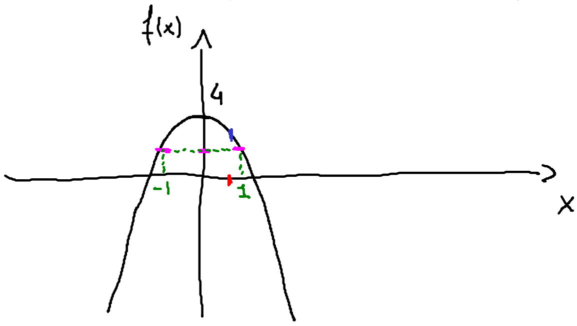
Mathematics 1 - Practice 3 09/30/2021

Ex 1

For each of the following functions, say if it is bijective and if so, compute the inverse.

1. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = 4 - x^2$ $\qquad x_{1,2} = \pm a \;\to\; f(x_{1,2}) = 4 - a^2$



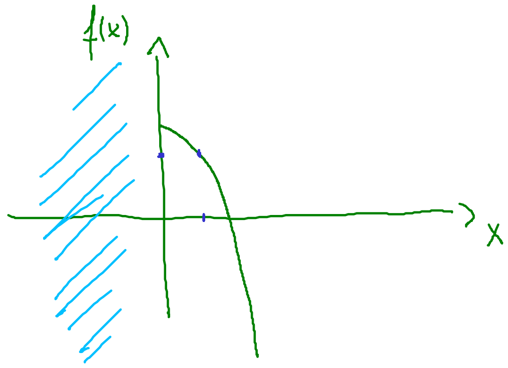
$x_1 = -1 \quad f(x_1) = 3$

$x_2 = 1 \quad f(x_2) = 3$

$f(x)$ is NOT bijective, hence it is NOT invertible

$f: [0, +\infty) \to (-\infty, 4]$

$f(x) = 4 - x^2$



$\downarrow$

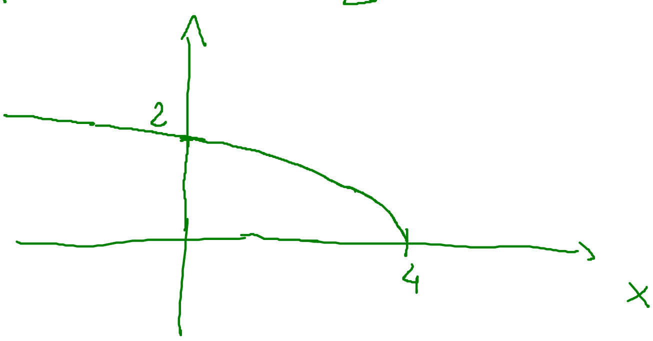
$y = 4 - x^2$

$y - 4 = -x^2 \qquad x^2 = 4 - y \qquad x = \sqrt{4-y}$

change of names

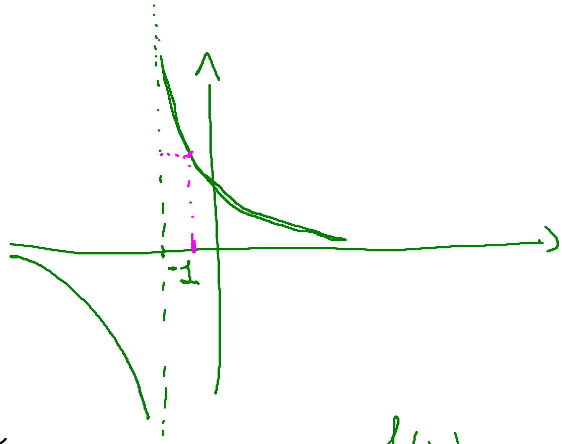
$f^{-1}(x) = \sqrt{4-x}$

$f: (-\infty, 4] \to [0, +\infty)$



2. $f : \mathbb{R} \setminus \{-1\} \longrightarrow \mathbb{R} \setminus \{0\}$

$$f(x) = \frac{1}{x+1}$$

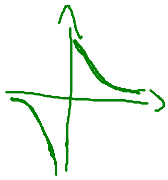


shift to the left

$f(x) \longrightarrow f(x+1)$

The function is bijective, we can compute the inverse

$\frac{1}{x}$



$$y = \frac{1}{x+1}$$

$$(x+1)y = 1 \qquad x+1 = \frac{1}{y} \qquad x = \frac{1}{y} - 1 \qquad \boxed{x = \frac{1-y}{y}}$$

$$\boxed{f^{-1}(y) = \frac{1-y}{y}}$$

Monotonic function	$\Rightarrow$	Invertible function
	BUT	
Invertible function	$\not\Rightarrow$	Monotonic function

Ex 2

Given functions $f(x)$ and $g(x)$, compute $f(g(x))$ and $g(f(x))$. Specify their domain and range.

