Mathematics 2

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Week 5 - Eigenvalues/eigenvectors, Definiteness of a matrix, Functions of 2 variables

University of Rome Tor Vergata

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Outline

1. Eigenvalues and Eigenvectors
2. Definiteness of a matrix
3. Functions of two variables
   - Domain and Range
   - Level curves
   - Continuity
   - First order partial derivatives

Vectors

Some properties we already know:

- A vector $v \in \mathbb{R}^n$ is just a matrix with 1 column and n rows
- Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have a clear geometric interpretation
- Vectors in $\mathbb{R}^n$ can be identified analogously as points and directions in the n-dimensional space, even though there is no natural spatial interpretation

Eigenvectors: Motivation

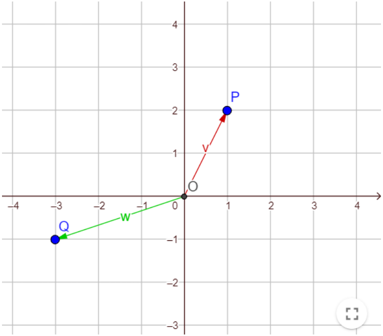
Suppose you are given

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and compute

$$\mathbf{w} = A\mathbf{v} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

Eigenvectors: Motivation



The matrix A

- changes the direction of the vector $\mathbf{v}$
- changes the length of the vector $\mathbf{v}$

This holds in any dimension

Eigenvectors: Motivation

However, there are vectors which are immune to direction changes, i.e. the matrix A only changes the length of the vector but not the direction:

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some constant λ.

These special vectors are called Eigenvectors of A and the multiplicative constant λ is called an Eigenvalue of A.

Eigenvalues and Eigenvectors

Definition

Let $A \in \mathcal{M}(n \times n)$ be a square matrix. An ***Eigenvector*** *of A is a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that*

$$A\mathbf{v} = \lambda \mathbf{v}$$

for some scalar $\lambda \in \mathbb{R}$. The value λ is called an ***Eigenvalue*** *of A*

Equivalently an Eigenvector is any not-trivial solution of the homogeneous system

$$(A - \lambda I_n)\mathbf{v} = \mathbf{0}$$

for every eigenvalue $\lambda \in \mathbb{R}$

How to compute Eigenvalues?

Consider the homogeneous system:

$$(A - \lambda I_n)\mathbf{x} = \mathbf{o}_n.$$

This system has non-trivial if and only if $\det(A - \lambda I_n) = 0$.

Eigenvalues are the solutions of the equation

$$\det(A - \lambda I_n) = 0$$

This is called the characteristic equation

How to compute Eigenvectors?

- For each eigenvalue we solve the system $(A - \lambda I_n)v = \mathbf{0}$
- All solutions of $(A - \lambda I_n)v = \mathbf{0}$ are the eigenvectors of A relative to the eigenvalue λ.

Compute all eigenvalues and eigenvectors of the matrix A

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$$

Solution:

Eigenvalues

- Let $A - \lambda I_2 = A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & -2 \\ -1 & -\lambda \end{pmatrix}$.
- We compute $\det \begin{pmatrix} 1-\lambda & -2 \\ -1 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2$
- Next we solve $\lambda^2 - \lambda - 2 = 0$, that is $\lambda_1 = -1$ and $\lambda_2 = 2$.
  These are the Eigenvalues.

Eigenvectors

- We start with $\lambda_1 = -1$ and compute $A - \lambda_1 I_2 = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$
- To find the eigenvectors corresponding to $\lambda_1 = -1$ we solve the system

$$\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- The system is consistent with ∞^1 solutions given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix}, \quad \forall t \in \mathbb{R}$$

These are all eigenvalues corresponding to $\lambda_1 = -1$.

Eigenvectors

- Next consider $\lambda_2 = 2$ and compute $A - \lambda_2 I_2 = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}$
- To find the eigenvectors corresponding to $\lambda_2 = 2$ we solve the system

$$\begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- The system is consistent with ∞^1 solutions given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2t \\ t \end{pmatrix}, \quad \forall t \in \mathbb{R}$$

These are all eigenvalues corresponding to $\lambda_2 = 2$.

