MATHEMATICS 1
ADDITIONAL EXERCISES N. 2

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Notation: log stands for the natural logatithm (i.e. the logarithm with the basis e)

1. Domain and Range

(1) Compute the domain of the following functions

$$(a)\quad f(x) = \frac{5-2x}{x^2}, \quad \textbf{Sol:} \quad D = \{x \in \mathbb{R} : x \neq 0\} = \mathbb{R} \setminus \{0\}$$

$$(b)\quad f(x) = \frac{x+2}{x^2+x+1}, \quad \textbf{Sol:} \quad D = \mathbb{R}$$

$$(c)\quad f(x) = \frac{3x-4}{5-x^2}, \quad \textbf{Sol:} \quad D = \left\{x \in \mathbb{R} : x \neq \pm\sqrt{5}\right\} = \mathbb{R} \setminus \left\{-\sqrt{5}, \sqrt{5}\right\}$$

$$(d)\quad f(x) = \sqrt{(x-2)(x-1)}, \quad \textbf{Sol:} \quad D = \{x \in \mathbb{R} : x \leq 1 \quad \text{or} \quad x \geq 2\} = (-\infty\,,\, 1] \cup [2,\, \infty\,)$$

$$(e)\quad f(x) = \sqrt{4x^2-9}, \quad \textbf{Sol:} \quad D = \left\{x \in \mathbb{R} : \, x \leq -\frac{3}{2} \quad \text{or} \quad x \geq \frac{3}{2}\right\} = \left(-\infty\,,\, -\frac{3}{2}\right] \cup \left[\frac{3}{2},\, \infty\right)$$

$$(f)\quad f(x) = \sqrt[3]{\frac{x}{x-3}}, \quad \textbf{Sol:} \quad D = \{x \in \mathbb{R} : x \neq 3\} = \mathbb{R} \setminus \{3\}$$

$$(g)\quad f(x) = \frac{x+4}{\sqrt[3]{x^2-1}}, \quad \textbf{Sol:} \quad D = \{x \in \mathbb{R} : x \neq \pm 1\} = \mathbb{R} \setminus \{-1, 1\}$$

$$(h)\quad f(x) = \sqrt{x} + \sqrt{x+1}, \quad \textbf{Sol:} \quad D = \{x \in \mathbb{R} : x \neq\geq 0\} = [0, +\infty)$$

$$(i)\quad f(x) = \frac{x+2}{x-\sqrt{x+2}},$$

$$\textbf{Sol:} \quad D = \{x \in \mathbb{R} : -2 \leq x < 2 \quad \text{or} \quad x > 2\} = [-2, +\infty) \setminus \{2\} = [-2,\, 2) \cup (2,\, \infty\,)$$

$$(j)\quad f(x) = \frac{1}{|x+1|-|x|}, \quad \textbf{Sol:} \quad D = \left\{x \in \mathbb{R} : x \neq -\frac{1}{2}\right\} = \mathbb{R} \setminus \left\{-\frac{1}{2}\right\}$$

$$(k)\quad f(x) = \sqrt{\frac{x}{x+7}}, \quad \textbf{Sol:} \quad D = \{x \in \mathbb{R} : \, x < -7 \quad \text{or} \quad x \geq 0\} = (-\infty\,,\, -7) \cup [0,\, \infty\,)$$

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(l) $f(x) = \dfrac{\cos(x)}{2\sin(x)-1}$,

Sol: $D = \left\{x \in \mathbb{R} : x \neq \dfrac{\pi}{6} + 2\pi n,\ x \neq \dfrac{5\pi}{6} + 2\pi n,\ n \in \mathbb{Z}\right\} = \mathbb{R} \setminus \left\{\dfrac{\pi}{6} + 2\pi n, \dfrac{5\pi}{6} + 2\pi n\right\}_{n\in\mathbb{Z}}$

