right.$$

10.
$$\begin{cases} -kx_1 + (k-1)x_2 + z = 1 \\ (k-1)x_2 + kx_3 = 1 \\ 2x_1 + x_3 = 5 \end{cases}$$

11.
$$\begin{cases} kx_1 + x_2 = -1 \\ 4x_1 - 2x_2 = -k \\ 6x_1 + 3x_2 = -3 \end{cases}$$

12.
$$\begin{cases} x + y + z = 0 \\ x + y + kz = k \\ x + (k-1)y = 0 \\ x + (k-1)y + kz = k \end{cases}$$

13.
$$\begin{cases} x - 2y = 0 \\ 2x + 2ky = 1 \\ x + ky = 1 \end{cases}$$

14.
$$\begin{cases} x + ky + kz = k \\ kx + z = 0 \\ x + ky + z = 2 \end{cases}$$

15.
$$\begin{cases} x + y + kz = 2 \\ x + 2y + 3z = k - 1 \\ 2x + ky - 2z = 3 \end{cases}$$