Overall

The matrix $A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$ has:

- Eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$
- Eigenvectors:

  for $\lambda_1 = -1$ the eigenvectors are $\left\{ \begin{pmatrix} t \\ t \end{pmatrix}, \ \forall t \in \mathbb{R} \right\}$

  for $\lambda_2 = 2$ the eigenvectors are $\left\{ \begin{pmatrix} -2t \\ t \end{pmatrix}, \ \forall t \in \mathbb{R} \right\}$

Definiteness of a matrix

Definition

Let $A \in \mathcal{M}(n \times n)$ be a square matrix. We say that A is

- *Positive Definite if all its eigenvalues are strictly positive: $\lambda_i > 0$ for all i*
- *Negative Definite if all its eigenvalues are strictly negative: $\lambda_i < 0$ for all i*
- *Indefinite if it has positive and negative eigenvalues*
- ***Positive Semidefinite** if all its eigenvalues $\lambda_i \geqslant 0$ for all i and at least one eigenvalue is equal to zero*
- *Negative Semidefinite if all its eigenvalues $\lambda_i \leqslant 0$ for all i and at least one eigenvalue is equal to zero*

Definition (Function of two variables)

A function of two variables with domain $D \subseteq \mathbb{R} \times \mathbb{R}$ *(or* $D \subseteq \mathbb{R}^2$*)*

$$f : D \to \mathbb{R}$$

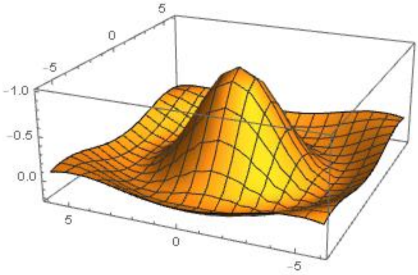
is a rule that assign to any point $(x, y) \in D$ *a real number* $f(x, y)$

Remark

- $\mathbb{R} \times \mathbb{R}$ *and* $\mathbb{R}^2$ *are two equivalent ways for denoting the same set, and we will use it indifferently.*
- *We also indicate a function of two variables with* $z = f(x, y)$*.*
- *The graph of a function of two variables is a surface.*

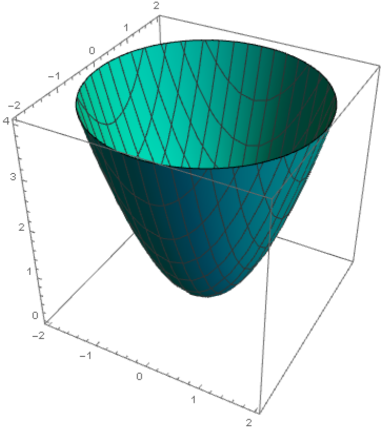
Examples: graph of a function of two variables

- $f(x,y) = \dfrac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$



Examples: graph of a function of two variables

- $f(x, y) = x^2 + y^2$



Definition (Domain and Range)

The domain D *of* f *is the set of all pairs* $(x, y) \in \mathbb{R} \times \mathbb{R}$ *for which the rule* $f(x, y)$ *makes sense*

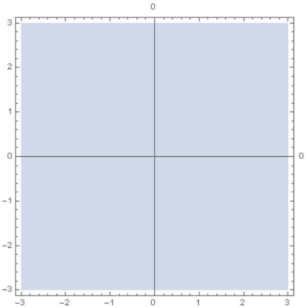
The range $C \subseteq \mathbb{R}$ *is the set of all values* $z = f(x, y)$

Remark: The domain is a subset of $\mathbb{R}^2$, and hence it can be drawn on the Cartesian plane.

Examples

- $f(x, y) = 2x + x^2y^2$

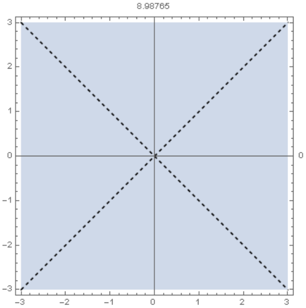
$$D = \mathbb{R} \times \mathbb{R}, \quad C = \mathbb{R}$$



Examples

- $f(x, y) = \frac{1}{x^2 - y^2}$

$$D = \{(x, y) \in \mathbb{R}^2 : y \neq x \text{ and } y \neq -x\}, \quad C = \mathbb{R} \setminus \{0\}$$



- $f(x, y) = \sqrt{xy}$

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R}, \quad \text{such that} \quad xy \geqslant 0\}$$

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Notice that $xy \geqslant 0$ if and only if either

- $x \geqslant 0,\ y \geqslant 0$
- or $x \leqslant 0,\ y \leqslant 0,$

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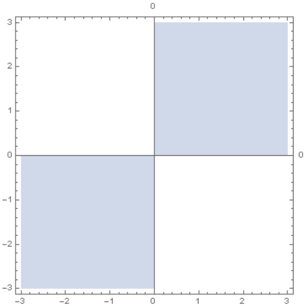
- $x \geqslant 0,\ y \geqslant 0$
- or $x \leqslant 0,\ y \leqslant 0$,

hence the domain is

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R} :$$
$$x \geqslant 0, y \geqslant 0 \text{ or } x \leqslant 0, y \leqslant 0\}$$