(m) $f(x) = e^{\sqrt{3x^2-4}}$, **Sol:** $D = \left\{x \in \mathbb{R} :\ x \leq\ -\dfrac{2}{\sqrt{3}} \quad \text{or} \quad x \geq \dfrac{2}{\sqrt{3}}\right\} = \left(-\infty\ ,\ -\dfrac{2}{\sqrt{3}}\right] \cup \left[\dfrac{2}{\sqrt{3}},\ \infty\right)$

(n) $f(x) = \dfrac{\log(1-x)}{2x-1}$, **Sol:** $D = \left\{x \in \mathbb{R} : x < \dfrac{1}{2} \quad \text{or} \quad \dfrac{1}{2} < x < 1\right\} = \left(-\infty\ ,\ \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2},\ 1\right)$

(o) $f(x) = \dfrac{\sqrt{3-x}}{\log(x^2+1)}$, **Sol:** $D = \{x \in \mathbb{R} :\ x < 0 \quad \text{or} \quad 0 < x \leq\ 3\} =\ (-\infty\ ,\ 0) \cup (0,\ 3]$

(p) $f(x) = \dfrac{\sqrt{e^{2x}+e^x+1}}{\sqrt{e^{2x}-1}}$, **Sol:** $D = \{x \in \mathbb{R} :\ x > 0\ \} = (0, +\infty)$

(q) $f(x) = \log\left(\dfrac{x+3}{x-1}\right)$, **Sol:** $D = \{x \in \mathbb{R} :\ x < -3 \quad \text{or} \quad x > 1\} = (-\infty\ ,\ -3) \cup (1,\ \infty\ )$

(r) $f(x) = \log(x+3) - \log(x-1)$, **Sol:** $D = \{x \in \mathbb{R} :\ x > 1\} = (1, +\infty)$

(s) $f(x) = \sqrt{\dfrac{\log(1-x)}{\log(x^2-9)}}$, **Sol:** $D = \left\{x \in \mathbb{R} : x < -\sqrt{10}\right\} = \left(-\infty\ ,\ -\sqrt{10}\right)$

(t) $f(x) = \log\left(\dfrac{x^2+4}{1-4x^2}\right)$, **Sol:** $D = \left\{x \in \mathbb{R} : -\dfrac{1}{2} < x < \dfrac{1}{2}\right\} = \left(-\dfrac{1}{2},\ \dfrac{1}{2}\right)$

(2) Compute the Range of the following functions

(a) $f(x) = \dfrac{5}{x^2}$, **Sol:** $R_f = \{y \geq 0\} = (0, +\infty)$

(b) $f(x) = x^2 - 3x + 1$, **Sol:** $R_f = \left\{y \geq -\dfrac{5}{4}\right\} = \left[-\dfrac{5}{4}, +\infty\right)$

(c) $f(x) = \sqrt{(x-2)(x-1)}$, **Sol:** $R_f = \{y \geq 0\} = [0, +\infty)$

(d) $f(x) = \sqrt{4x^2-9}$, **Sol:** $R_f = \{y \geq 0\} = [0, +\infty)$

(e) $f(x) = \sqrt{x} + \sqrt{x+1}$, **Sol:** $R_f = \{y \geq 1\} = [1, +\infty)$

(f) $f(x) = e^{\sqrt{3x^2-4}}$, **Sol:** $R_f = \{y \geq 1\} = [1, +\infty)$

(g) $f(x) = \log(x+3)$, **Sol:** $R_f = \mathbb{R}$

(h) $f(x) = \dfrac{1}{x^2+5}$, **Sol:** $R_f = \left\{0 < y \leq \dfrac{1}{5}\right\} = \left(0, \dfrac{1}{5}\right]$

(i) $f(x) = x^4 + x^2 - 3$, **Sol:** $R_f = \{y \geq -3\} = [-3, +\infty)$

Detailed solutions:

$$(a) \quad f(x) = \frac{5}{x^2}$$

For every $x \in \mathbf{R}$, $x^2 \geq 0$. Then, $\frac{5}{x^2} \geq 0$. This implies that the range is $R_f = \{y \in \mathbb{R} : y \geq 0\}$.

