MATHEMATICS 1
ADDITIONAL EXERCISES N. 1

KATIA COLANERI

1. Review on Equations and Inequalities

(1) Solve the following equations:

(a) $(1-3x)(x+2) = (2x+1)\left(\frac{1}{3}x + \frac{2}{3}\right) - \frac{x^2-4}{3}$

Sol:

$$x = -2,\ x = 0$$

(b) $\frac{(x-1)(x+2)}{3} - \frac{2}{5}(x^2-4x) = \frac{8(1-3x)}{15}$

Sol:

$$x = \frac{53-\sqrt{2737}}{2},\ x = \frac{53+\sqrt{2737}}{2}$$

(c) $(x-2)(x+2) = \frac{8}{3}x - 5$

Sol:

$$x = \frac{4+\sqrt{7}}{3},\ x = \frac{4-\sqrt{7}}{3}$$

(d) $\frac{7x-1}{x^2+4x+4} + \frac{4}{x^2-4} - \frac{3x^2+5x-2}{x^3-4x+2x^2-8} = 0$

Sol:

$$x = 3,\ x = 1$$

(e) $\frac{2}{x-2} + \frac{3x}{x^2-4} + 1 = \frac{16}{3x-6} - \frac{x+5}{3x^2-12}$

Sol:

$$x = 3,\ x = -3$$

(f) $x^3(x^4+1) + \frac{2}{x} = \frac{x^4+33}{16x}$

Sol:

$$x = \frac{1}{2},\ x = -\frac{1}{2}$$

Katia Colaneri, Department of Economics and Finance, University of Rome Tor Vergata, Via Columbia 2, 00133 Rome, Italy.

E-mail address: katia.colaneri@uniroma2.it.

(g) $\left(\dfrac{1}{x+3}\right)^2 - \dfrac{7}{x+3} - 18 = 0$
Sol:
$$x = -\frac{7}{2},\ x = -\frac{26}{9}$$

(h) $\sqrt{x^2+4} = x+1$
Sol:
$$x = \frac{3}{2}$$

(i) $2x-5 = \sqrt{4x^2-x-1}$
Sol:
$$\nexists\, x \in \mathbb{R}$$

(j) $\sqrt{2x+3} - 1 = \sqrt{8x+5}$
Sol:
$$x = -\frac{11}{18}$$

(k) $\sqrt{7x+1} - 2\sqrt{x+1} = -1$
Sol:
$$x = 0$$

(2) Solve the following equations with exponentials:

(a) $3^{2x+1} = 1$
Sol:
$$x = -\frac{1}{2}$$

(b) $e^{2x+1} = e^{\frac{1}{x}}$
Sol:
$$x = \frac{1}{2},\ x = -1$$

(c) $e^{2x} + 3e^x - 4 = 0$
Sol:
$$x = 0$$

(d) $\dfrac{e^{2x}+1}{e^x-1} + \dfrac{e^x+1}{2} = 1$
Sol:
$$x \in \mathbb{R}$$

(e) $\dfrac{1}{e^{2x+2}} - \dfrac{1}{e^x} + 1 = 0$
Sol:
$$x = \log\left(\frac{e^2+\sqrt{e^4-4e^2}}{2e^2}\right),\ x = \log\left(\frac{e^2-\sqrt{e^4-4e^2}}{2e^2}\right)$$

(3) Solve the following equations with logarithms:

(a) $\log(3x+4)=0$
Sol:
$$x=-1$$
(b) $\log(x+1)=\log(2x)$
Sol:
$$x=1$$
(c) $2\log(x-1)+\log(x-2)=0$
Detailed Solution:
$$\begin{gathered}2\log(x-1)+\log(x-2)=0\\ \log\left((x-1)^2(x-2)\right)=\log(1)\\ (x-1)^2(x-2)=1\\ (x^2-2x+1)(x-2)-1=0\\ x^3-4x^2+5x-3=0\end{gathered}$$
This is a third degree polynomial the solution of which is $x\approx\ 2.46557\ldots$. For the purpose of this exercise it would be enough to stop at the latest equation, since the solution cannot be computed analytically.
(d) $\log^2(2x+1)-4\log(2x+1)+3=0$
Sol:
$$x=\frac{e^3-1}{2},\ x=\frac{e-1}{2}$$