1. $f : \mathbb{R} \to \mathbb{R} \quad f(x) = 2x+1$

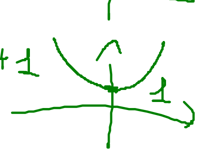
   $g : \mathbb{R} \to [0, +\infty) \quad g(x) = x^2$

$f(g(x))$

$2x^2$

$$f(g(x)) = 2g(x) + 1 = 2x^2 + 1$$

$2x^2+1$



$$f \circ g : \mathbb{R} \longrightarrow [1, +\infty)$$

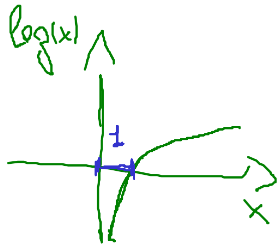
$$g(f(x))$$

$$g(f(x)) = f(x)^2 = (2x+1)^2$$

$$g \circ f : \mathbb{R} \longrightarrow [0, +\infty)$$

2. $f : [-1, 1] \to [0, 1]$ $\quad f(x) = \sqrt{1-x^2}$

$g : (0, +\infty) \to \mathbb{R}$ $\quad g(x) = \log(x)$



$$f(g(x))$$

$$f(g(x)) = \sqrt{1-g(x)^2} = \sqrt{1-[\log(x)]^2}$$

$\log(x^2) = 2\log(x)$	$\log^2(x) = (\log(x))^2$

DOMAIN

$\boxed{x > 0}$ C.E for log

$1 - \log^2(x) \geq 0$ C.E for $\sqrt{}$

$\log^2(x) - 1 \leq 0$ $\quad \underbrace{(\log(x)+1)}_{A}\underbrace{(\log(x)-1)}_{B} \leq 0$

A $\quad \log(x)+1 \geq 0 \quad \log(x) \geq -1 \quad e^{\log(x)} \geq e^{-1}$

$\boxed{x \geq \frac{1}{e}}$

B $\log(x) - 1 \geq 0$ $\quad \log(x) \geq 1 \quad e^{\log(x)} \geq e$

$\boxed{x \geq e}$

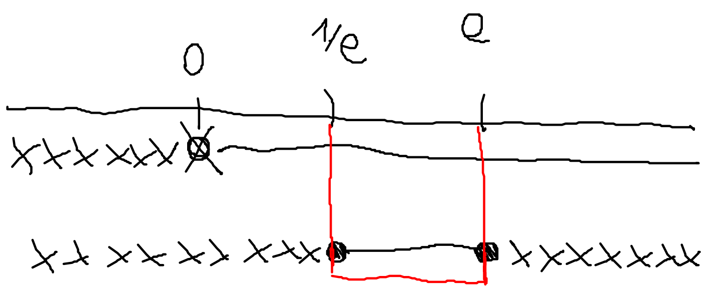
		$1/e$		e	
A $\log(x) + 1$	$-$		$+$		$+$
B $\log(x) - 1$	$-$		$-$		$+$
	$+$		$-$		$+$

$\boxed{\frac{1}{e} \leq x \leq e}$

"Total" C.E.

$x > 0$

$\frac{1}{e} \leq x \leq e$



$\boxed{1/e \leq x \leq e}$

$$f(g(x)) : \left[1/e, e\right] \longrightarrow \left[0, 1\right]$$

Domain $\quad$ Range

x $\quad$ y, $f(x)$

$g(f(x))$

$$g(f(x)) = \log(f(x)) = \log\left(\sqrt{1-x^2}\right)$$

Domain

$\sqrt{1-x^2} > 0 \implies$ A $\boxed{x \neq \pm 1}$ C.E. for the log argument

$1 - x^2 \geq 0$ C.E. for the $\sqrt{}$ argument

$(x-1)(x+1) \leq 0$

		-1		1	
$x-1$	$-$		$-$		$+$
$x+1$	$-$		$+$		$+$
	$+$		$-$		$+$

B

$\boxed{-1 \leq x \leq 1}$

A & B $\implies$ $\boxed{-1 < x < 1}$

$$g \circ f : (-1, 1) \longrightarrow (-\infty, 0]$$

$$\log(\dots\dots) \longrightarrow (-\infty, +\infty)$$

$$\log\left(\sqrt{\dots\dots}\right) \longrightarrow (-\infty, +\infty)$$

$$\log\left(\sqrt{1-x^2}\right) \longrightarrow (-\infty, 0]$$

1 2 3 6

$$1-x^2 \leq 1 \Rightarrow \sqrt{1-x^2} \leq 1 \Rightarrow \boxed{\log\left(\sqrt{1-x^2}\right) \leq \log(1)}$$