$$(b) \quad f(x) = x^2 - 3x + 1$$

Notice that $y = x^2 - 3x + 1$ represents a parabola, it is convex because the coefficient of x^2 is positive. Then the minimum value is attained at the vertex $V = \left(\frac{3}{2}, -\frac{5}{4}\right)$. Hence $y \geq -\frac{5}{4}$. See also the plot. Then we can say that the range is $R_f = \left\{y \in \mathbb{R} : y \geq -\frac{5}{4}\right\}$

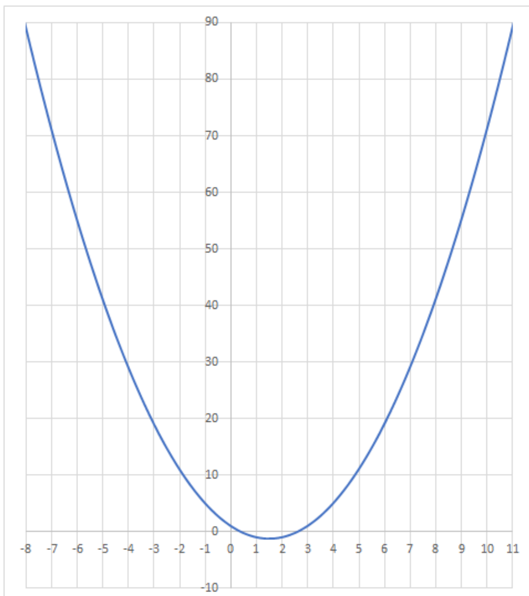


FIGURE 1. Exercise 2 (b)

$$(c) \quad f(x) = \sqrt{(x-2)(x-1)}$$

The domain of this function is $D = (-\infty, 1] \cup [2, +\infty)$. For every $x \in D$, the argument of the square root is positive. Hence, since the square root of a positive number is positive we get that $y = \sqrt{(x-2)(x-1)} \geq 0$. This implies that the range is $R_f = \{y \in \mathbb{R} : y \geq 0\}$

$$(d) \quad f(x) = \sqrt{4x^2 - 9}$$

The domain of this function is $D = \left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{3}{2}, +\infty\right)$. For every $x \in D$, the argument of the square root is positive. Hence, since the square root of a positive number is positive we get that $y = \sqrt{4x^2 - 9} \geq 0$. This implies that the range is $R_f = \{y \in \mathbb{R} : y \geq 0\}$

$$(e) \quad f(x) = \sqrt{x} + \sqrt{x+1}$$

The domain of this function is $D = [0, +\infty)$. For every $x \in D$, $\sqrt{x} \geq 0$ and $\sqrt{x+1} \geq 1$. Then $y = \sqrt{x} + \sqrt{x+1} \geq 0 + 1 = 1$. This implies that the range is $R_f = \{y \in \mathbb{R} : y \geq 1\}$

$$(f) \quad f(x) = e^{\sqrt{3x^2-4}}$$

The domain of this function is $D = \left(-\infty, -\frac{2}{\sqrt{3}}\right] \cup \left[\frac{2}{\sqrt{3}}, +\infty\right)$. For every $x \in D$, the argument of the square root is positive. Hence, since the square root of a positive number is positive we get that $y = e^{\sqrt{3x^2-4}} \geq e^0 = 1$. This implies that the range is $R_f = \{y \in \mathbb{R} : y \geq 1\}$

$$(g) \quad f(x) = \log(x+3)$$

The domain of this function is $D = (-3, +\infty)$. For all $x \in D$ the argument of the logarithm is larger or equal than zero and hence the logarithm exists. Moreover it can take any value in $\mathbb{R}$ (positive or negative), that is $y = \log(x+3) \in \mathbb{R}$. Hence the range is $R_f = \mathbb{R}$

$$(h) \quad f(x) = \frac{1}{x^2+5}$$

Recall that for every $x \in \mathbf{R}$, $x^2 \geq 0$ and hence $x^2+5 \geq 5$. Therefore $\frac{1}{x^2+5} \leq \frac{1}{5}$. Moreover, it is immediate to see that $\frac{1}{x^2+5} \geq 0$, because both numerator and denominator are positive. Hence $0 \leq \frac{1}{x^2+5} \leq \frac{1}{5}$. Put in other words, the range is $R_f = \left\{y \in \mathbb{R} : 0 \leq y \leq \frac{1}{5}\right\}$

$$(i) \quad f(x) = x^4 + x^2 - 3$$

Recall that $x^4 \geq 0$ and also $x^2 \geq 0$. Then $y = x^4 + x^2 - 3 \geq 0 + 0 - 3 = -3$. This implies that $R_f = \{y \in \mathbb{R} : y \geq -3\}$