(4) Solve the following trigonometric equations:
(a) $\sin(3x)=\dfrac{1}{2}$
Sol:
$$x=\frac{\pi}{18}+\frac{2\pi n}{3},\ x=\frac{5\pi}{18}+\frac{2\pi n}{3}$$
$\forall n\in\mathbb{Z}$
(b) $2\sin(x)-\sqrt{3}=0$
Sol:
$$x=\frac{\pi}{3}+2\pi n,\ x=\frac{2\pi}{3}+2\pi n$$
$\forall n\in\mathbb{Z}$
(c) $\cos(2x)=\cos(x)$
Sol:
$$x=\frac{4\pi n}{3},\ x=\frac{2\pi}{3}+\frac{4\pi n}{3}$$
$\forall n\in\mathbb{Z}$
(d) $4\sin^2(x)-1=0$
Sol:
$$x=\frac{\pi}{6}+\pi n,\ x=\frac{5\pi}{6}+\pi n$$
$\forall n\in\mathbb{Z}$

(e) $2\sin^2(x) + \sqrt{3}\sin(x) = 0$
Sol:
$$x = \pi n,\ x = \frac{4\pi}{3} + 2\pi n,\ x = \frac{5\pi}{3} + 2\pi n$$
$\forall n \in \mathbb{Z}$
(f) $\cos^2(x) + \cos(x) = 0$
Sol:
$$x = \frac{\pi}{2} + \pi n,\ x = \pi + 2\pi n$$
$\forall n \in \mathbb{Z}$
(g) $\cos^2(x) - 3\cos(x) + 2 = 0$
Sol:
$$x = 2\pi n$$
$\forall n \in \mathbb{Z}$
(h) $\sin(x) - \cos(x) = 0$
Sol:
$$x = \frac{\pi}{4} + \pi n$$
$\forall n \in \mathbb{Z}$
(i) $\sin(x) - \sqrt{3}\cos(x) = 0$
Sol:
$$x = \frac{\pi}{3} + \pi n$$
$\forall n \in \mathbb{Z}$

(5) Solve the following inequalities:
(a) $(2x-5)(x+9) - (x+3)(x-9) < 0$
Sol:
$$\frac{-\sqrt{433}-19}{2} < x < \frac{\sqrt{433}-19}{2}$$
(b) $\dfrac{x-2}{2} - \dfrac{5}{3} > \dfrac{x}{2} - \dfrac{2x+x^2}{3}$
Sol:
$$x < -4 \quad \text{or} \quad x > 2$$
(c) $(2x^2-8)(3x+5) \leq 0$
Sol:
$$x \leq -2 \quad \text{or} \quad -\frac{5}{3} \leq x \leq 2$$
(d) $x^3 - x^2 - 6x < 0$
Sol:
$$x < -2 \quad \text{or} \quad 0 < x < 3$$
(e) $\dfrac{x^2}{x-2} > x + 1$
Sol:
$$x < -2 \quad \text{or} \quad x > 2$$

(f) $\dfrac{x}{x-2} - 2 \geq \dfrac{-x+3}{x+1}$
Sol:
$$x < -1 \quad \text{or} \quad 2 < x \leq 5$$

(g) $\dfrac{x^2+5x+6}{x^2-4x+4} \leq 1$
Sol:
$$x \leq -\frac{2}{9}$$

(h) $\dfrac{x(x-1)(x^2-3)}{(x+1)(x-4)} > 0$
Sol:
$$x < -\sqrt{3} \quad \text{or} \quad -1 < x < 0 \quad \text{or} \quad 1 < x < \sqrt{3} \quad \text{or} \quad x > 4$$