$\log(1) = 0$

$(1-x^2) \geq 0$

Ex 3

Given the function $h = f(g(x))$ determine

a) Domain of h

b) functions f and g

c) domain of f and g

1. $h(x) = \sqrt{\log(x)}$ $\qquad D_h = [1, +\infty)$

$f(x) = \sqrt{x}$ $\qquad D_f = [0, +\infty)$

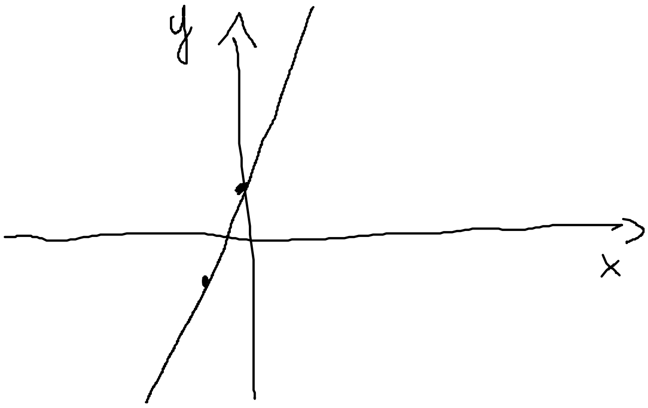
$g(x) = \log(x)$ $\qquad D_g = (0, +\infty)$

2. $h(x) = \dfrac{1}{e^x - 3}$ DO IT ON YOUR OWN!