2. Even/Odd functions

(1) For each of the following functions say if they are even, odd or neither

$$(a) \quad f(x) = \frac{2x^2+1}{4x^4}$$

Detailed solution: The domain of this function is $D = \mathbb{R} \setminus \{0\}$.
For all $x \in D$, we have that $-x \in D$. Moreover,

$$f(-x) = \frac{2(-x)^2+1}{4(-x)^4} = \frac{2x^2+1}{4x^4} = f(x)$$

Hence the function is even.

$$(b) \quad f(x) = \frac{1}{x^3+x}$$

Detailed solution: The domain of this function is $D = \mathbb{R} \setminus \{0\}$.
For all $x \in D$, we have that $-x \in D$. Moreover,

$$f(-x) = \frac{1}{(-x)^3+(-x)} = -\frac{1}{x^3+x} = -f(x)$$

Hence the function is odd.

$$(c) \quad f(x) = \frac{x^3}{x^4+1}$$

Detailed solution: The domain of this function is $D = \mathbb{R}$.
For all $x \in D$, we have that $-x \in D$. Moreover,

$$f(-x) = \frac{(-x)^3}{(-x)^4+1} = -\frac{x^3}{x^4+1} = -f(x)$$

Hence the function is odd.

$$(d) \quad f(x) = \sqrt{25 - x^2}$$

Detailed solution: The domain of this function is $D = (-\infty, -5] \cup [5, +\infty)$.
For all $x \in D$, we have that $-x \in D$. Moreover,

$$f(-x) = \sqrt{25 - (-x)^2} = \sqrt{25 - x^2} = f(x)$$

Hence the function is even.

$$(e) \quad f(x) = \frac{x^4 - x^3}{x^4+1},$$

Detailed solution: The domain of this function is $D = \mathbb{R}$.
For all $x \in D$, we have that $-x \in D$. However,

$$f(-x) = \frac{(-x)^4 - (-x)^3}{(-x)^4+1} = \frac{x^4 + x^3}{x^4+1}$$

Hence the function is not even nor odd.

$$(f) \quad f(x) = |x+5|,$$

Detailed solution: The domain of this function is $D = \mathbb{R}$.
For all $x \in D$, we have that $-x \in D$. However,

$$f(-x) = |-x+5|$$

Hence the function is not even nor odd.

$$(g) \quad f(x) = \log(x+3),$$

Detailed solution: The domain of this function is $D = (-3, +\infty)$.
Then there exists $x \in D$, such that $-x \notin D$. For example pick $x = 5$, then $-x = -5$ which is not in the domain. Hence the function is not even nor odd.

$$(h) \quad f(x) = \frac{x^3}{4|x|+3},$$

Detailed solution: The domain of this function is $D = \mathbb{R}$.
For all $x \in D$, we have that $-x \in D$. Moreover,

$$f(-x) = \frac{(-x)^3}{4|-x|+3} = -\frac{x^3}{4|x|+3} = -f(x)$$

Hence the function is odd.