(6) Solve the following inequalities with exponentials:

(a) $3^{2x+1} > 1$
Sol:
$$x > -\frac{1}{2}$$

(b) $2^{2-x} \leq 8$
Sol:
$$x \geq -1$$

(c) $2e^{x+3} > 5$
Sol:
$$x > \log\left(\frac{5}{2}\right) - 3$$

(d) $\dfrac{e^x - 1}{x-1} \leq 0$
Sol:
$$0 \leq x < 1$$

(e) $(e^x - 2)(e^{-x} + 1) \geq 0$
Sol:
$$x \geq \log(2)$$

(f) $\dfrac{3e^{2x}}{4 - e^x} \geq 1$
Sol:
$$0 \leq x < 2\log(2)$$

(7) Solve the following inequalities with logarithms:

(a) $\log(x^2 - 4) \geq 0$
Sol:
$$x \leq -\sqrt{5} \quad \text{or} \quad x \geq \sqrt{5}$$

(b) $\log(x+1) \leq \log(x-1)$

Sol:

$$\nexists\, x \in \mathbb{R}$$

(c) $\log(4x^2-1)\log(4-x) \leq 0$

Sol: (Notice that there was a typo in the text, a + instead of a $\cdot$, I apologize for this).

$$-\sqrt{\frac{1}{2}} \leq\ x < -\frac{1}{2} \quad \text{or} \quad \frac{1}{2} < x \leq \sqrt{\frac{1}{2}} \quad \text{or} \quad 3 \leq\ x < 4$$

(d) $\log^2(x+2) - 2\log(x+2) - 3 \leq 0$

Sol:

$$\frac{1}{e} - 2 \leq\ x \leq\ e^3 - 2$$

(e) $\log(2x+1) - \log(3-x) < 2$

Sol:

$$-\frac{1}{2} < x < \frac{299}{102}$$

(f) $\log^2(1-3x) - 13\log(1-3x) + 36 \leq 0$

Sol:

$$\frac{-e^9+1}{3} \leq\ x \leq\ \frac{-e^4+1}{3}$$

(8) Solve the following trigonometric inequalities:

(a) $4\sin^2(x) - 1 > 0$

Sol:

$$\frac{\pi}{6} + 2\pi n < x < \frac{5\pi}{6} + 2\pi n \quad \text{or} \quad -\frac{5\pi}{6} + 2\pi n < x < -\frac{\pi}{6} + 2\pi n,$$

$\forall n \in \mathbb{Z}$

(b) $\sin(x) - \sqrt{3}\cos(x) > 0$

Sol:

$$\frac{\pi}{3} + 2\pi n < x < \frac{4\pi}{3} + 2\pi n,$$

$\forall n \in \mathbb{Z}$

(c) $\sin(x) + \cos(x) + 1 < 0$

Sol:

$$-\pi + 2\pi n < x < -\frac{\pi}{2} + 2\pi n,$$

$\forall n \in \mathbb{Z}$

(d) $\dfrac{2\sin^2(x) - 1}{\cos(x)} < 0$

Sol:

$$-\frac{\pi}{4} + 2\pi n < x < \frac{\pi}{4} + 2\pi n \quad \text{or} \quad \frac{\pi}{2} + 2\pi n < x < \frac{3\pi}{4} + 2\pi n \quad \text{or} \quad \frac{5\pi}{4} + 2\pi n < x < \frac{3\pi}{2} + 2\pi n,$$

$\forall n \in \mathbb{Z}$

(e) $\dfrac{\sin(x)}{\sin(x)+1} > 1$
Sol:

$$\nexists x \in \mathbb{R},$$

equivalently we can write $\emptyset$.