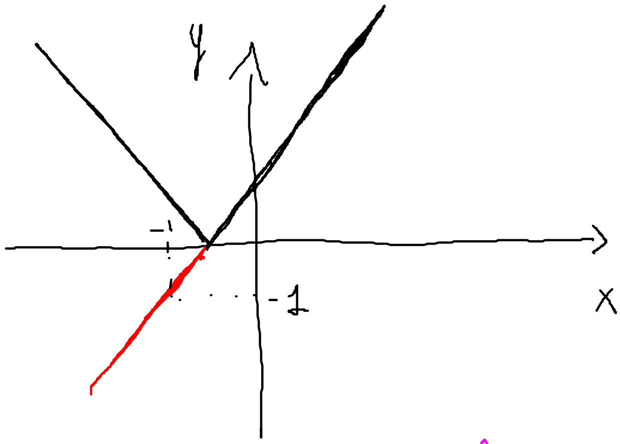
Ex 4

Draw the graph of $f(x)$, $|f(x)|$, $-f(x)$, $f(x)+a$, $f(x+a)$, $f(-x)$

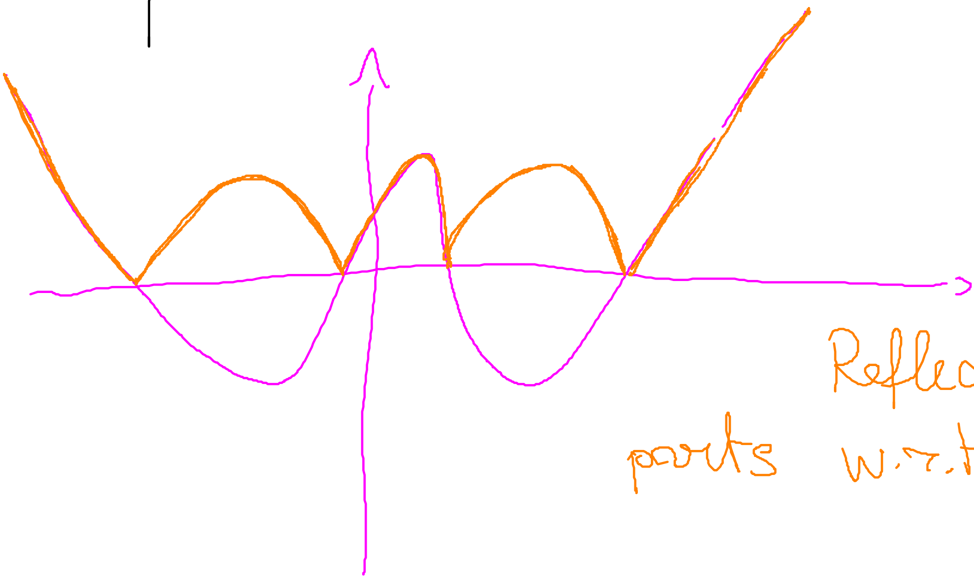
1. $f(x) = 2x + 1$



$f(x)$



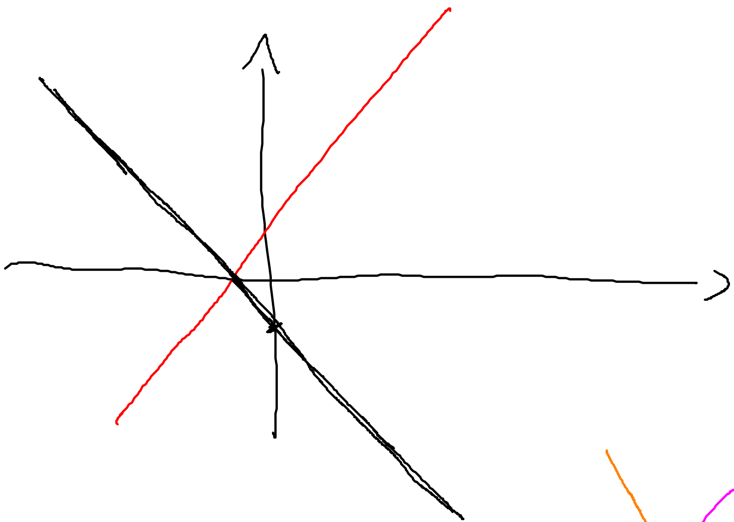
$|f(x)|$



$f(x)$

$|f(x)|$

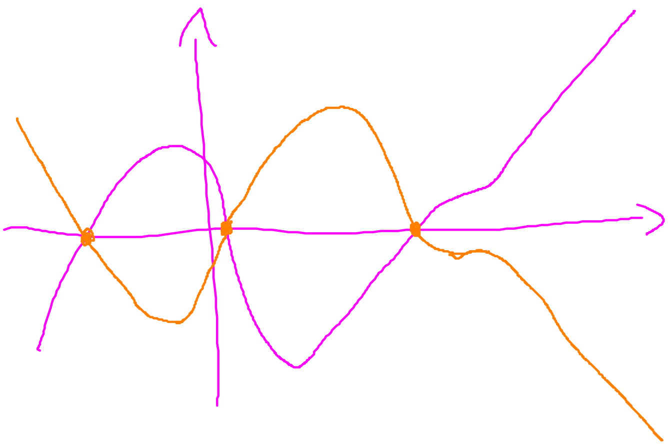
Reflect the negative parts w.r.t the x-axis