3. Increasing/decreasing functions

(1) Show, using the definition, that the following functions are increasing/decreasing on the indicated interval

(a) $f(x) = x^2 + 4$, is strictly increasing on $I = (0, +\infty)$
Detailed solution: The domain of this function is $D = \mathbb{R}$, hence $I \subset D$.
Let $x_1, x_2 \in I$ (hence x_1 and x_2 are both positive). We want to show that if $x_2 > x_1$ then $f(x_2) > f(x_1)$.
By applying then properties of operations on real numbers we have that

$$x_2 > x_1 \Rightarrow x_2^2 > x_1^2 \Rightarrow x_2^2 + 4 > x_1^2 + 4$$

(b) $f(x) = \frac{1}{x-3}$, is strictly decreasing on $I = (-\infty, 3)$
Detailed solution: The domain of this function is $D = \mathbb{R} \setminus \{3\}$, hence $I \subset D$.
Let $x_1, x_2 \in I$. We want to show that if $x_2 > x_1$ then $f(x_2) < f(x_1)$.
By applying then properties of operations on real numbers we have that

$$x_2 > x_1 \Rightarrow x_2 - 3 > x_1 - 3 \Rightarrow \frac{1}{x_2 - 3} < \frac{1}{x_1 - 3}$$

(c) $f(x) = \sqrt{x+1}$, is strictly increasing on $I = (-1, +\infty)$
Detailed solution: The domain of this function is $D = [-1, +\infty)$, and hence $I \subset D$.
Let $x_1, x_2 \in I$. We want to show that if $x_2 > x_1$ then $f(x_2) > f(x_1)$.
By applying then properties of operations on real numbers we have that

$$x_2 > x_1 \Rightarrow x_2 + 1 > x_1 + 1 \Rightarrow \sqrt{x_2 + 1} > \sqrt{x_1 + 1}$$

(d) $f(x) = -x^3 - 1$ is strictly decreasing on $\mathbb{R}$
Detailed solution: The domain of this function is $D = \mathbb{R} = I$.
Let $x_1, x_2 \in I$. We want to show that if $x_2 > x_1$ then $f(x_2) < f(x_1)$.
By applying then properties of operations on real numbers we have that

$$x_2 > x_1 \Rightarrow x_2^3 > x_1^3 \Rightarrow -x_2^3 < -x_1^3 \Rightarrow -x_2^3 - 1 < -x_1^3 - 1$$

4. Injective/Surjective/Bijective functions

(1) For each of the following functions say if they are injective, surjective, bijective

(a) $f : [0, +\infty) \to \mathbb{R}$, such that $f(x) = \sqrt{x}$

Detailed solution: *Injective*: Let $x_1, x_2 \in [0, +\infty)$. We use the equivalent formulation of injectivity and we show that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

$$f(x_2) = \sqrt{x_1}, \qquad f(x_2) = \sqrt{x_2}$$

Then we have

$$\sqrt{x_1} = \sqrt{x_2}$$

We can elevate both sides of the equality to the power 2, because both x_1 and x_2 are positive, and we get

$$x_1 = x_2$$

Therefore the function is injective.

Surjective: The function is not surjective because for instance if we pick $-1 \in \mathbb{R}$, there is no value of $x \in [0, +\infty)$ such that $\sqrt{x} = -1$

Bijective: The function is not bijective because, in particular, is not surjective.

(b) $f : [0, +\infty) \to [0, +\infty)$, such that $f(x) = \sqrt{x}$

Detailed solution: *Injective*: Let $x_1, x_2 \in [0, +\infty)$. We sue the equivalent formulation of injectivity and we show that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

$$f(x_2) = \sqrt{x_1}, \qquad f(x_2) = \sqrt{x_2}$$

Then we have

$$\sqrt{x_1} = \sqrt{x_2}$$

We can elevate both sides of the equality to the power 2, because x_1 and x_2 are positive and we get

$$x_1 = x_2$$

Therefore the function is injective.

Surjective: It is clear that for all $y \in [0, +\infty)$, the equation $y = \sqrt{x}$ has a solution which is given by $x = y^2$. The resulting value of x is positive, then $x \in D$, and therefore the function is surjective.

Bijective: The function is bijective.

(c) $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = x^3 + 1$

Detailed solution: *Injective*: Let $x_1, x_2 \in [0, +\infty)$. We sue the equivalent formulation of injectivity and we show that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

$$f(x_1) = x_1^3 + 1, \qquad f(x_2) = x_2^3 + 1$$

Then we have

$$x_1^3 + 1 = x_2^3 + 1$$
$$x_1^3 = x_2^3$$
$$x_1 = x_2$$

In the last step we have just taken the third root on both sides of the equality. Then the function is injective.