(f) $\dfrac{\sin(2x)+\cos(x)}{\cos(2x)+1} \geq 0$
Sol:

$$2\pi n \leq \ x < \frac{\pi}{2} + 2\pi n \quad \text{or} \quad \frac{7\pi}{6} + 2\pi n \leq \ x < \frac{3\pi}{2} + 2\pi n \quad \text{or} \quad \frac{11\pi}{6} + 2\pi n \leq \ x \leq \ 2\pi + 2\pi n,$$

$\forall n \in \mathbb{Z}$

(9) Solve the following equalities and inequalities with absolute values:

(a) $|x^2 + 3x| = -2$
Sol: $\nexists x \in \mathbb{R}$, equivalently we can write $\emptyset$ or the equation is impossible.

(b) $|x^2 + 3x| > -2$
Sol: $\forall x \in \mathbb{R}$, or simply $\mathbb{R}$

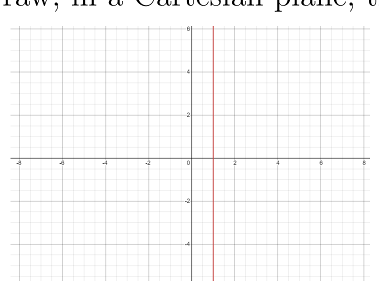
(c) $|7x + 3| = 2 - x$
Sol: $x = -\frac{5}{6} \quad \text{or} \quad x = -\frac{1}{8}$

(d) $|2x - 5| = x$
Sol: $x = \frac{5}{3} \quad \text{or} \quad x = 5$

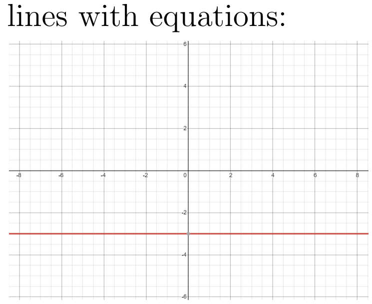
(e) $\left|\dfrac{x^2+2}{x+2}\right| \leq 2$
Sol: $-\sqrt{3} + 1 \leq \ x \leq \ 1 + \sqrt{3}$, equivalently we can write $[-\sqrt{3} + 1, \ x \leq \ 1 + \sqrt{3}]$

2. Equation of a line

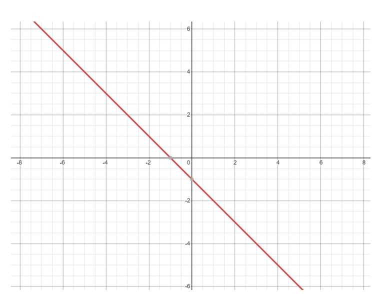
(1) Draw, in a Cartesian plane, the lines with equations:



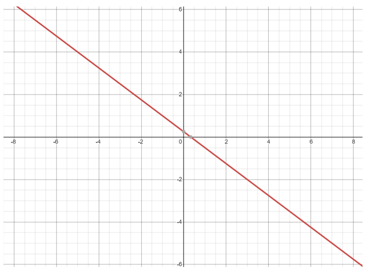
$x = 1$



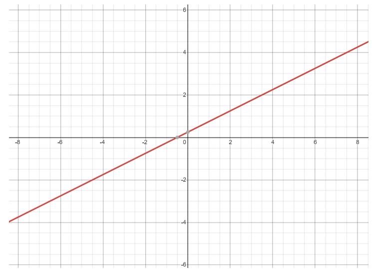
$y = -3$



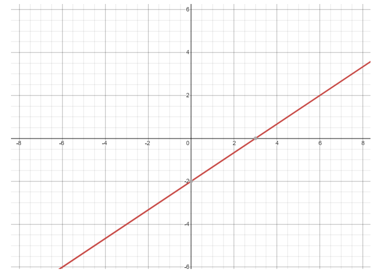
$x + y + 1 = 0$



$3x + 4y - 1 = 0$



$y = \frac{1}{2}x + \frac{1}{4}$



$2x - 3y = 6$

(2) Compute the equations of the lines through points:
  (a) $A = (1, 1)$, $B = (1, 3)$.
    Sol: $x = 1$
  (b) $A = (1, 1)$, $B = (3, 1)$.
    Sol: $y = 1$
  (c) $A = (-1, 5)$, $B = (2, 0)$.
    Sol: $y = -\frac{5}{3}x + \frac{10}{3}$
(3) Compute the point-slope equations of the lines:
  (a) $P = (1, -1)$, $m = \frac{1}{2}$.
    Sol: $y = \frac{1}{2}(x - 1) - 1$
  (b) $P = (0, 3)$, $m = -3$.
    Sol: $y = -3x + 3$
  (c) $P = (-2, 0)$, $m = 0$.
    Sol: $y = 0$

3. Set theory and Logic

(1) For each of the following sets, provide a representation in mathematics terms
  (a) **Sol:** $\{n \in \mathbb{N} : n/2 \in \mathbb{N}\}$, alternatively $\{n \in \mathbb{N} : n = 2k \forall k \in \mathbb{N}\}$;
  (b) **Sol:** $\{z \in \mathbb{Z} : -1 \le z \le 3\}$, alternatively $\{-1, 0, 1, 2, 3\}$;
  (c) **Sol:** $\{q \in \mathbb{Q} : -1 \le z \le 3\}$;
  (d) **Sol:** $\{q \in \mathbb{R} : -1 \le z \le 3\}$, alternatively $[-1, 3]$;
(2) Given $A = \{a, b, c, d\}$, say if the following propositions are true or false
  (a) $b \in A$
    Sol: True
  (b) $\{a, b\} \in A$
    Sol: False: the correct proposition would be $\{a, b\} \subset A$, since $\{a, b\}$ is a set
  (c) $\{a, b\} \subset A$
    Sol: True
  (d) $d \notin A$
    Sol: False

(e) $\emptyset \subset A$
Sol: True: the empty set is always a subset of any set. Indeed, this is the set with no elements.

(3) Let U be the universal set, A and B subsets of U. Determine, in each of the following cases $A \cup B$, $A \cap B$, A^c, B^c, $(A \cap B)^c$

(a) $U = \{-2, -1, 0, 1, 2\}$, $A = \{-1, 0\}$, $B = \{-2, 0, 1\}$
Sol:

$$
\begin{aligned}
A \cup B &= \{-2, -1, 0, 1\}, \\
A \cap B &= \{0\}, \\
A^c &= \{-2, 1, 2\}, \\
B^c &= \{-1, 2\}, \\
(A \cap B)^c &= \{-2, -1, 1, 2\}
\end{aligned}
$$

(b) $U = \mathbb{N}$, $A = \{n \in \mathbb{N} : n \leq 10\}$, $B = \{n \in \mathbb{N} : n \geq 2\}$
Sol:

$$
\begin{aligned}
A \cup B &= \mathbb{N}, \\
A \cap B &= \{n \in \mathbb{N} : 2 \leq n \leq 10\}, \\
A^c &= \{n \in \mathbb{N} : n \geq 11\} = \{n \in \mathbb{N} : n > 10\}, \\
B^c &= \{n \in \mathbb{N} : n \leq 1\} = \{0, 1\}, \\
(A \cap B)^c &= \{n \in \mathbb{N} : n \leq 1 \quad \text{or} \quad n \geq 11\}
\end{aligned}
$$