$f(x)$

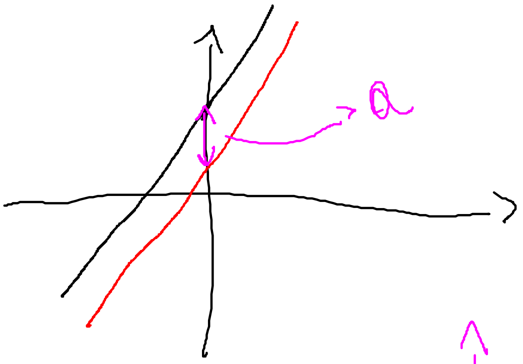
$-f(x) = -2x - 1$

The function is reflected w.r.t x-axis



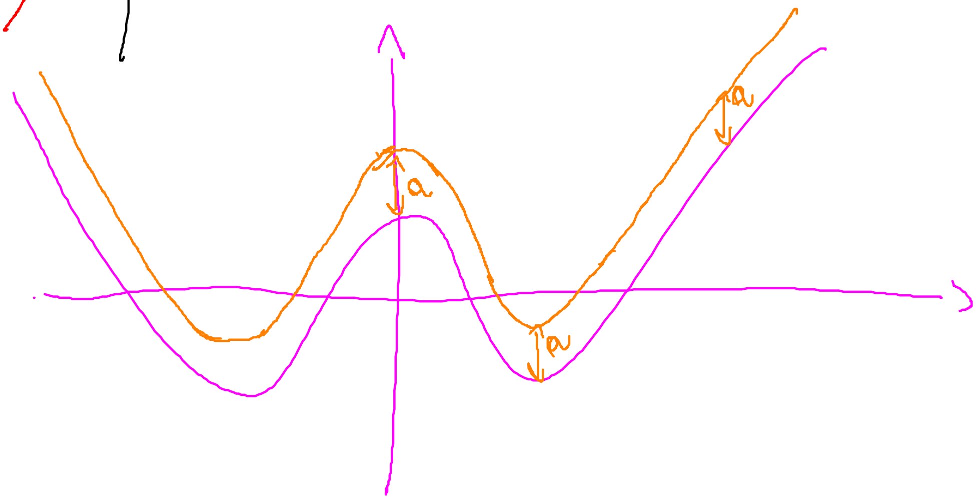
$f(x)$

$-f(x)$



$f(x)$

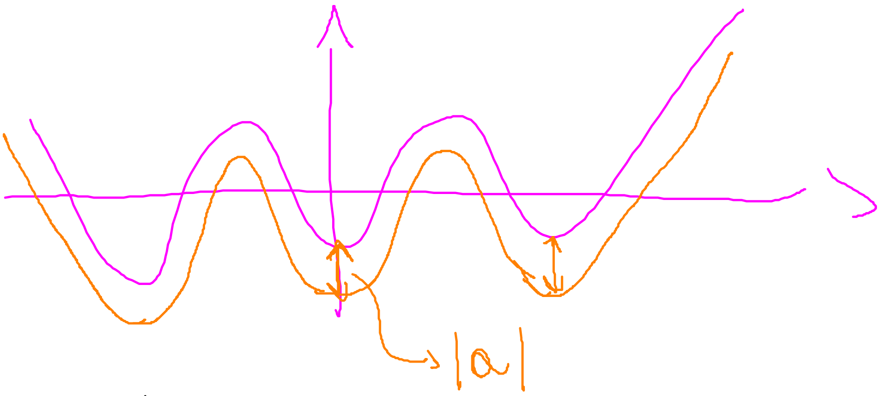
$f(x)+a = 2x+1+a$



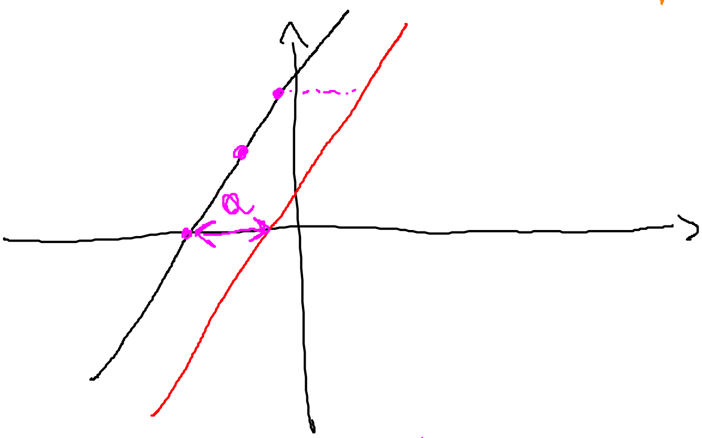
$f(x)$

$f(x)+a$

$a>0$

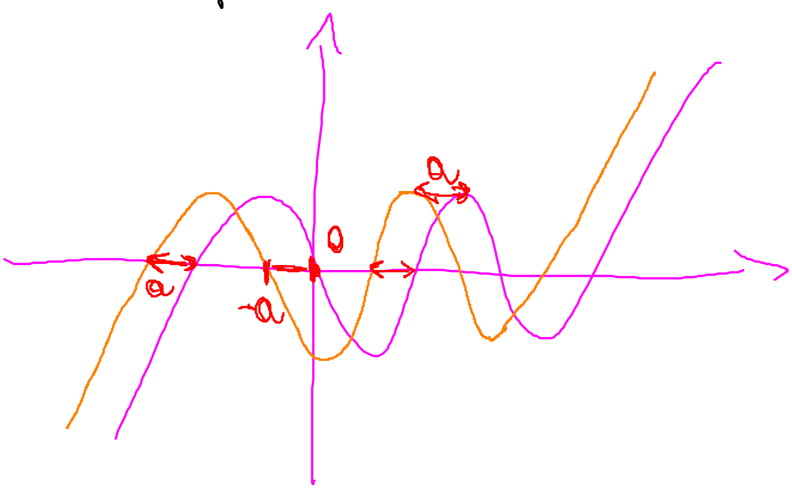