Surjective: It is clear that for all $y \in \mathbb{R}$, the equation $y = x^3 + 1$ has a solution which is given by $x = \sqrt[3]{y-1}$. The resulting value of x is a real number (could be either positive or negative), then $x \in D$, and therefore the function is surjective.

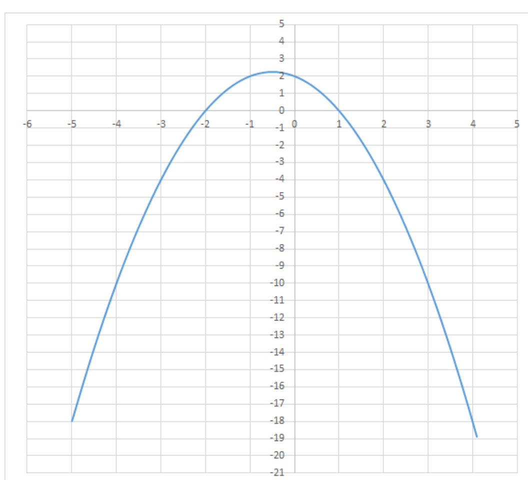
Bijective: The function is bijective.

(d) $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = -x^2 - x + 2$ **Detailed solution:** *Injective*: The function is not injective, because for instance if we pick $x_1 = 0$ and $x_2 = 1$, then $f(x_1) = 0 + 0 + 2 = 2$ and $f(x_2) = -1 + 1 + 2 = 2$. Therefore we have found two points $x_1 \neq x_2$ with the same image $f(x_1) = f(x_2)$.

Surjective: The function is not surjective, because for instance, the equation $-x^2 - x + 2 = 3$ does not have a solution. Indeed,

$$-x^2 - x + 2 = 3$$
$$-x^2 - x - 1 = 3$$
$$x = \frac{1 \pm \sqrt{1-4}}{-2}$$

hence the equation does not have any solution. It is also clear from the plot of the function that this is not surjective:



Bijective: The function is not bijective.

(e) $f : \left(-\infty, \frac{9}{4}\right] \to \mathbb{R}$, such that $f(x) = -x^2 - x + 2$
Detailed solution: *Injective*: The function is not injective. Cfr the exercise above.

Surjective: The function is not surjective, because for instance, the equation $-x^2 - x + 2 = 3$ does not have a solution. Cfr the exercise above.

Bijective: The function is not bijective.

(f) $f : \mathbb{R} \to [0, 1]$, such that $f(x) = 1 - \sin^2(x)$
Detailed solution: *Injective*: The function is not injective because it is periodic (then it repeats after 2π).
Surjective: The function is surjective. Indeed for every $y \in [0, 1]$, the equation $y = 1 - \sin^2(x)$ has at least one solution.

Bijective: The function is not bijective.

(g) $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{0\}$, such that $f(x) = \frac{1}{x-1}$
Detailed solution: *Injective*: Let $x_1, x_2 \in [0, +\infty)$. We sue the equivalent formulation of injectivity and we show that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

$$f(x_1) = frac1x_1 - 1, \qquad f(x_2) = frac1x_2 - 1$$

Then we have

$$\frac{1}{x_1 - 1} = \frac{1}{x_2 - 1} \Rightarrow \frac{x_2 - 1}{(x_1 - 1)(x_2 - 1)} = \frac{x_1 - 1}{(x_1 - 1)(x_2 - 1)}$$
$$\Rightarrow x_2 - 1 = x_1 - 1$$
$$\Rightarrow x_2 = x_1$$

Therefore the function is injective.

Surjective: It is clear that for all $y \in \mathbb{R} \setminus 0$, the equation $y = \frac{1}{x-1}$ has a solution which is given by $x = \frac{1}{y} + 1$. The resulting value of x cannot be equal to 1 because $\frac{1}{y} \neq 0$, then $x \in D$, and therefore the function is surjective.

Bijective: The function is bijective.