Mathematics 2, AY 2023-2024
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Additional Exercises n.5

1 Linear systems

Ex 1. Solve the following linear systems

1.
$$\begin{cases} x_1 - x_2 + x_4 = 1 \\ x_3 - x_4 = -1 \\ x_1 - x_2 + 2x_3 - x_4 = 1 \end{cases}$$

2.
$$\begin{cases} x_1 + 3x_2 - x_3 = 4 \\ 2x_1 + x_2 + 3x_3 = 3 \\ x_1 + 2x_2 = 3 \end{cases}$$

3.
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ -2x_1 + 3x_3 = 3 \\ x_1 + 3x_2 = 0 \end{cases}$$

4.
$$\begin{cases} 2x_1 + 3x_2 + x_3 = 0 \\ x_1 - 2x_2 = 1 \\ 3x_1 + x_2 + x_3 = 0 \end{cases}$$

Ex 2. Solve the following parametric linear systems, as k changes

1.
$$\begin{cases} -x_1 + x_2 + x_3 = 2 \\ x_1 - x_2 = 1 \\ kx_1 - 2x_2 - 2x_3 = k \end{cases}$$

2.
$$\begin{cases} x_1 - x_2 - x_3 = 0 \\ 3x_1 + x_2 + 2x_3 = 0 \\ 4x_1 + kx_2 = k \end{cases}$$

3.
$$\begin{cases} x_1 - kx_2 = 1 \\ 2x_1 - x_2 + x_3 = k \\ x_1 + x_2 = 2 \end{cases}$$

4.
$$\begin{cases} 2x_1 + 3x_2 = 1 \\ -kx_1 + 2x_2 = -2 \\ 2x_1 - 2x_2 = k \end{cases}$$

5.
$$\begin{cases} x_1 + kx_2 = 1 \\ -x_1 + x_2 + x_3 = 2 \\ x_1 + kx_2 - 2kx_3 = 2 \end{cases}$$

6.
$$\begin{cases} x_1 - kx_2 + x_3 = 0 \\ x_1 + x_2 = 1 \\ x_1 + x_3 = 2k \end{cases}$$

7.
$$\begin{cases} x_1 + kx_2 = k \\ -x_1 - x_2 = 1 \\ x_1 + x_2 + kx_3 = -k - 1 \end{cases}$$

8.
$$\begin{cases} 2x_1 + (k-1)x_2 = 2k \\ -kx_1 - x_2 = 1 \\ kx_1 + x_2 = k + 2 \end{cases}$$

9.
$$\begin{cases} x_1 + x_2 + kx_3 = 0 \\ x_1 + kx_2 = 0 \\ kx_1 = 0 \end{cases}$$

10.
$$\begin{cases} -kx_1 + (k-1)x_2 + z = 1 \\ (k-1)x_2 + kx_3 = 1 \\ 2x_1 + x_3 = 5 \end{cases}$$

11.
$$\begin{cases} kx_1 + x_2 = -1 \\ 4x_1 - 2x_2 = -k \\ 6x_1 + 3x_2 = -3 \end{cases}$$

12.
$$\begin{cases} x + y + z = 0 \\ x + y + kz = k \\ x + (k-1)y = 0 \\ x + (k-1)y + kz = k \end{cases}$$

13.
$$\begin{cases} x - 2y = 0 \\ 2x + 2ky = 1 \\ x + ky = 1 \end{cases}$$

15.
$$\begin{cases} x + y + kz = 2 \\ x + 2y + 3z = k - 1 \\ 2x + ky - 2z = 3 \end{cases}$$

14.
$$\begin{cases} x + ky + kz = k \\ kx + z = 0 \\ x + ky + z = 2 \end{cases}$$

2 Eigenvalues, Eigenvectors and definiteness of a matrix

Ex 3. Say if the vector $\mathbf{v}$ is an eigenvector of the matrix A. If so, determine the eigenvalue.

1. $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

4. $A = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 1 \\ 2 & -3 & -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

5. $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -6 & 2 & -6 & -6 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$

6. $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -6 & 2 & -6 & -6 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$

Ex 4. For each of the following matrices

   i. Compute all eigenvalues;

   ii. Compute all eigenvectors;

   iii. Determine the definiteness of the matrix A

1. $A = \begin{pmatrix} 2 & -1 \\ -2 & \frac{1}{2} \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 1 \\ 6 & -1 \end{pmatrix}$

3. $A = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}$

4. $A = \begin{pmatrix} 0 & -4 \\ 2 & 0 \end{pmatrix}$

5. $A = \begin{pmatrix} -1 & -6 \\ 2 & -1 \end{pmatrix}$

6. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

7. $A = \begin{pmatrix} -\frac{1}{3} & 4 & -\frac{8}{3} \\ 0 & 1 & 0 \\ -\frac{4}{3} & 4 & -\frac{5}{3} \end{pmatrix}$

8. $A = \begin{pmatrix} -1 & 4 & -8 \\ 0 & 1 & 0 \\ -4 & 4 & -5 \end{pmatrix}$

9. $A = \begin{pmatrix} -1 & 3 & -2 \\ 0 & 1 & 0 \\ -2 & 3 & -1 \end{pmatrix}$

10. $A = \begin{pmatrix} -1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

3 Functions of several variables: Domain and level curves

Ex 5. Compute and draw the domain of the following functions of two variables.

1. $f(x, y) = \dfrac{x + y}{x - 2y}$

2. $f(x, y) = \ln(xy + 3)$

3. $f(x, y) = e^{\frac{x^2+1}{y-2}}$

4. $f(x, y) = \ln(x^2 + y^2 - 1)$

5. $f(x, y) = \ln(1 - x^2 - y^2)$

6. $f(x, y) = x^2 + y^2 + \sqrt{2y(x + 1)}$

7. $f(x, y) = \dfrac{\sqrt{4 - (x^2 + y^2)}}{\sqrt{y - 2x + 1}}$

8. $f(x, y) = \ln(xy)$

9. $f(x, y) = \sqrt{x^2 + (y + 2)^2 - 9}$

10. $f(x, y) = \dfrac{1}{x^2 + y^2 - 9} + \ln(e - y - x^2)$

11. $f(x, y) = \dfrac{e^{\sqrt{3x+5y}}}{\sqrt{2x^2 + y^2 + 1}}$

12. $f(x, y) = \sqrt{x - y^2}$

13. $f(x, y) = \dfrac{1}{(x - 3)^2 + (y - 1)^2 - 4}$

Ex 6. Compute and draw the level curves of the following functions.

1. $f(x, y) = x^2 + y^2 - 1$

2. $f(x, y) = 3x + 4y$

3. $f(x, y) = \ln(x^2 + y^2)$

4. $f(x, y) = \dfrac{x^2 + 1}{y - 1}$

5. $f(x, y) = \sqrt{y - 2x^2 + x}$

6. $f(x, y) = e^{(x-1)^2+y^2}$

7. $f(x, y) = y - x^2 + 2x$

8. $f(x, y) = 2x - y + 1$

9. $f(x, y) = \dfrac{y - 1}{2x + 2}$

10. $f(x, y) = e^{xy}$

11. $f(x, y) = \sqrt{x - y^2}$