$a<0$



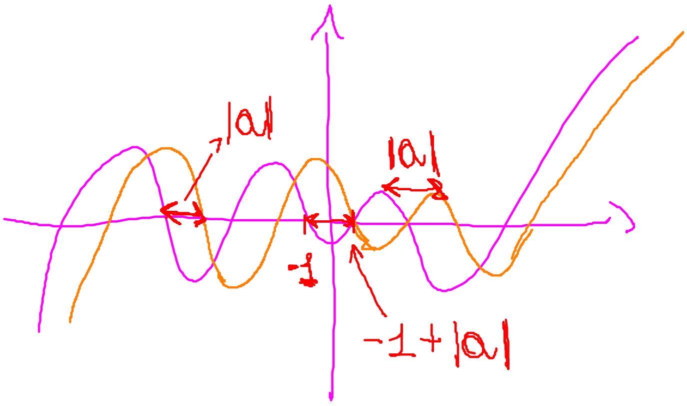
$f(x)$

$f(x+a) = 2(x+a)+1$



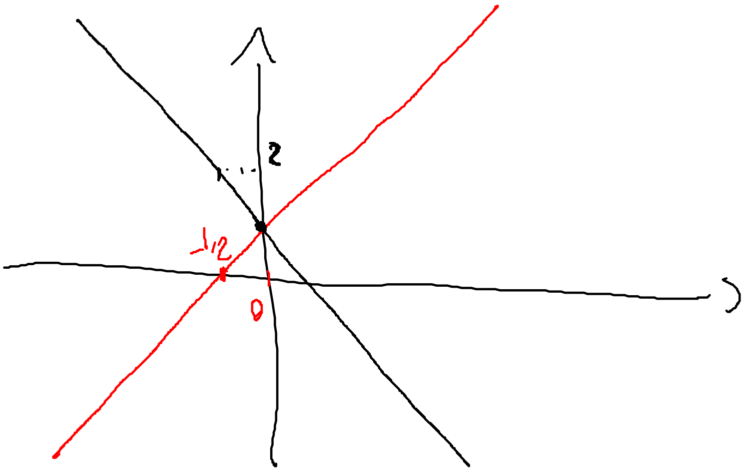
$f(x)$

$f(x+a)$



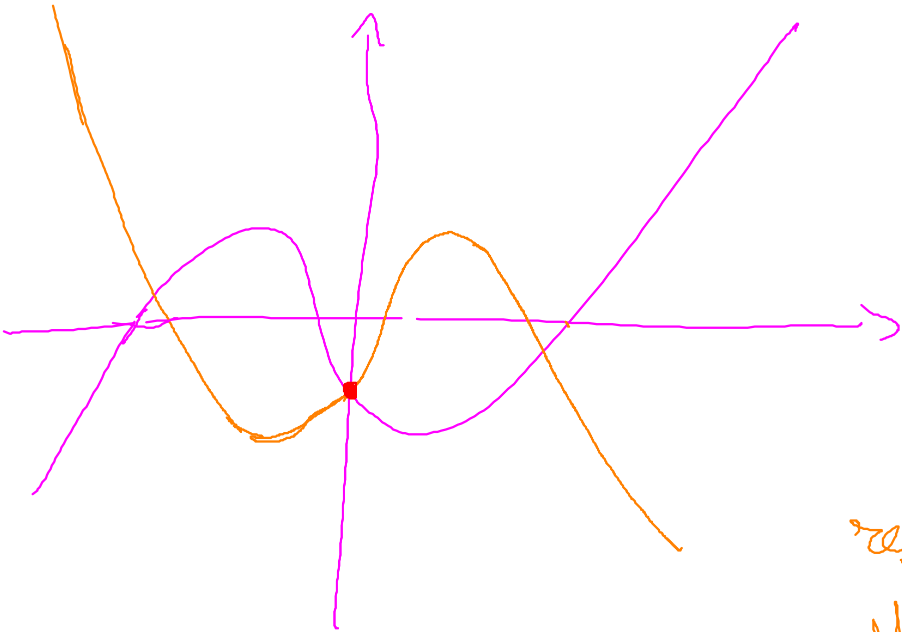
$f(x)$

$f(x+a)$ $\quad a<0$



$f(x) = 2x+1$

$f(-x) = -2x+1$



$f(x)$

$f(-x)$

The function is reflected w.r.t. the y-axis.

What happens with $f(|x|)$?