- $f(x, y) = \sqrt{xy}$

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geqslant 0, y \geqslant 0 \text{ or } x \leqslant 0, y \leqslant 0\}, \quad C = [0, +\infty)$$

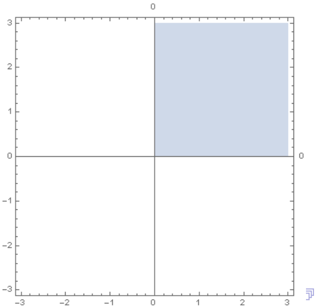


Examples

A function $F(x, y) = Ax^a y^b$, with A, a, b constant, is called a Cobb-Douglas function (very important in Economics!)

- $f(x, y) = 3\sqrt{x}\sqrt{y}$

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R}, \ \text{ such that } \ x \geqslant 0 \text{ and } y \geqslant 0\}, \quad C = [0, +\infty)$$

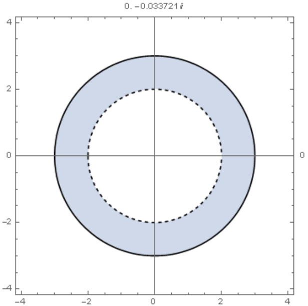


Examples

- $$f(x,y) = \frac{2}{\sqrt{x^2+y^2-4}} + \sqrt{9-x^2-y^2}$$

  $$D = \left\{(x,y) \in \mathbb{R} \times \mathbb{R}, \text{ such that } 4 < x^2+y^2 \leqslant 9\right\}, \quad C = (0,+\infty)$$

Graphically:

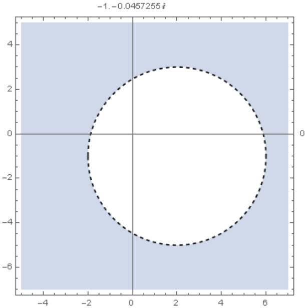


Examples

- $f(x,y) = \log((x-2)^2 + (y+1)^2 - 16)$

  $D = \left\{(x,y) \in \mathbb{R} \times \mathbb{R}, \text{ such that } (x-2)^2 + (y+1)^2 > 16\right\}, \quad C = \mathbb{R}$

Graphically:

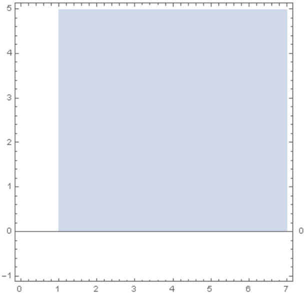


Examples

- $f(x, y) = \sqrt{x-1} + \sqrt{y}$

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R}, \quad \text{such that } x \geqslant 1 \text{ and } y \geqslant 0\}, \qquad C = [0, +\infty)$$

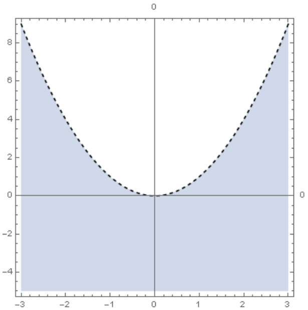
Graphically:



Examples

- $f(x, y) = \log(x^2 - y)$

$$D = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}, \quad \text{such that } y < x^2\right\}$$



Level curves

- A level curve is the contour curve that results from the intersection of the surface described by $z = f(x, y)$ and the plane $z = c$.
- Hence a level curve at height c represents the set of all points (x, y) in which the function $f(x, y)$ assumes the same value c.

Definition

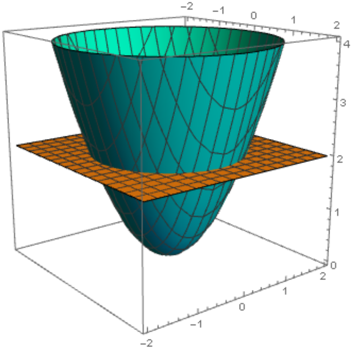
Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a function of two variables. The level curves of* f *are all curves with equation*

$$\boxed{f(x,y)=c},$$

for all $c \in \mathbb{R}$ *such that this equation makes sense.*

Examples

- $f(x, y) = x^2 + y^2$



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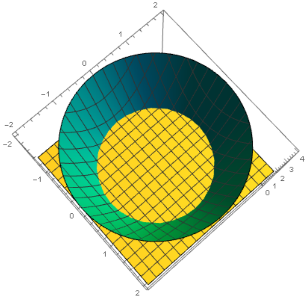
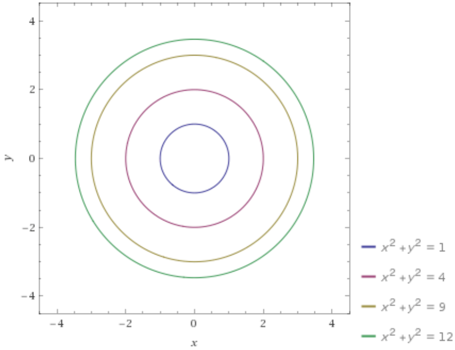