(c) $U = \mathbb{Z}$, $A = \{z \in \mathbb{Z} : 0 \leq z \leq 3\}$, $B = \{z \in \mathbb{Z} : -7 \leq zeq - 1\}$
Sol:

$$
\begin{aligned}
A \cup B &= \{z \in \mathbb{Z} : -7 \leq z \leq 3\}, \\
A \cap B &= \emptyset, \\
A^c &= \{z \in \mathbb{Z} : z < 0 \quad \text{or} \quad z \geq 4\}, \\
B^c &= \{z \in \mathbb{Z} : z \leq -8 \quad \text{or} \quad z \geq 0\}, \\
(A \cap B)^c &= \mathbb{Z}
\end{aligned}
$$

(d) $U = \mathbb{R}$, $A = \{x \in \mathbb{R} : x^2 \leq 4\}$, $B = \{x \in \mathbb{R} : x^2 > 1\}$
Sol: Notice first that the sets A and B correspond to $A = [-2, 2]$ and $B = (-\infty, -1] \cup [1, +\infty)$

$$
\begin{aligned}
A \cup B &= \mathbb{R}, \\
A \cap B &= [-2, -1] \cup [1, 2], \\
A^c &= (-\infty, -2) \cup (2, +\infty), \\
B^c &= (-1, 1), \\
(A \cap B)^c &= (-\infty, -2) \cup (-1, 1) \cup (2, +\infty)
\end{aligned}
$$

(4) Given sets A and B compute $A \setminus B$ and $B \setminus A$. Do you get the same set?

(a) $A = \{-3, -2, -1, 4, 5, 6\}$, $B = \{-3, -2, 0, 2, 6, 8\}$

Sol:

$$A \setminus B = \{-1, 4, 5\}, \quad B \setminus A = \{0, 2, 8\}$$

(b) $A = \{x \in \mathbb{R} : -4 \leq x \leq 4\}$, $B = \{x \in \mathbb{R} : x > 1\}$

Sol: Notice that sets A and B are also equal to $A = [-4, 4]$, $B = (1, +\infty)$

$$A \setminus B = [-4, 1], \quad B \setminus A = (4, +\infty)$$

(5) Translate in mathematical terms the following statements

(a) Every number in the set A has the opposite in A

Sol: $\forall x \in A, \exists -a \in A$

(b) There is only one number in A which is odd

Sol: $\exists! x \in A : x$ is odd.

Equivalently we can write $\exists! x \in A : \; x/2 \notin \mathbb{Z}$

(c) There are at least two numbers in A whose sum is equal to zero

Sol: $\exists x, y \in A : \; x + y = 0$

(6) For each of the following pair of propositions say if $P \Rightarrow Q$, $Q \Rightarrow P$, $P \Leftrightarrow Q$

(a) P: $x^2 = 4$; Q: $x = 2$.

Sol: $Q \Rightarrow P$

(b) P: $x^3 = 8$, Q: $x = 2$.

Sol: $P \Leftrightarrow Q$

(c) P: x is larger than 3; Q: x^2 is larger than 9.

Sol: $P \Rightarrow Q$

(d) P: x and y are negative; Q: $x + y$ is negative.

Sol: $P \Rightarrow Q$

(e) P: $(x^2 + 1)(y - 2) > 0$; Q: $y > 2$.

Sol: $P \Leftrightarrow Q$

(7) Say which of the following sets is open, closed, not open nor closed, bounded, unbounded

(a) $\{x \in \mathbb{R} : -4 \leq x \leq 4\}$.

Sol: This is equal to $[-4, 4]$, closed and bounded

(b) $\{x \in \mathbb{R} : x > 1\}$.

Sol: This is equal to $(-1, +\infty)$, open and unbounded

(c) $\{x \in \mathbb{R} : x^2 > 3\}$.

Sol: This is equal to $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, +\infty)$, open and unbounded

(d) $\{x \in \mathbb{R} : 2 < x^2 \leq 4\}$.

Sol: This is equal to $[-2, \sqrt{2}) \cup (\sqrt{2}, 2]$, not open nor closed and bounded

(e) $\{x \in \mathbb{R} : -1 < x < 5\}$.

Sol: This is equal to $(-1, 5)$, open and bounded