Level curves are circles centered at the origin

$$x^2 + y^2 = c, \quad \boxed{c>0}$$

and the origin $O = (0, 0)$ for $\boxed{c=0}$.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$x^2+y^2=1$
$x^2+y^2=4$
$x^2+y^2=9$
$x^2+y^2=12$

Level curves in Economics

Economists use level curves to study production functions and utility functions. We now make an example with the production function.

Example

Let $f(x, y)$ *be a production function. Here* x *and* y *represents the inputs (typically* x *is the capital and* y *is the labour).*

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Level curves are

$$f(x, y) = c, \quad \forall\, c \ \textit{ constant}$$

These curves are called ***isoquants*** *("same quantity"): they represent values of inputs (capital and labour) that lead to the same level of production.*

Example: the Cobb-Douglas Function

- $f(x, y) = xy$
  - x represent the units of capital
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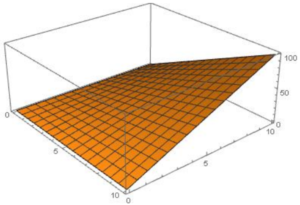

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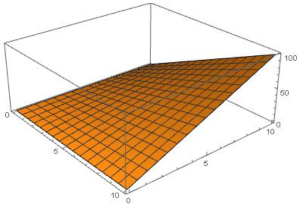
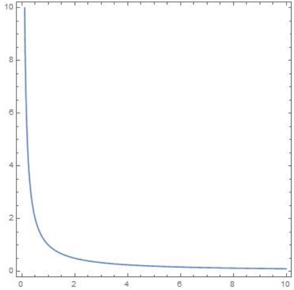


Isoquants of production are hyperboles

$$xy = c \Rightarrow y = \frac{c}{x}$$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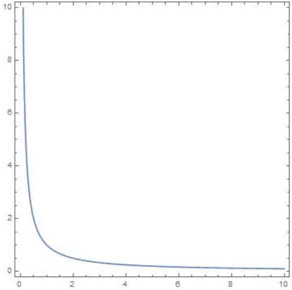
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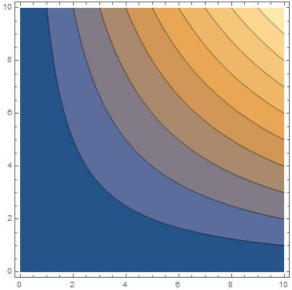
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Continuity: When is a function of two variables continuous?

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Definition

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a function of two variables. The function* f *is continuous at a point* $(x_0, y_0) \in D$ *if and only is*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

exists, it is finite and it coincides with $f(x_0, y_0)$*.*

What does it mean $(x, y) \to (x_0, y_0)$? In means that (x, y) approaches (x_0, y_0) from any direction!

First order partial derivative with respect to x.

Definition

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. We say that f is differentiable with respect to x at a point (x_0, y_0) is

$$\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

exists and it is finite.

Definition

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. We say that f is differentiable in D with respect to x if it is differentiable with respect x at all points $(x_0, y_0) \in D$.

First order partial derivative with respect to y.

Definition

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. We say that f is differentiable with respect to y at a point (x_0, y_0) is

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

exists and it is finite.

Definition

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. We say that f is differentiable in D with respect to y if it is differentiable with respect y at all points $(x_0, y_0) \in D$.

Notation

- We denote by $\frac{\partial f}{\partial x}(x, y) = D_x(f(x, y)) = f_x(x, y)$ the partial derivative of f with respect to the variable x.
- We denote by $\frac{\partial f}{\partial y}(x, y) = D_y(f(x, y)) = f_y(x, y)$ the partial derivative of f with respect to the variable y.

Definition

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a function of two variables. If* $f(x, y)$ *has both partial derivatives with respect to* x *and* y *we can define the* ***gradient*** *of* f *as the vector*

$$\nabla f(x, y) = \begin{pmatrix} f_x(x, y) \\ f_y(x, y) \end{pmatrix}$$